Lecture (Jan. 22nd, 2019)

Figure 9-1

The instantaneous probability of failure ρ, often called hazard rate, varies with time as indicated. This tends to greatly complicate the computation. Fortunately, with little error, for the mid-life period between the rapid burn-in and burn-out intervals, ρ is assumed time invariant ($\bar{\rho}$) as shown in the figure

For this region, the law of reliability simplifies to:

$$R = e^{-\bar{\rho}\int_0^t dt} = e^{-\bar{\rho}\cdot t}$$

The hazard rate is the number of failures occurring per hour for each survived unit. Therefore, its reciprocal σ is the number of survival hours to the next failure or the meantime to failure.

$$\sigma = \frac{1}{\bar{\rho}}$$

and: $$R = e^{-\frac{t}{\sigma}}$$

<u>Reliability of Complex Systems</u>

Systems are made of many components, all interrelated and most of them contributing to the unreliability of the system as a unit. The joint reliability is predicted from individual component probabilities as follows:

Series Systems

Systems consisting of several components so connected and independent that if one part fails the entire system fails, are series systems.

In a series system the reliability is the product of the reliabilities of each component.

$$\therefore R_s = R_1 \cdot R_2 \cdot R_3 \ldots R_n = \prod_1^n R_i$$

$$= e^{-(\bar{\rho}_1 + \bar{\rho}_2 + \bar{\rho}_3 + \cdots + \bar{\rho}_n)t} = e^{-\sum_1^n \bar{\rho}_i t}$$

The unreliability or probability of failure of such a system is:

$$Q_s = 1 - e^{-\sum_1^n \bar{\rho}_i t}$$

Parallel Redundant Systems

When very high system reliabilities are required, duplicate components and even entire duplicate circuits become desirable, so that if the first fails, the second will carry on.

	$R = e_1^{-\overline{\rho_1}t}$	
Begin	$R = e_2^{-\overline{\rho_2}t}$	End
	$R = e_3^{-\overline{\rho_3}t}$	

This parallel reliability is referred to as parallel redundancy, because all units operate simultaneously. The probability that one of the two parallel components will survive is the sum of the probabilities of the three outcomes; neither of components A and B fails, A fails but not B, and B fails but not A.

$$\therefore R_p = e_1^{-\overline{\rho_1}t} + e_2^{-\overline{\rho_2}t} - e_1^{-(\overline{\rho_1}+\overline{\rho_2})t} = R_1 + R_2 - R_1R_2$$

The probability of failure is Q_p:

$$Q_p = \left(1 - e_1^{-\overline{\rho_1}t}\right)\left(1 - e_2^{-\overline{\rho_2}t}\right)$$

The probability of survival for n components is simpler to compute via unreliability:

$$Q_p = Q_{p1} \cdot Q_{p1} \cdot \dots \cdot Q_{pn} = \prod_{i=1}^{n} Q_i$$

and: $R_p = 1 - Q_p$

Stand-by Systems

When it is impractical to operate a system with parallel branches, and yet some assurance of continued operation is necessary, stand-by units become advisable. Such a system may be regarded as a simple system with multiple lives.

	Stand-by Unit	
Begin	Primary Unit	End

This obeys a principle known as $Poisson's\ distribution$ which yields:

$$e^{-\bar{\rho}t}\left[1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!} + \dots + \frac{(\bar{\rho}t)^n}{n!}\right] = 1$$

If one stand-by unit is present, the system reliability would be:

$$R_B = e^{-\bar{\rho}t}(1 + \bar{\rho}t)$$

And with two stand-by units:

$$R_B = e^{-\bar{\rho}t}\left(1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!}\right)$$

And for n stand-by units:

$$R_B = e^{-\bar{\rho}t}\left[1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!} + \dots + \frac{(\bar{\rho}t)^n}{n!}\right]$$

Example: A series-parallel system is made of components as depicted in the figure. The probabilities for each component, all for the same period of time, is as indicated in the box. Compute the reliability of the system.

