Lecture (Feb. 26th, 2019)

General Three-Dimensional Stresses

With the exception of contact stress, most maximum stress states occur under plane (2-D) stress conditions.

In the presence of 3-D stresses, the three principal stresses are found by solving the following equilibrium equation:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \quad (3\text{–}15)$$

The root of which are:

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

The three principal shear stresses are:

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \quad (3\text{–}16)$$

Hooke's Law

$$F = k\delta$$

Or:

$$\delta = \frac{F}{k}$$

Where:
F = The applied force
k = Spring rate (elastic zone)
δ = Resulting spring deflection

We also have:

$$\sigma = \frac{F}{A} \; or \; \frac{P}{A} \quad (1)$$

$$\varepsilon = \frac{\delta}{L} \quad (2)$$

$$\sigma = \varepsilon E \; or \; \varepsilon = \frac{\sigma}{E} \quad (3)$$

Substituting (1) and (3) into (2):

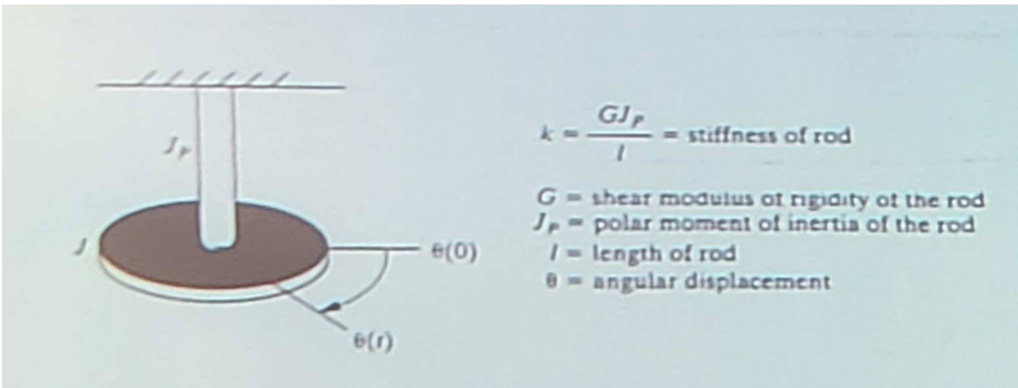
$$\delta = \frac{Fl}{AE} = \frac{Pl}{AE} = \frac{F}{k}$$

Or:

$$k = \frac{AE}{l}$$

Where:
$A =$ cross-section
$E =$ modulus of elasticity
$l =$ length

<u>Torsion</u>



Hooke's Law:

$$T = k_t \theta$$

Or:

$$\theta = \frac{T}{k_t}$$

$$\tau = \frac{Tr}{J_p} \quad (1)$$

$$\gamma = \frac{r\theta}{l} \quad (2)$$

$$\tau = \gamma G \quad or \quad \gamma = \frac{\tau}{G} \quad (3)$$

Substituting (1) and (3) into (2):

$$\frac{Tr}{J_p G} = \frac{r\theta}{l}$$

Or:

$$\theta = \frac{TL}{J_p G} = \frac{T}{k_t}$$

$$k_t = \frac{J_p G}{L}$$

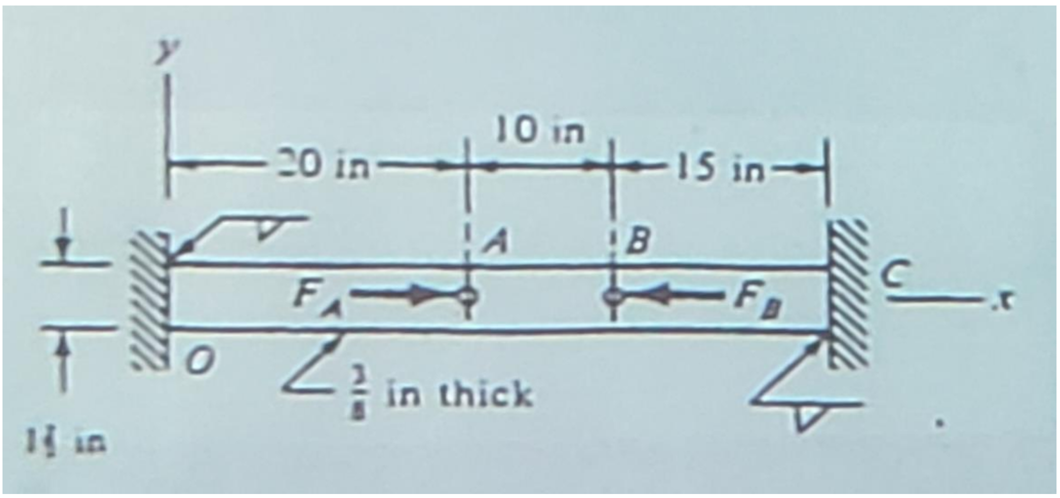
Statically Indeterminate Problems

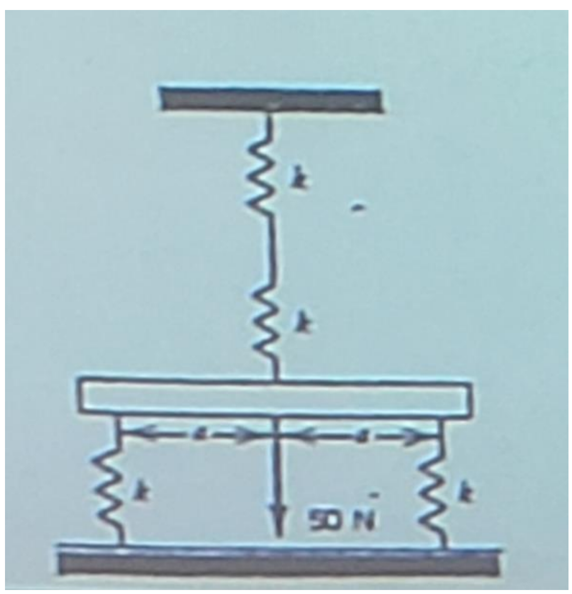
Statically indeterminate systems are characterized by the presence of more supports or members than the minimum required for the equilibrium of the structure. For such situations, the deformations of the parts must be taken into consideration.

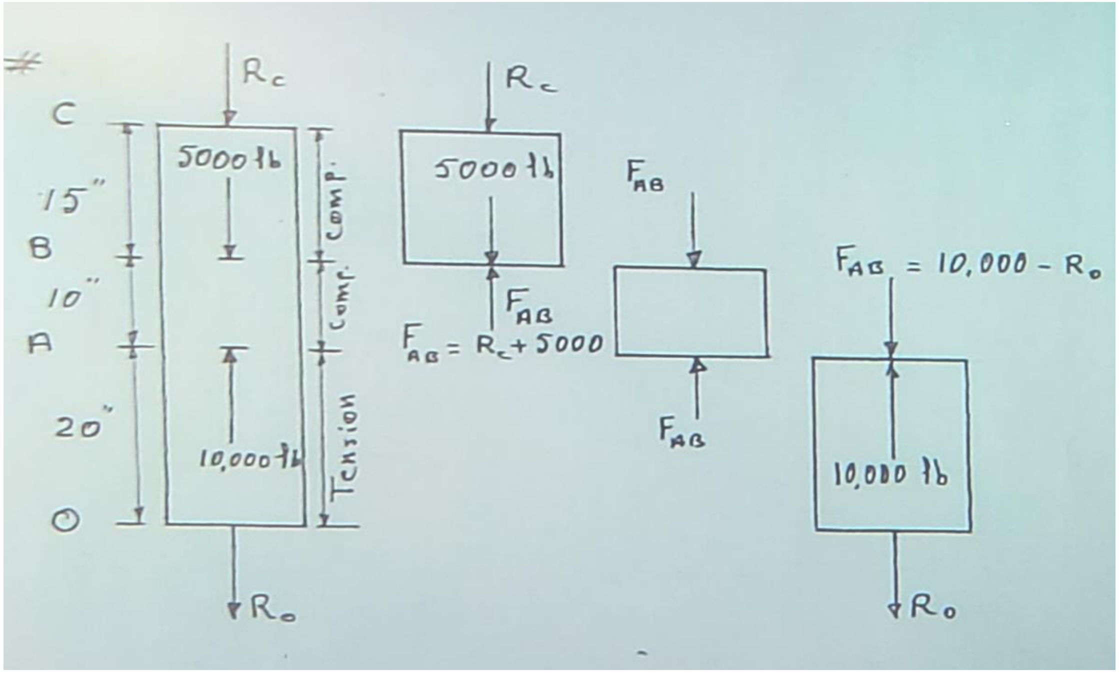
Example:

a- The figure shows a $\frac{3}{8}$ x $1\frac{1}{2}$ in rectangular steel bar welded to fixed supports as each end. The bar is axially loaded by the forces $F_A = 10{,}000\ lb$ and $F_a = 5{,}000\ lb$ acting on pins at A and B. Assuming that the bar will not buckle laterally, find the reactions at the fixed supports.

