Lecture (Mar. 12th, 2019)

$$U_s = \frac{1+v}{6E}\left(2S_{yp}^2\right)$$

$$\therefore (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2S_{yp}^2$$

Introducing the design factor n_d we have,

$$\sigma' = \sigma_{eq} = \frac{S_{yp}}{n_d}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{\frac{1}{2}}$$

Where:

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{\frac{1}{2}}$$

IS known at the $Von\ Mises\ Stress$

For plane stress, $\sigma_3 = 0$,

$$\sigma' = (\sigma_1 + \sigma_2 + \sigma_1\sigma_2)^{\frac{1}{2}}$$

Which is the equation of an eclipse.

Note that in the case of pure shear, $\sigma_1 = -\sigma_2$ or $3\sigma_1^2 = S_{yp}^2$, and $\sigma_1 = 0.577S_{yp}$ while the maximum shear stress theory assumes $\sigma_1 = 0.5S_{yp}$

In terms of the rectangular stress components we can write σ' as

$$\sigma' = \frac{1}{\sqrt{2}}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_x - \sigma_z)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{\frac{1}{2}}$$

And for $2 - D$ stress:

$$\sigma' = \left(\sigma_x^2 - \sigma_{x_y} + \sigma_y^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

<u>Coulomb-Mohr Theory (For Ductile Materials)</u>

This theory can be used to predict failure for materials whose strength in tension and compression are not equal. If states that:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

Where either yield strength or ultimate strength can be used.

Incorporating the design factor n_d:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n_d}$$

For plane stress, if the two nonzero principal stresses are $\sigma_A \geq \sigma_B$ then:

If $\sigma_A \geq \sigma_B \geq 0$, then $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$

$$\therefore \sigma_A = \frac{S_t}{n_d}$$

If $\sigma_A \geq 0 \geq \sigma_B$, then $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$

$$\therefore \frac{\sigma_A}{S_t} - \frac{S_t}{S_c} = \frac{1}{n_d}$$

If $0 \geq \sigma_A \geq \sigma_B$, then $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$

$$\therefore \sigma_B = -\frac{S_c}{n_d}$$

Note that for pure shear τ, $\sigma_1 = -\sigma_3 = \tau$

The torsional yield strength occurs when $\tau_{max} = S_{sy}$

Substituting into:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

We get:

$$\frac{S_{sy}}{S_{yt}} + \frac{S_{sy}}{S_{yc}} = 1$$

$$S_{sy}S_{yc} + S_{sy}S_{yt} = S_{yt}S_{yc}$$

$$S_{sy} = \frac{S_{yt}S_{yc}}{S_{yt} + S_{yc}}$$

Reading Assignment:
Example 5-1
Example 5-2

<u>Maximum-Normal-Stress Theory (For Brittle Materials)</u>

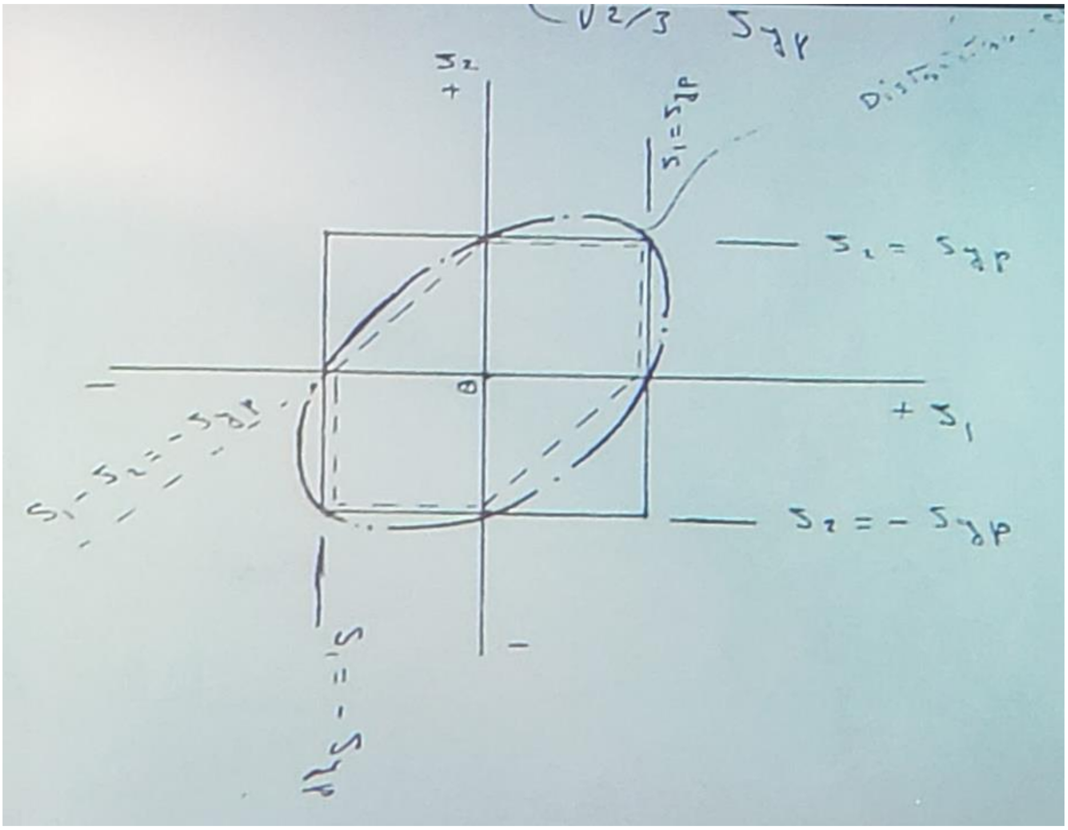
According to this theory, failure occurs at a point in a body when one of the principal stresses at that point equals the critical stress for that material.

If:

$$|\sigma_1| > |\sigma_2| > |\sigma_3|$$

Then:

$$\sigma_1 = \frac{S_{ut}}{n_d}$$



<u>Brittle Coulomb-Mohr Theory:</u>

$$\sigma_A = \frac{S_{ut}}{n_d} \quad ; \quad For\ \sigma_A \geq \sigma_B \geq 0$$

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n_d} \quad ; \quad For\ \sigma_A \geq 0 \geq \sigma_B$$

$$\sigma_B = -\frac{S_{uc}}{n_d} \quad ; \quad For\ 0 \geq\ \sigma_A \geq \sigma_B$$

Modified Coulomb-Mohr Theory:

$$\sigma_A = \frac{S_{ut}}{n_d}$$

$$For\ \sigma_A \geq \sigma_B \geq 0\ and\ \sigma_A \geq 0 \geq \sigma_B\ and\ \left|\frac{\sigma_B}{\sigma_A}\right| \leq 1$$

$$\frac{\sigma_A(S_{uc} - S_{ut})}{S_{uc}S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n_d}$$

$$For\ \sigma_A \geq 0 \geq \sigma_B\ and\ \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

$$\sigma_B = -\frac{S_{uc}}{n_d}$$

$$For\ 0 \geq \sigma_A \geq \sigma_B\ and\ \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

Reading Assignment:
Example 5-3
Example 5-4
Example 5-5

<u>Fatigue Failure (Variable Loading)</u>

Reading Assignment:
Sections 6.1 to 6.6

Fluctuating Stresses

Although most fluctuating stresses in machinery are sinusoidal in nature due to rotating elements, some irregular patterns do occur. However, regardless of its shape, if a pattern exhibits a single maximum and a single minimum force, its shape is not important, but the peaks are important. Let F_{max} be the largest force and F_{min} be the smallest force. Then a steady component, F_m, and an alternating component, F_a, can be constructed.

$$F_m = \frac{F_{max} + F_{min}}{2} \quad ; \quad F_a = \left|\frac{F_{max} - F_{min}}{2}\right|$$

(TODO – Picture)

Where:
σ_{min} = minimum stress
σ_{max} = maximum stress
σ_a = stress amplitude = $(\sigma_{max} - \sigma_{min})/2$
σ_m = mean stress or midrange stress = $(\sigma_{max} + \sigma_{min})/2$
σ_r = stress range = $2\sigma_a$
σ_s = steady, or state stress