Begin	0.96	0.98	0.92	0.95	End
	0.96	0.98	0.92	0.95	

Solution:

The reliability of each series branch (since they are the same) is:

$$R_s = R_1 \cdot R_2 \cdot R_3 \cdot R_4$$
$$R_s = (0.96)(0.98)(0.92)(0.95) = 0.85$$

The reliability of the two parallel branches, and therefore for the system, is

$$R_p = 2R_s - R_1R_2 = (2)(0.82) - (0.82)(0.82)$$
$$R_p = 0.97$$

Example: Calculate the reliability of a system with two stand-by units if each has a mean life to failure of $100\ hr$, for a period of $10\ hr$. Compute this with the reliability for the system after one stand-by unit is removed; after both stand-by units are removed.

Solution: We have:

$$\bar{\rho} = \frac{1}{m} = \frac{1}{100} = 0.01 = hazard\ rate$$

i) System reliability with two stand-by units:

$$R_B = e^{-\bar{\rho}t}\left(1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!}\right)$$

$$R_B = e^{-(0.01)(10)}\left[1 + (0.01)(10) + \frac{\left((0.01)(10)\right)^2}{2!}\right] = 0.9998453$$

ii) System reliability with one stand-by unit:

$$R_B = e^{-(0.01)(10)}[1 + (0.01)(10)] = 0.9953211$$

iii) System reliability with no stand-by:

$$R_B = e^{-(0.01)(10)} = 0.9048374$$

<u>Relating Design Factor to Reliability</u>

Stress and strength are statistical in nature. In the probability density functions for stress σ and strength S shown in the figure below, the mean values of stress and strength are $\bar{\sigma} = \mu_\sigma$ and $\bar{S} = \mu_s$ respectively.

$\therefore The\ average\ designfactor\ \bar{n}_d$:

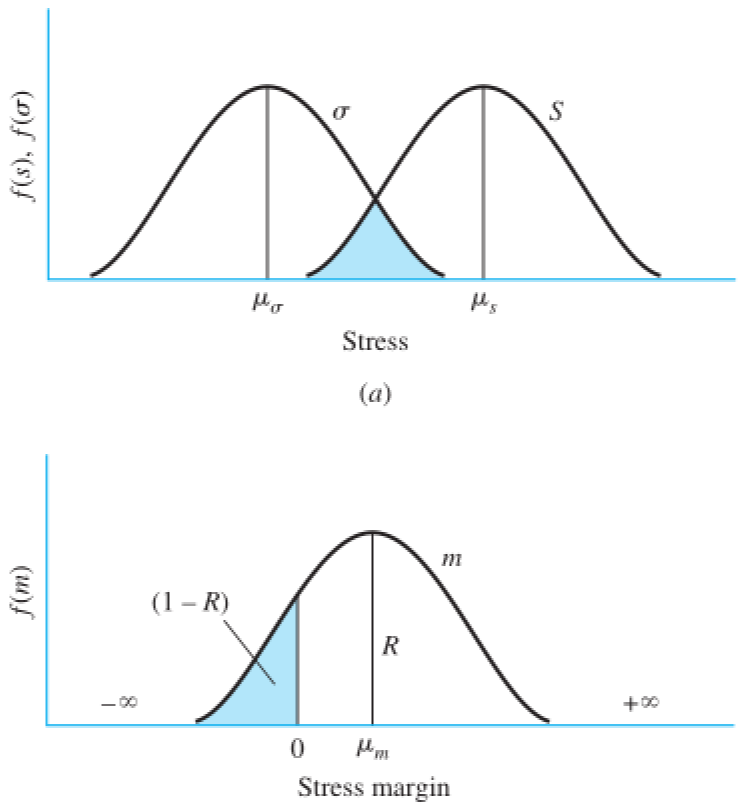
$$\bar{n}_d = \frac{\mu_s}{\mu_\sigma}$$

The margin of safety m at any value of σ and s is:

$$m = S - \sigma$$

The average margin of safety $\bar{m}$ is:

$$\bar{m} = \mu_s - \mu_\sigma$$



Lecture (Jan. 24th, 2019)

For the overlap in Fig. (a) $\sigma > S$ and the margin of safety is negative. The reliability that a part will perform without failure, R, is the area of the margin of safety distribution (Fig. (b)) for $m > 0$.

