Lecture (Jan. 29th, 2019)

Minimization by Method of Lagrange Multipliers

Given a function

$$U(x_1, x_2, \dots , x_n)$$

We wish to find its minimum value & co-ordinates of $x_1, x_2 ---$ etc. at this minimum. Let the function be subject to the constraints

$$\varphi_1(x_1, x_2 --- \ x_n) = 0$$

$$\varphi_2(x_1, x_2 --- \ x_n) = 0$$

Procedure:

(1) Form a new function

$$U(x_1, x_2, \dots , x_n) = U(x_1, x_2, \dots , x_n) + \lambda_1\varphi_1(x_1, x_2, \dots , x_n) + \lambda_2\varphi_2(x_1, x_2, --- x_n) \dots \ etc.$$

Let there be N variable & M constraints. We now treat the λ's as variables and write the $(m + n)$ equations.

$$\frac{\partial U^*}{\partial x_1} = 0$$

$$\frac{\partial U^*}{\partial x_2} = 0$$

$$etc.$$

$$\frac{\partial U^*}{\partial \lambda_1} = 0$$

Final step: solve the set of algebraic equations (2) for the variables $x_1, x_2, --- x_n$. The function $U(x_1, x_2)$ will be minimum at this point.

NOTE: The method of Lagrange Multipliers is generally good for handling problems where constraints on the variables exist.

Assignment #2: 3.4, 3.5, 3.6, 3.9, 3.10, 3.14

Midterm is February 12th, unless we can find a room, and then it will be on the Saturday.

Load and Stress Analysis

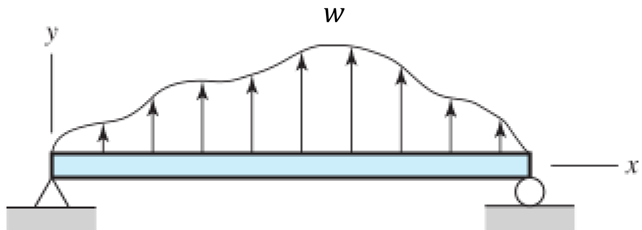
Beams

1. Loads Distributed Along a Line

Let's assume that the function w describing a particular distributed load is known. The graph of w is called the loading curve. The force acting on an element dx of the line is wdx.

The total force F is:

$$F = \int_L w\,dx$$



The moment about the origin due to the force exerted on the element dx is $xwdx$, so the total moment about the origin due to the distributed load is:

$$M = \int_L xw\,dx$$

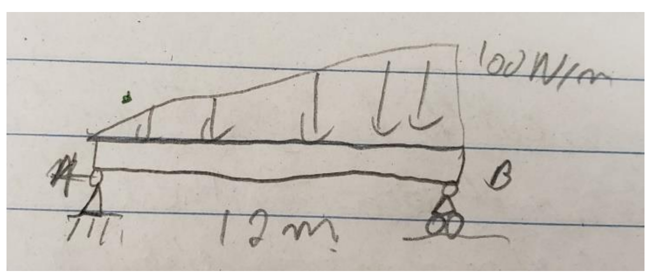
Or:

$$M = \bar{x}F = \int_L xw\,dx$$

Where F is the equivalent load if placed at the position:

$$\bar{x} = \frac{\int_L xw\,dx}{\int_L w\,dx}$$

Example: The beam is subjected to a triangular distributed load whose value at B is $100\ N/m$. Determine the reactions at A and B.



First method:

$$w = \frac{100}{12}x \quad (N/m)$$

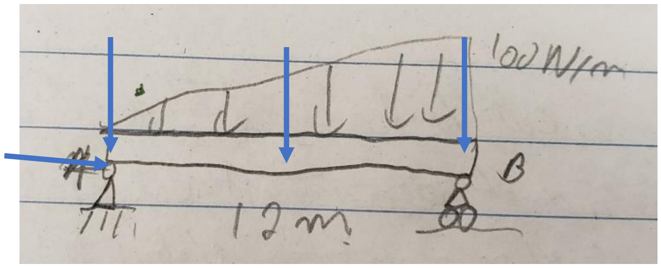
The total load is:

$$F = \int_L w\,dx = \int_0^{12} \left(\frac{100}{12}\right) x\,dx = 600\ N$$

The clockwise moment about A due to the load is:

$$M_A = \int_L xw\,dx = \int_0^{12} \left(\frac{100}{12}\right) x^2\,dx = 4800\ N \cdot m$$