b- A very stiff horizonal bar, supported by four identical spring as shown is subjected to a center load of $50\ N$. What load is applied to each spring?







$$\therefore R_c + 5{,}000 = 10{,}000 - R_o$$

$$\therefore R_c + R_o = 5000 \quad (1)$$

Also:

$$\delta_{OA} = \delta_{AB} + \delta_{BC}$$

Or:

$$\frac{20R_o}{AE} = \frac{10(R_c + 5000)}{AE} + \frac{15R_c}{AE}$$

$$20R_o = 50{,}000 + 25R_c$$

From (1): $R_o = 5000 - R_c$

Substituting in (2): $20(5000 - R_c) = 50{,}000 + 25R_c$

$$or\ R_c = 1111.11\ lb$$

$$and\ R_o = 5000 - 1111.111 = 3888.888\ lb$$

$$R_c \cong 1111\ lb$$
$$R_o \cong 3889\ lb$$

Upper springs each deflect only half as much as lower springs, hence carry only half the load.

Let $L =$ load carried by each lower spring.

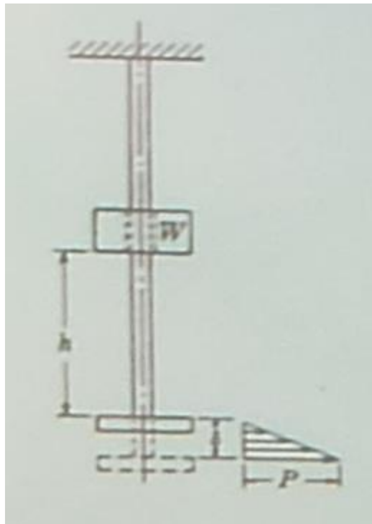
$$2L + \frac{L}{2} = 50\ N$$

$$\therefore L = 20\ N$$

Lower springs carry $20\ N$ each.

Upper springs carry $10\ N$ each.

Stresses Due to Shock and Impact Loading



Where:
W = falling weight, lb
h = height of free fall, in
δ = deflection, in
P = impact load, lb
$C = P/\delta$ = lb/in of deflection

Energy balance:

$$\frac{1}{2}P\delta = W(h+\delta)$$

$$P = 2\frac{W}{\delta}(h+\delta)$$

$$\frac{P}{W} = 2\left(\frac{h}{\delta}+1\right)$$

But:

$$\delta = \frac{P}{c}$$

$$\therefore \frac{P}{W} = 2\left(\frac{hC}{P}+1\right)$$

$$P^2 = 2W(hC+P)$$

$$P^2 - 2WP - 2WhC = 0$$

$$P = \frac{2W \pm \sqrt{4W^2 + 8WhC}}{2}$$

$$P = W\left(1 + \sqrt{1 + \frac{2hC}{W}}\right)$$

$$\frac{P}{W} = 1 + \sqrt{1 + \frac{2hC}{W}}$$

For a bar in Tension:

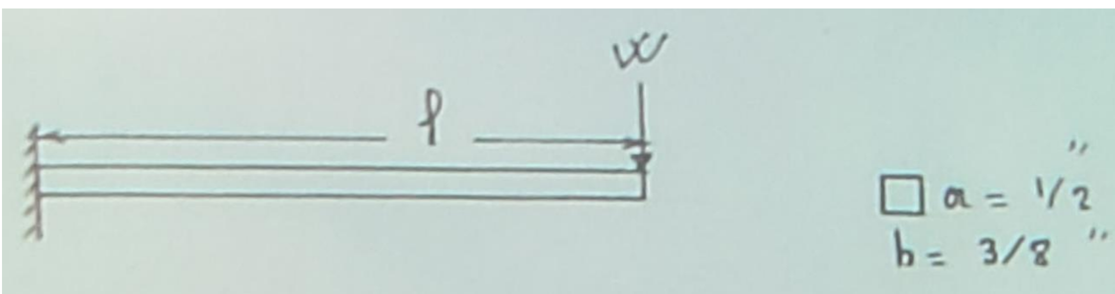
$$\delta = \frac{PL}{AE}$$

$$\therefore C = \frac{P}{\delta} = \frac{P}{PL/AE} = \frac{AE}{L}$$

Special case: if the load is applied instantaneously without velocity of approach then:

$$P = 2W$$

Example 9:



Where:
$W = 8\ lb$
$a = 1/2\ in$
$b = 3/8\ in$
$l = 12\ in$

Find the max bending stress in the beam

a — load applied gradually
b — load dropped from a distance of 3/16 in

Solution:

$$S_{max} = \frac{M}{I/c}$$

$$\frac{I}{c} = \frac{ba^2}{6} = \frac{\left(\frac{3}{8}\right)(0.5)^2}{6} = 0.0156\ in^3$$

a) $M = (8)(12) = 96\ lb - in$

$$\sigma_{max} = \frac{96}{0.0156} \cong 6154\ psi$$

b) Impact stress

$$\delta = \frac{Pl^3}{3EI}$$

$$I = \frac{ba^3}{12} = \frac{\left(\frac{3}{8}\right)\left(\frac{1}{2}\right)^3}{12} = 0.0039\ in^4$$

$$C = \frac{P}{\delta} = \frac{3EI}{L^3} = \frac{(3)(30)(10^6)(0.0039)}{(12)^3} = 204\ lb/in$$

$$P = W\left(1 + \sqrt{1 + \frac{2hC}{W}}\right) = (8)\left(1 + \sqrt{1 + \frac{(2)(204)(3/16)}{(8)}}\right)$$

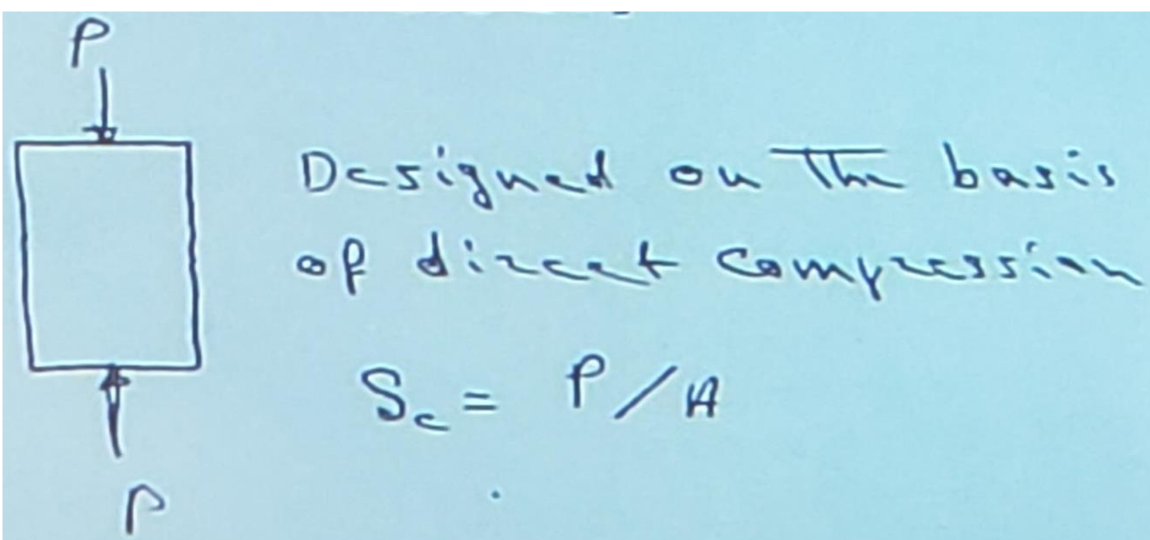
$$P = 34.64\ lb$$

$$\sigma_{max} = \frac{M}{I/C} = \frac{(34.64)(12)}{(0.0156)} = 26646\ psi$$

Lecture (Feb. 28th, 2019)