Reliability is then the probability that $m > 0$.

$$R = p\,(S > \sigma) = p\,(\{S - \sigma\} > 0) = p\,(m > 0)$$

Noting that for normal distributions:

$$\mu_m = \mu_s - \mu_\sigma$$

And:

$$\hat{\sigma}_m = (\hat{\sigma}_s^2 + \hat{\sigma}_\sigma^2)^{\frac{1}{2}}$$

We write:

$$Z_m = \frac{(m - \mu_m)}{\hat{\sigma}_m}$$

To find the probability that $m > 0$ we substitute $m = 0$ in Z_m.

$$Z_m = \frac{(0 - \mu_m)}{\hat{\sigma}_m} = \frac{-\mu_m}{\hat{\sigma}_m} = -\frac{(\mu_s - \mu_\sigma)}{(\hat{\sigma}_s^2 + \hat{\sigma}_\sigma^2)^{\frac{1}{2}}}$$

Dividing all terms by μ_σ:

$$Z_m = -\frac{\left(\frac{\mu_s}{\mu_\sigma} - 1\right)}{\left(\frac{\hat{\sigma}_s^2}{\mu_\sigma} + \frac{\hat{\sigma}_\sigma^2}{\mu_\sigma}\right)^{\frac{1}{2}}} = -\frac{\bar{n}_d - 1}{\left(\frac{\hat{\sigma}_s^2}{\mu_\sigma}\frac{\mu_s}{\mu_2} + \frac{\hat{\sigma}_\sigma^2}{\mu_\sigma}\right)^{\frac{1}{2}}} = -\frac{\bar{n}_d - 1}{\left(\bar{n}_d^2\frac{\hat{\sigma}_s^2}{\mu_s^2} + \frac{\hat{\sigma}_\sigma^2}{\mu_\sigma}\right)^{\frac{1}{2}}}$$

Introducing the terms $C_s = \sigma_s/\mu_s$ and $C_\sigma = \sigma_\sigma/\mu_\sigma$

$$Z_m = -\frac{\bar{n}_d - 1}{(\bar{n}_d^2 C_s^2 + C_\sigma^2)^{\frac{1}{2}}}$$

Solving for n_d:

$$n_d = \frac{1 + [1 - (1 - Z^2 C_s^2)(1 - Z^2 C_\sigma^2)]^{\frac{1}{2}}}{1 - Z^2 C_s^2} \quad \textit{where } R > 0.5$$

$$n_d = \frac{1 - [1 - (1 - Z^2 C_s^2)(1 - Z^2 C_\sigma^2)]^{\frac{1}{2}}}{1 - Z^2 C_s^2} \quad \textit{where } R \le 0.5$$

Where Z refers to Z_m.

Important note: Comparing Fig(b) to Tab. A-10

$$R = 1 - \Phi_{(z)} \quad z \leq 0$$

$$R = \Phi_{(z)} \qquad z > 0$$

<u>Optimizing by Differentiation</u>

When all functional constrains can be involved in a single criterion function, the parameters are readily optimized. The derivative of the criterion function with respect to each parameter is set to zero.

The n equations are then solved simultaneously for the optimum parametric values. Of course, these must be established consistent with any regional limitations that may apply.

If the criterion is expressible in terms of a single significant parameter, the mathematical problem reduces to finding where the stope is zero.

Example: A rectangular tank with its base twice as long as wide is to have a volume of $12\ ft^2$. Determine the most economical dimensions, if the bottom sheet material costs $20¢/ft^2$ and the sides $10¢/ft^2$.

Solution: Cost of bottom = $a \ \times 2a \ \times 2a = 40a^2¢$

And the four sides cost = $2 \times a \times b \times 10 + 2 \times 2ab \times 10 = 60ab¢$

The total cost = $C = 40a^2 + 60ab$

$$V = 2a \times a \times b = 12 \quad \text{or} \quad b = 6/a^2$$

$$c = 40a^2 + \frac{360}{a}$$

$$\frac{dc}{da} = 80a - \frac{360}{a^2} = 0$$

From which $a = 1.65'$ and $b = 2.2'$

And the most economical tank is:

$$1.65\ ft \ \times 3.30\ ft \ \times 2.20\ ft$$

The cost is:

$$C = 40a^2 + \frac{360}{a} = 40(1.65)^2 + \frac{360}{1.65}$$

$$C \cong 3.27$$

<u>Optimization by Dual Variables</u>

This method consists of the replacement of the generalized function by a dual problem that results in the simultaneous solutions of a system of linear equation. If the number of unknowns exceeds the number of equations that can be written, the method will not yield a solution. The dual problem derives from a particular treatment of arithmetic and geometric mean expressions.