From the equilibrium equations:

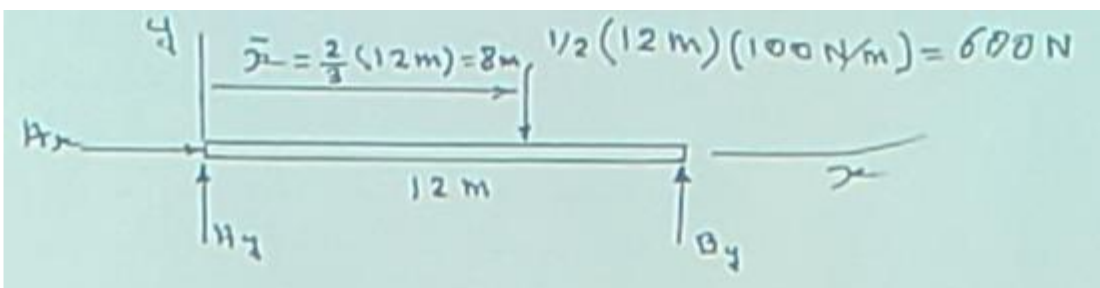


$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = A_y + B_y - 600 = 0$$

$$-\sum M_A = 12B_y - 4800 = 0$$

$$\therefore A_x = 0 \quad ; \quad A_y = 200\ N \quad ; \quad B_y = 400\ N$$

Second Method



$$F = \left(\frac{1}{2}\right) \cdot (12\ m) \cdot \left(100 \frac{N}{m}\right) = 600\ N$$

$$\bar{x} = \left(\frac{2}{3}\right) \cdot (12\ m) = 8\ m$$

$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = A_y + B_y - 600 = 0$$

$$-\sum M_A = 12B - 8 \cdot 600 = 0$$

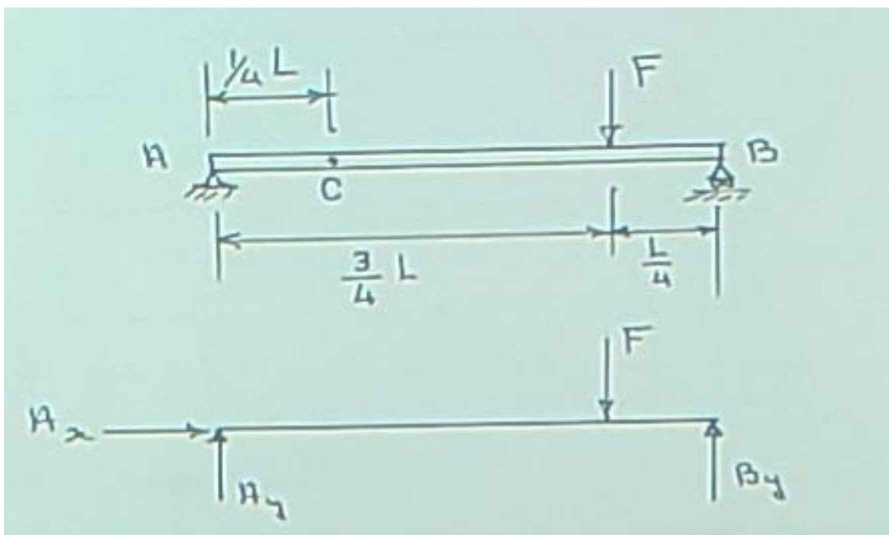
$$\therefore A_x = 0 \quad ; \quad A_y = 200\ N \quad ; \quad B_y = 400\ N$$

2. Internal Forces and Moments in Beams

Determining the internal forces and moment at a particular cross section of a bream typically involves three steps:

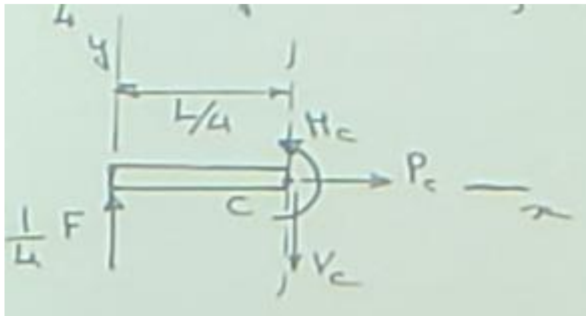
1. Determine the external forces and moments – Draw free-body diagram of the beam and determine the reactions at tits supports. If the beam is a member of a structure, you must first analyze the structure.
2. Draw the free-body diagram of part of the beam – cut the beam at the point at which you want to determine the internal forces and moment and draw the free-body diagram of one of the resulting parts. You can choose the part with the simplest free-body diagram, if your cut divides a distributed load, don't represent the distributed load by an equivalent force until after you have obtained your free-body diagram.
3. Apply the equilibrium equations – use the equilibrium equations to determine the axial force P, the shear force V, and the bending moment M.

Example: Determine the internal forces and moment at C.



$$\sum F_x = A_x = 0 \quad ; \quad \sum M_A = LB_y - F\left(\frac{3}{4}L\right) = 0 \quad ; \quad B_y = \left(\frac{3}{4}\right)F$$

$$\sum F_y = \left(\frac{3}{4}\right)F + A_y - F = 0 \quad ; \quad A_y = \left(\frac{1}{4}\right)F$$

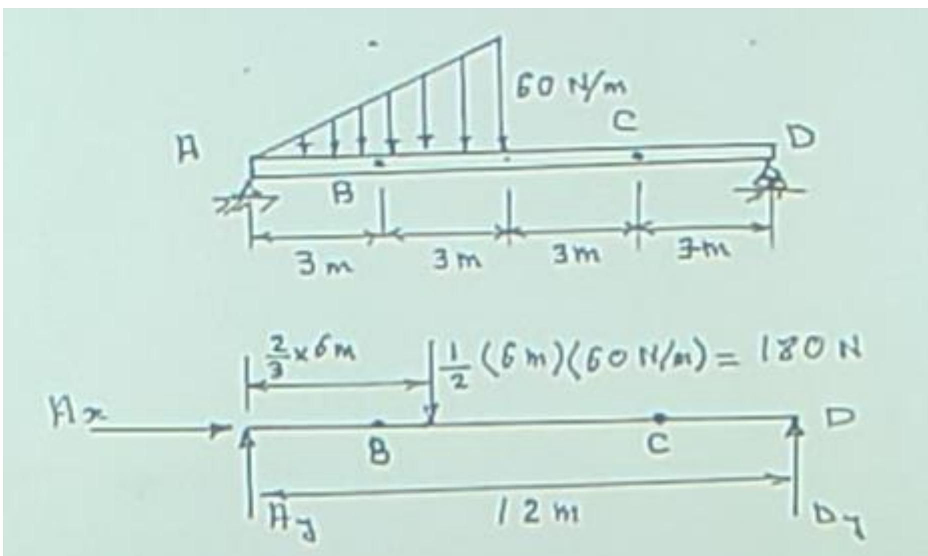


$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = \left(\frac{1}{4}\right)F - V_c = 0 \quad ; \quad V_c = \left(\frac{1}{4}\right)F$$

$$\sum M_C = M_c - \left(\frac{1}{4}\right)\left(\frac{F}{4}\right) = 0 \quad ; \quad M_c = \left(\frac{1}{16}\right)FL$$

$$P_c = 0 \quad ; \quad V_c = \left(\frac{1}{4}\right)F \quad ; \quad M_c = \left(\frac{1}{16}\right)FL$$