3.7 Instability Considerations

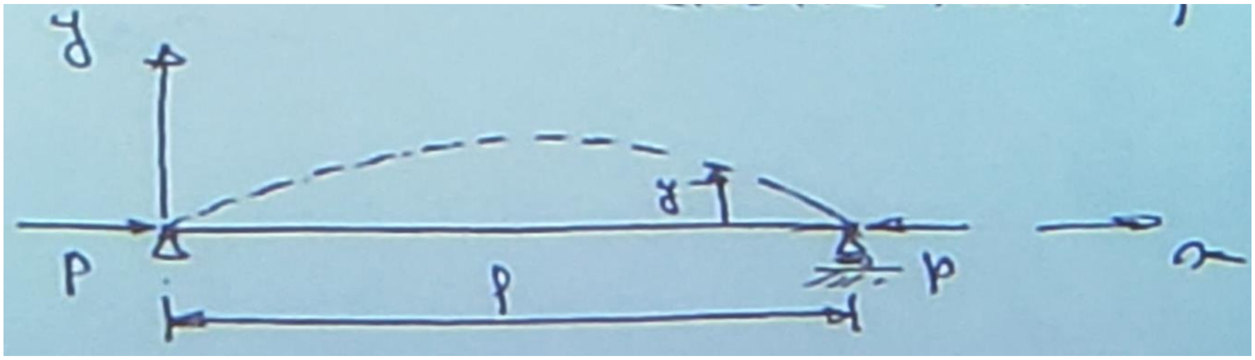
- Short compressive members



- <u>Euler Analysis</u> (Slender columns, i.e. long with small cross-section)

$$\frac{B}{\rho^2} > 2$$

Elastic instability; deformation within the elastic limit



$$M = -P\ y(x)$$

But from strength of material

$$M = EI\frac{d^2y}{dx^2} = EI\ y''(x)$$

$$\therefore\ EI\ y''(x) + P\ y(x) = 0 \quad D.E.$$

Let

$\alpha^2 = \dfrac{P}{EI}$

Then

$y''(x) + \alpha^2\ y(x) = 0$

Solution

$y(x) = A\sin(\alpha x + P)$

B.C. $\quad y = 0\ \ at\ \ x = 0$

and $\quad y = 0\ \ at\ \ x = \rho$

$$\therefore For\ x = 0 \quad 0 = A\sin(0 + \beta) \quad \therefore P = 0$$

$$For\ x = \rho \quad 0 = A\sin\alpha\rho$$

$\therefore sin\alpha\rho = 0 \quad since\ A \neq 0$

$But\ \alpha^2 = \dfrac{P}{EI}$

$\therefore (\alpha\rho)^2 = n^2\pi^2 = \dfrac{P}{EI}\rho 2$

$P = \dfrac{n^{2\pi^2 EI}}{\rho^2}$

$P_{cont} = \frac{2\pi^2 EI}{\rho^2}$

Solving for other boundary conditions it can be proven that:

$F_{crit} = \frac{n\pi^2 EI}{\rho^2}$

Where:
$n =$ end fixity coefficient (Figure 4-18, page 196, Table 4-2 Page 199)

Introducing the quantity

$$B = \frac{S_y \rho^2}{n\pi^2 E}$$

And replacing I by $\rho^2 A$

Then:

$$F_{crit} = \frac{n\pi^2 AE}{(L/\rho^2)} = \frac{S_y A \rho^2}{B}$$

- J.B. Johnson Formula (less slender formula)

$$\frac{B}{\rho^2} < 2$$

(plastic instability; allowable stresses exceeded)