- Key Factors in Fatigue Failure

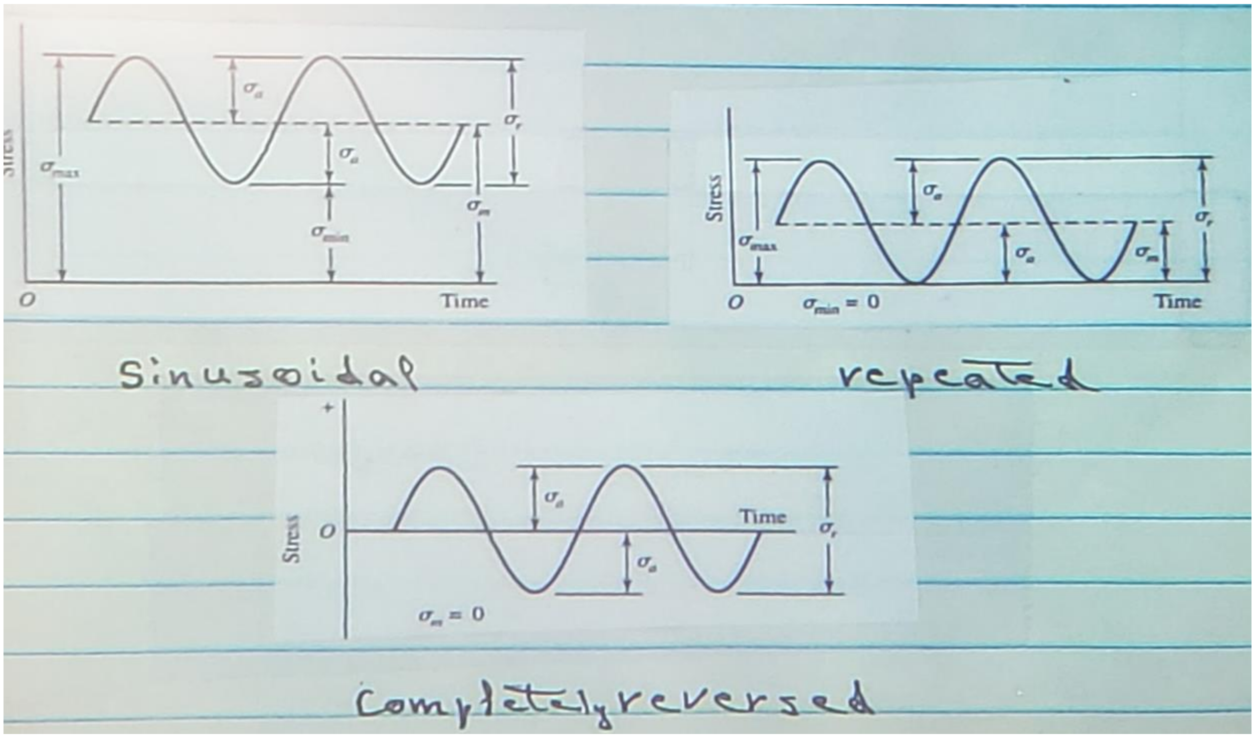
1 – A maximum stress of sufficient magnitude