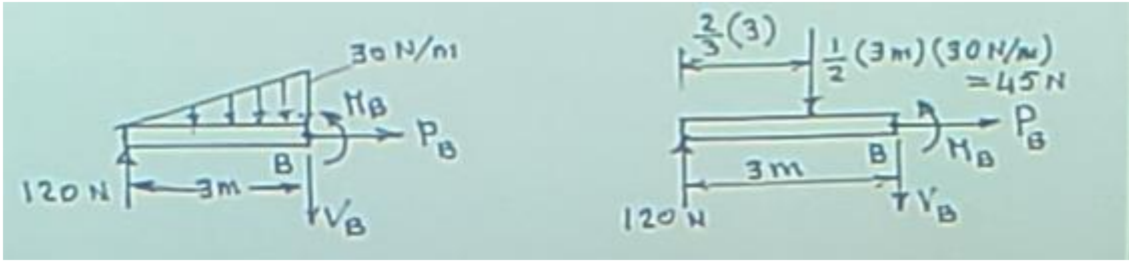
Example: Determine the internal forces and moment at (a) and B and (b) at C



$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = A_y + D_y - 180 = 0$$

$$\sum M_A = 12D_y - 4(180) = 0$$

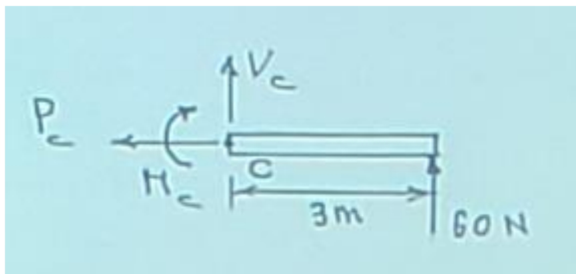
$$A_x = 0 \quad ; \quad A_y = 120\ N \quad ; \quad D_y = 60\ N$$



$$\sum F_x = P_B = 0 \quad ; \quad \sum F_y = 120 - 45 - V_B = 0$$

$$\sum M_B = M_B + (1)(45) - (3)(120) = 0$$

$$P_B = 0 \quad ; \quad V_B = 75\ N \quad ; \quad M_B = 315\ N \cdot m$$



$$\sum F_x = -P_C = 0$$

$$\sum F_y = V_c + 60 = 0$$

$$\sum M_C = -M_C + (3)(60) = 0$$

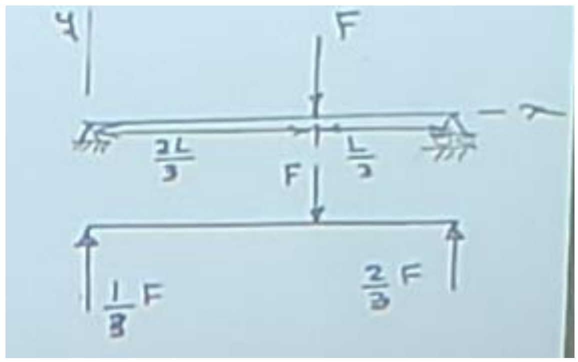
Solving we obtain:

$$P_C = 0 \quad ; \quad V_C = -60 \quad ; \quad M_C = 180\ N \cdot m$$

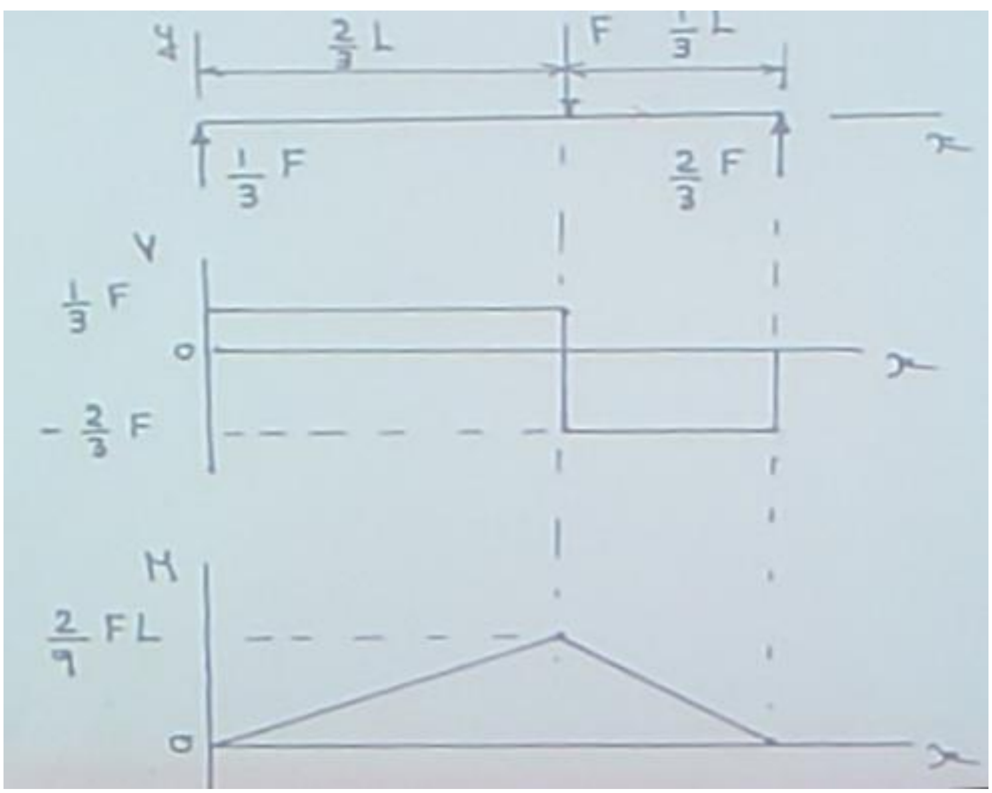
Lecture (Jan. 31st, 2019)

3. Shear Force and Bending Moment Diagrams

The shear force and bending moment diagrams are simply the graphs of V and M respectively, as functions of x. They show the changes in the shear force and bending moment that occur along the beam's length as well as their maximum and minimum values.



$$\left.\begin{aligned} P &= 0 \\ V &= \frac{1}{3}F \\ M &= \frac{1}{3}Fx \end{aligned}\right\} \quad 0 < x < \frac{2}{3}L$$



3. <u>Relations between Distributed Load, Shear Force, and Bending Moment</u>

$$\frac{dv}{dx} = -w$$

$$\frac{dM}{dx} = v$$

If we define $q(x)$ as the load intensity with units of force per unit length and is positive in the positive y −direction. Then,

$$q = -w = \frac{dv}{dx} = \frac{d^2M}{dx^2}$$

And:

$$V = \int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q\,dx$$

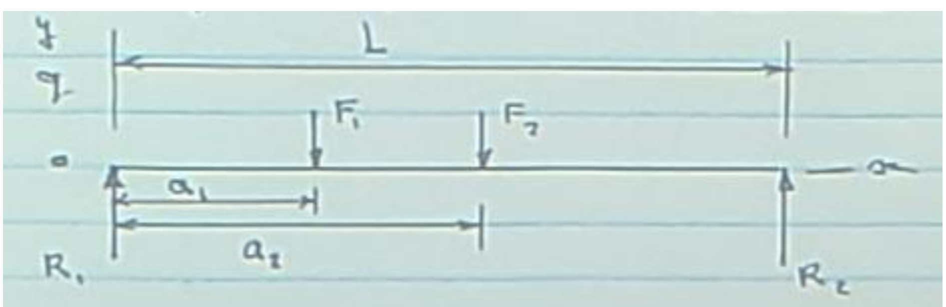
And:

$$M = \int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V\,dx$$

Singularity Functions

Singularity functions are shown in Table 3.1 constitute an easy means of integrating across discontinuities. Consequently, they are used to write general expressions for shear force and bending moment in beams in the presence of concentrated forces and moments.

Example – Derive expressions for the loading, shear force, and bending moment of the beams as shown.