$\therefore Experimental\ results$

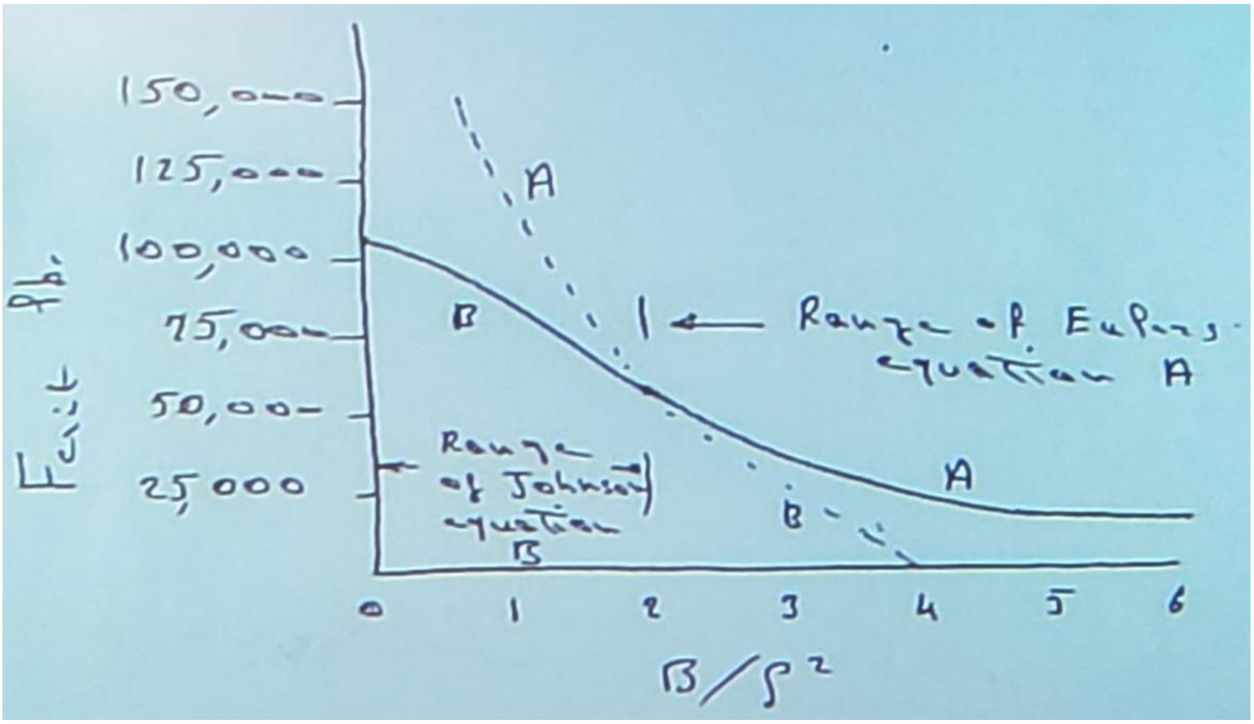
$F_{crit} = AS_y \left(1 - \frac{S_y \rho^2}{4n\pi^2 E \rho^2}\right)$

introducing B:

$F_{crit} = AS_y \left(1 - \frac{B}{4\rho^2}\right)$

If F_{crit} (Euler) equated to F_{crit} (Johnson) we find:

$$\frac{B}{\rho^2} = 2$$



$\therefore if \frac{B}{\rho^2} < 2$ $\boldsymbol{Use\ Johnson}$

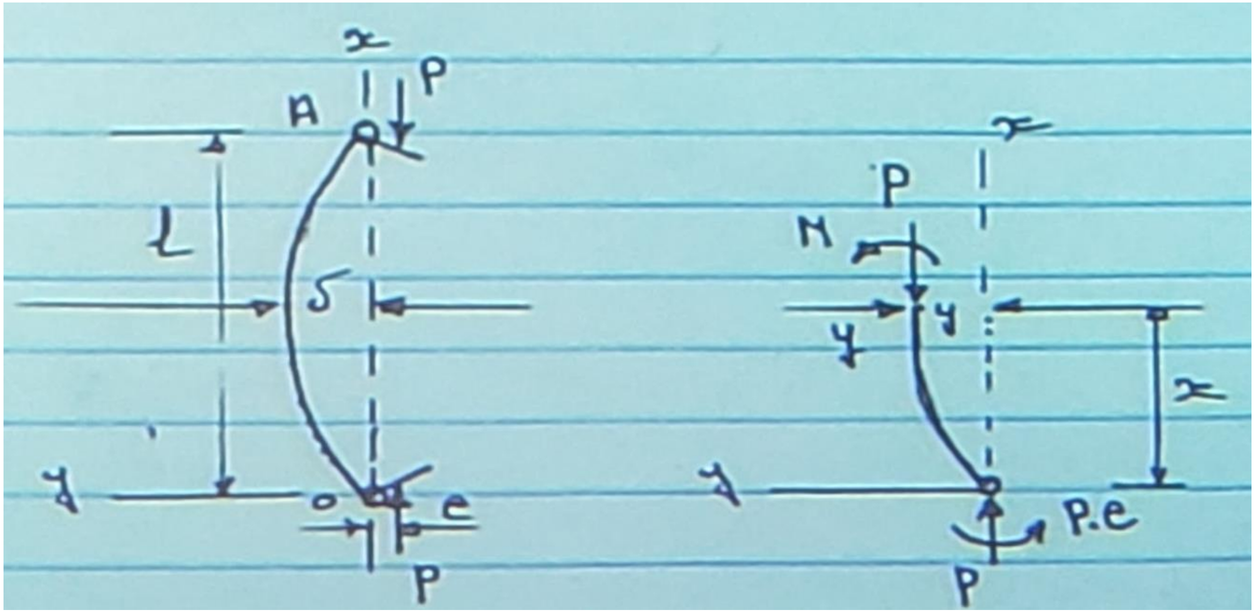
(most machine members are in this range)