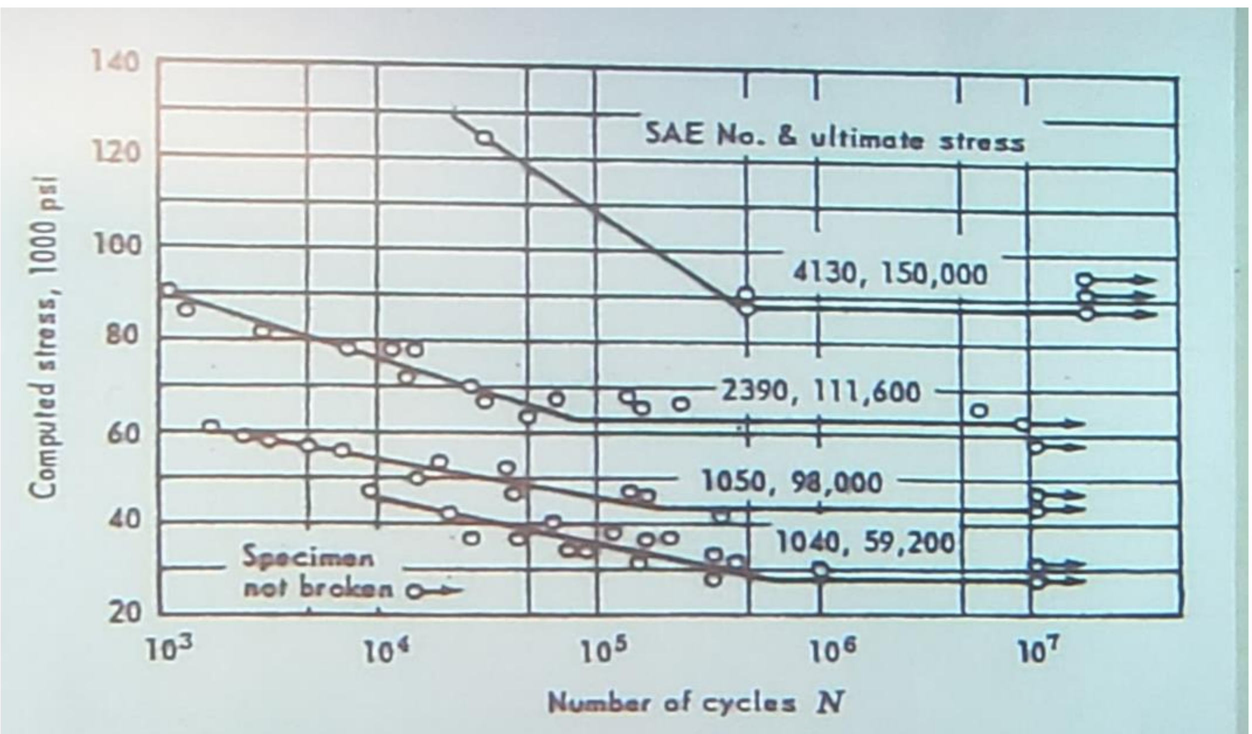
2 – An applied stress fluctuation of large enough magnitude