Solution

$$q \;=\; R_1 < x >^{-1} - F_1 < x - a_1 >^{-1} - F_2 < x - a_2 >^{-1} + R_2 < x - L >^{-1}$$

Have:

$$\int_{V_A}^{V_B} dV = \int_{x_A}^{x_B} q\,dx = V_B - V_a$$

And:

$$V = 0 \; at \; x = -\infty$$

$$\therefore V = \int_{-\infty}^{x} q\,dx = R_1 < x >^0 - F_1 < x - a_1 >^0 - F_2 < x - a_2 >^0 + R_2 < x - L >^0$$

Also have:

$$V = 0 \; at \; x > L$$

$$\therefore R_1 - F_1 - F_2 + R_2 = 0 \; - \; Equation\,(1)$$

The bending moment:

$$\int_{M_A}^{M_B} dM = \int_{x_A}^{x_B} V\,dx = M_B - M_A$$

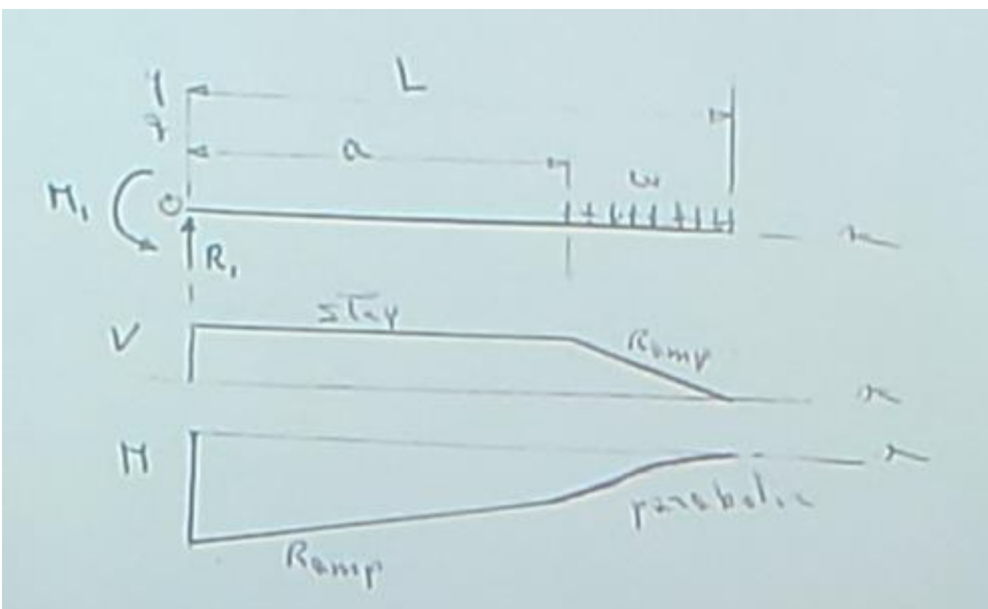
$$\therefore M = \int_{-\infty}^{x} V\,dx = R_1 < x >^1 - F_1 < x - a_1 >^1 - F_2 < x - a_2 >^1 + R_2 < x - L >^1$$

And:

$$R_1 L - F_1(L - a_1) - F_2(L - a_2) = 0 \; - Equation\,(2)$$

$$(1)\; and \; (2) \; are \; solved \; for \; R_1 \; and \; R_2$$

Example – The figure shows the loading diagram for a beam cantilevered at 0 and having a uniform load w acting on the position $a \leq x \leq L$. Derive the shear force and moment sections. M_1 and R_1 are the support reactions.



Solution – The loading function is:

$$q = -M_1 < x >^{-2} + R_1 < x >^{-1} - w < x - a >^{0}$$

First integration to obtain V:

$$V = \int_{-\infty}^{x} q\, dx = -M, x >^{-1} + R_1 < x >^{0} - \frac{w}{2} < x - a >^{1}$$

Second Integration:

$$M = \int_{-\infty}^{x} V\, dx = -M_1 < x >^{0} + R_1 < x >^{-1} - \frac{w}{2} < x - a >^{2}$$