If $a_1 + a_2 + a_3 + \cdots + a_n = 1$, the expression concerned with the weighted arithmetic mean is:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

And that for the weighted geometric mean is:

$$(x_1^{a1})(x_2^{a2})(x_3^{a3}) \ldots (x_n^{an})$$

The inequality (the arithmetic average is greater than the geometric) becomes an equality only if all of the x-terms are, in addition to the above, equal. The inequality is best written in the following form. If $a_i x_i = u_i$, then

$$u_1 + u_2 + \cdots + u_n \geq \left(\frac{u_1}{a_1}\right)^{a1} \left(\frac{u_2}{a_2}\right)^{a2} \ldots \left(\frac{u_n}{a_n}\right)^{an}$$

Example: Four hundred (400) yd^3 of sand must be ferried across a river. The sand is to be shipped across in open containers of length L, width W, and height h. The bottom and sides of the container cost $\$10/yd^2$ and the ends $\$20/yd^2$. Each round trip on the ferry costs $10¢$. The containers are assumed to have no salvage value after the transfer. Minimize the transfer cost.

Solution: The total transportation cost is:

$$\begin{aligned} &C = (\$10)(bottom) \\ &+(\$10)(side)(2\ sides) \\ &+(\$20)(end)(two\ ends) \\ &+(total\ volume\ of\ sand/volume\ of\ container)(cost\ of\ trip) \end{aligned}$$

$$C = 10lw + 20lh + 40wh + \left(\frac{400}{lwh}\right)\left(\frac{10}{100}\right)$$

The dual function is:

$$C(a) = \left(\frac{10lw}{a_1}\right)^{a_1} \left(\frac{20lh}{a_2}\right)^{a_2} \left(\frac{40wh}{a_3}\right)^{a_3} \left(\frac{40}{a_4 lwh}\right)^{a_4}$$

To satisfy a minimum C, the dual variables must conform to:

$$(lw)^{a_1}(lh)^{a_2}(wh)^{a3}\left(\frac{1}{lwh}\right)^{a_n} = 1$$

And eliminating the variables with the dual function becomes:

$$C(a) = \left(\frac{10}{a_1}\right)^{a_1} \left(\frac{20}{a_2}\right)^{a_2} \left(\frac{40}{a_3}\right)^{a_3} \left(\frac{40}{a_4}\right)^{a_4}$$

In order that the dual variables relation be satisfied, the sum of the exponents for each variable must equal zero. Thus,

$$\begin{aligned}
&For\ l, \quad a_1 + a_2 + 0 - a_4 = 0\\
&For\ w, \quad a_1 + 0 + a_3 - a_4 = 0\\
&For\ h, \quad 0 + a_2 + a_3 - a_4 = 0
\end{aligned}$$

And $a_1 + a_2 + a_3 + a_4 = 1$ for the inequality to become and equality.

Solving simultaneously,

$a_1 = 1/5$
$a_2 = 1/5$
$a_3 = 1/5$
$a_4 = 2/5$

And $C(a) = \left(\frac{10}{1/5}\right)^{1/5}\left(\frac{20}{1/5}\right)^{1/5}\left(\frac{40}{1/5}\right)^{1/5}\left(\frac{40}{2/5}\right)^{2/5}$

The minimum cost is $100

To obtain the design parameters, observe that the exponent values yield the proportionate cost of each contributing cost.

Thus for:

$$\begin{aligned}
a_1&: \quad 10lw = \frac{1/5}{5/5} \times 100 = 20\\
a_2&: \quad 20lh = \frac{1/5}{5/5} \times 100 = 20\\
a_3&: \quad 40wh = \frac{1/5}{5/5} \times 100 = 20\\
a_4&: \quad \frac{40}{lwh} = \frac{2/5}{5/5} \times 100 = 40
\end{aligned}$$

Solving the above, the optimum parameter values are:

$$l = 2yd \quad ; \quad w = 1\ yd \quad ; \quad h = 1/2\ yd$$