$\therefore if \frac{B}{\rho^2} > 2$ $\boldsymbol{Use\ Euler}$

Start with Johnson. Find B/ρ^2 – if < 2 $O.K.$ if not go to Euler

Where:
F_{crit} = critical load causing failure, lb
A = cross-sectional area, in^2
I = moment of inertia of area, in^4
L = length of column, in
ρ = least radius of gyration of cross-section, in
n = end-fixity coefficient, see Figure 4-18 and Table 4-2
E = modulus of elasticity, psi
S_y = yield point of material, psi
$B = S_y L^2 / n\pi^2 E$

Columns with Eccentric Loading



IT can be shown that at $x = l/2$, the deflection:

$$\delta = y_{\frac{l}{2}} = e\left[\sec\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) - 1\right]$$

The maximum bending moment occurs at $\rho/2$ and is:

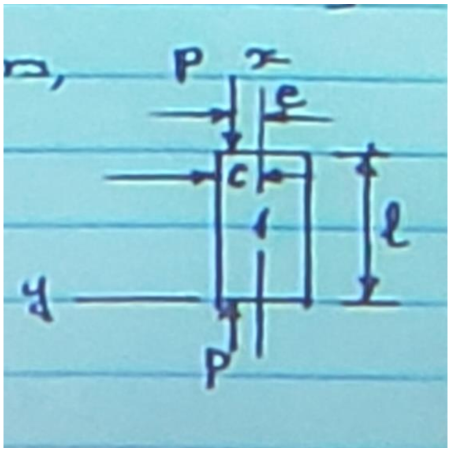
$$M_{max} = P(e + \delta) = P \cdot e \cdot \sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

The maximum compressive stress:

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{A\rho^2} = \frac{P}{A}\left[1 + \frac{ec}{\rho^2} sec \frac{l}{2\rho}\sqrt{\frac{P}{EA}}\right]$$

In the case of short compressive members, valid if $\left(\frac{l}{\rho}\right) \leq 0.282 \ \left(\frac{AE}{P}\right)^{\frac{1}{2}}$

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{P \cdot e \cdot c \cdot A}{I \cdot A} = \frac{P}{A}\left(1 + \frac{ec}{\rho^2}\right)$$

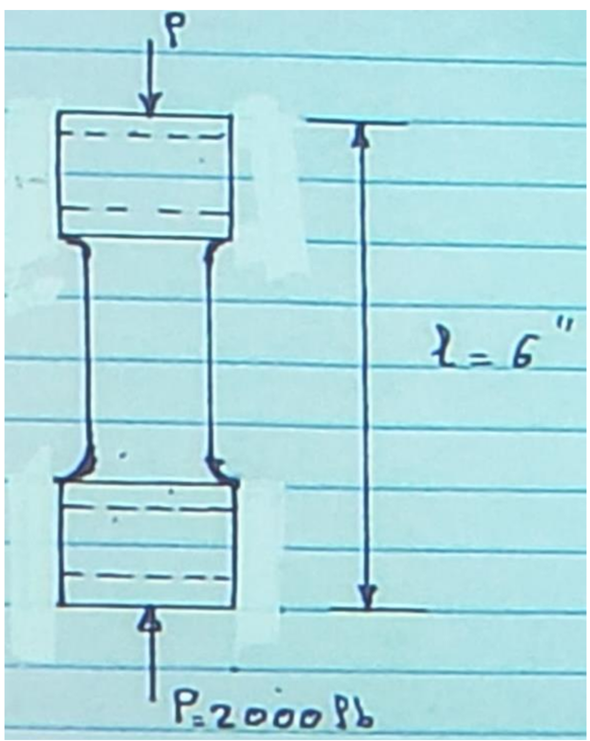


Where:

$$\rho^2 = \frac{I}{A}$$

Example:
A strut of circular cross-section
Material is SAE10HR Steel
Diameter of loading pin 0.5"
Allowable bearing pressure at pin = $10{,}000\ psi$
$l = 6"$
$P = 2{,}000\ lb$

Using a design factor of 1.5 determine the dimensions for the strut



Solution:
From Table A-20, $S_y = 42{,}000\ psi$
From Table A-23, $E = 29 \cdot 10^6\ psi$
From Table 4-2, $n = 1$

$$B = \frac{S_y l^2}{n\pi^2 E} = \frac{(42 \cdot 10^3)(6^2)}{(1)\pi^2(29 \cdot 10^6)} = 0.00528$$

$$F_{crit} = n_d P = (1.5)(2{,}000) = 3{,}000\ lb$$

Starting with Johnson's equation, assuming $B/\rho^2 < 2$:

$$F_{crit} = AS_y\left(1 - \frac{B}{4\rho^2}\right) \quad ; \quad A = \frac{\pi d^2}{4} \quad ; \quad \rho = \frac{d}{4}$$