For x slightly larger than L:

$$V = M = 0$$

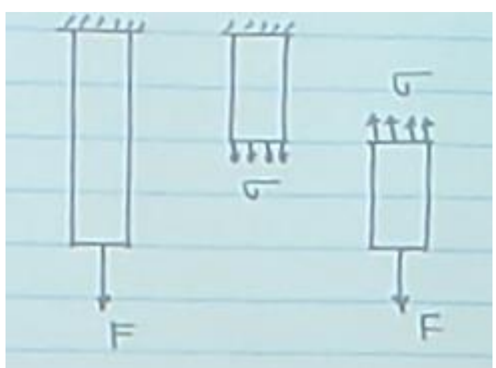
$$\therefore V_{x>L} = -M_1(0) + R_1 - w(L - a) = 0 \quad - \quad Equation\ (1)$$

And:

$$M_{x>L} = -M_1 + R_1 L - \frac{w}{2}(L - a)^2 = 0 \quad - \quad Equation\ (2)$$

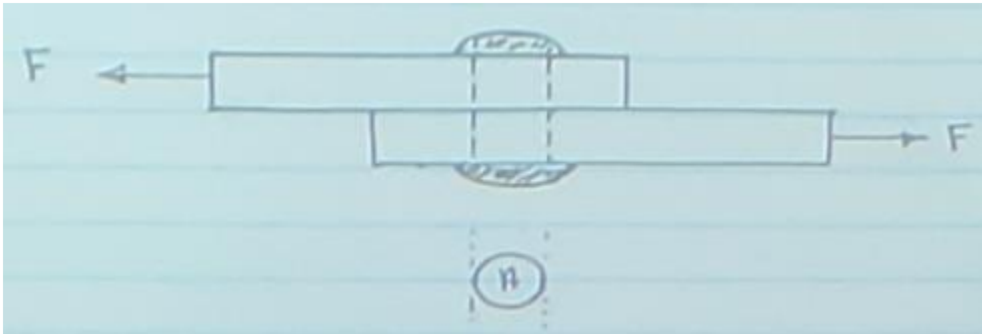
- *Pure Tension or Compression*

$$\sigma = \frac{F}{A}$$



- *Pure Shear Stress*

$$\tau = \frac{F}{A}$$



Elastic Strain

- *Strain*

$$\varepsilon = \frac{\delta}{L}$$

From Hooke's Law:

$$\sigma = E\varepsilon$$

$$\tau = G\gamma$$

Where:

$$E - modulus\ of\ elasticity$$
$$\varepsilon - strain$$
$$G - shear\ modulus$$
$$\gamma - shear\ strain$$

And:

$$\therefore\ \frac{F}{A} = E\frac{\delta}{L}$$

And:

$$\delta = \frac{FL}{AE}$$

- Poisson's Ratio μ or v

$$\mu = -\frac{lateral\ strain}{axial\ strain}$$

The three elastic constants are related by:

$$E = 2G(1 + \mu)$$

Stress-Strain Relations

- *Uniaxial Stress*

$$\varepsilon_1 = \frac{\sigma_1}{E} \quad ; \quad \varepsilon_2 = -\mu \cdot \varepsilon_1 \quad ; \quad \varepsilon_3 = -\mu \cdot \varepsilon_1$$

- *Biaxial Stress*

$$\varepsilon_1 = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \mu \frac{\sigma_1}{E}$$

$$\varepsilon_3 = \frac{-\mu\sigma_1}{E} - \frac{\mu\sigma_2}{E}$$

Solving for σ_1 and σ_2:

$$\sigma_1 =$$

$$\sigma_2 =$$

- *Triaxial Stress*

$$\varepsilon_1 = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} - \mu \frac{\sigma_3}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \mu \frac{\sigma_1}{E} - \mu \frac{\sigma_3}{E}$$

$$\varepsilon_3 = \frac{\sigma_3}{E} - \mu \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E}$$

Or:

$$\sigma_1 = \frac{Ee_1(1-\mu) + \mu E(\varepsilon_2 + \varepsilon_3)}{1 - \mu - 2\mu^2}$$

$$\sigma_2 = \frac{Ee_2(1-\mu) + \mu E(\varepsilon_1 + \varepsilon_3)}{1 - \mu - 2\mu^2}$$

$$\sigma_3 = \frac{Ee_3(1-\mu) + \mu E(\varepsilon_1 + \varepsilon_2)}{1 - \mu - 2\mu^2}$$