Substituting and solving for d^2:

$$d^2 = \left(\frac{4F_{crit}}{\pi S_y}\right) + 4B$$

$$d^2 = \frac{(4)(3{,}000)}{(\pi)(42{,}000)} + (4)(0.00528) = 0.112\ in^2$$

$$d = 0.335\ in$$

Using standard $3/8\ in$ we check for B/ρ^2

$$\frac{B}{\rho^2} = \frac{16B}{d^2} = \frac{(16)(0.00528)}{\left(\frac{3}{8}\right)^2} = 0.601$$

$$\therefore \frac{B}{\rho^2} < 2; \quad Johnson's\ Formula\ Justified$$

For the eye, $S = P/td$

$$10{,}000 = 2{,}000/(t \cdot 0.5)$$

$$t = \frac{0.2}{0.5} = 0.4\ in$$

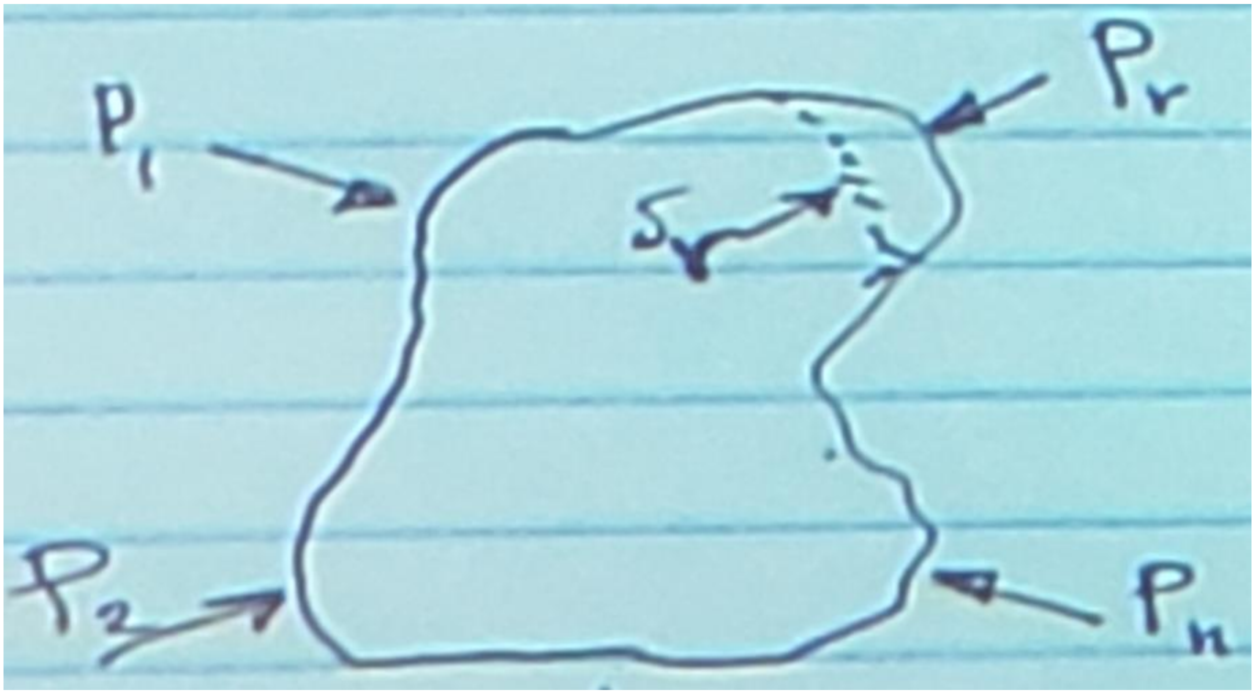
Use $1/2\ in$ to allow for machining of the faces of the eye.

<u>Beam Deflection</u>

- Castigliano's Theorem

The displacement corresponding to any force of a system of forces acting on an elastic body can be determined by taking the partial derivative of the elastic strain energy with respect to that force.

$$\frac{\delta u}{\delta P_r} = \delta_r$$



P_1
P_r
S_r
P_2
P_n