3 – A sufficient number of cycles of the applied stress

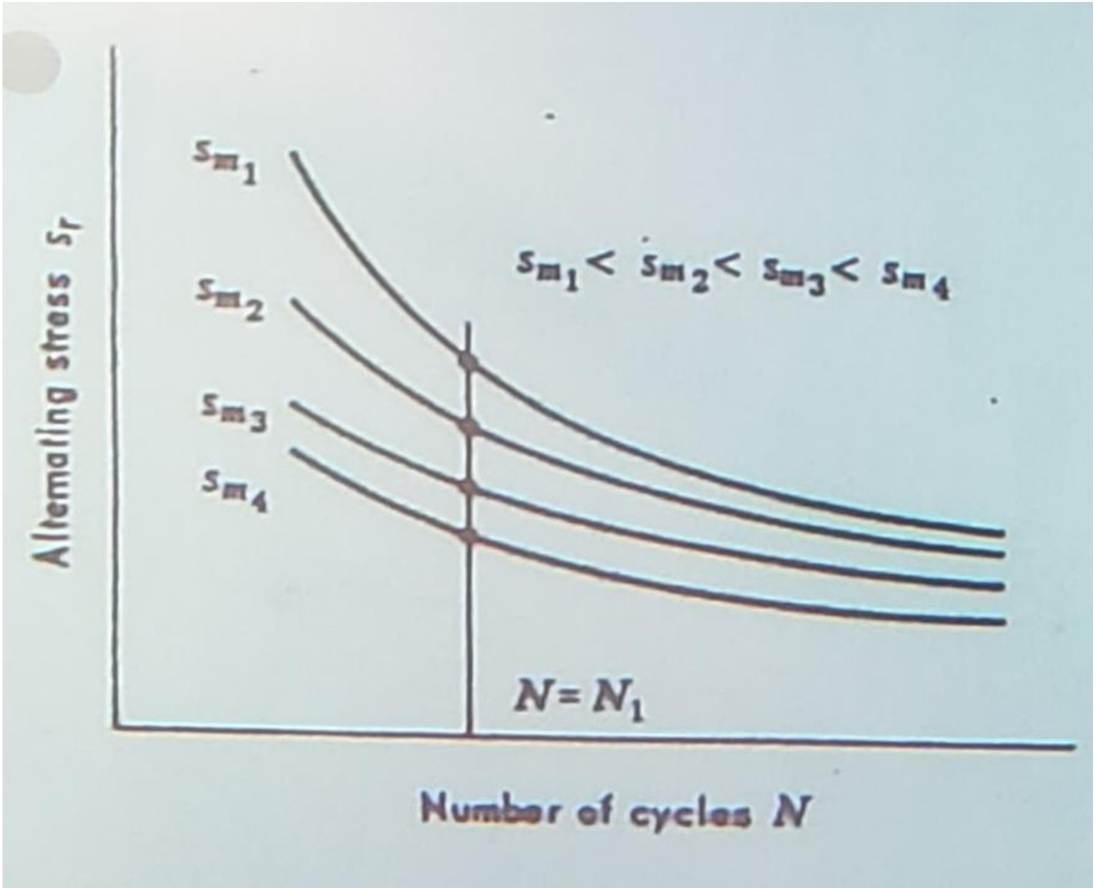
Fatigue design procedure

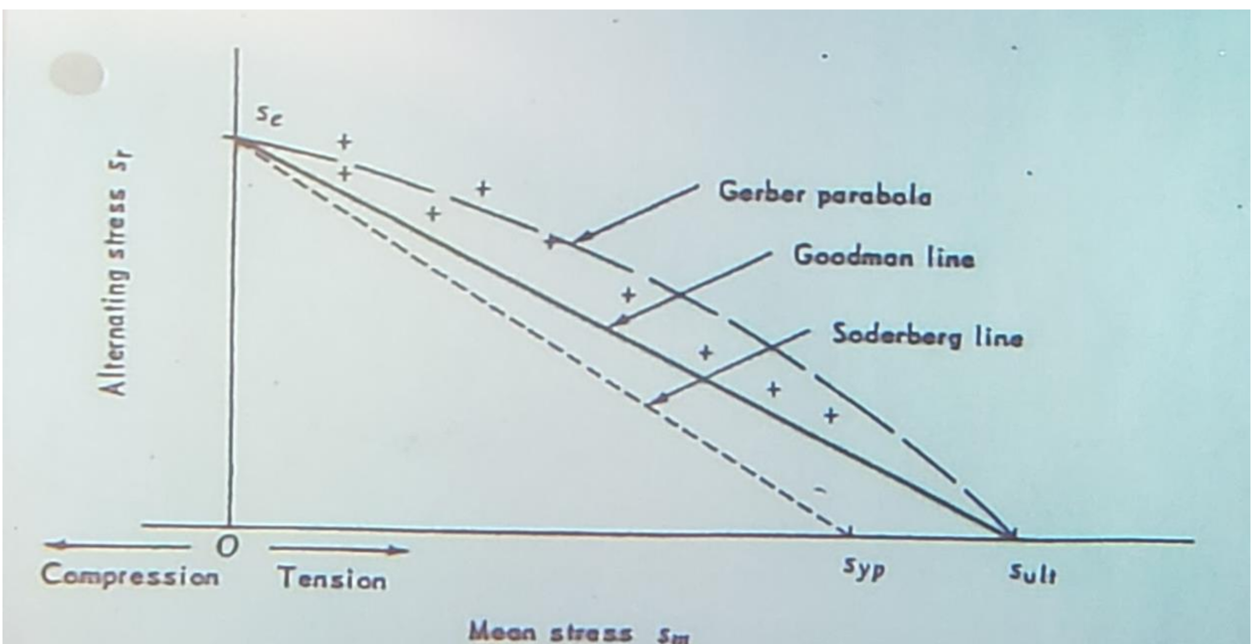
One of the most common methods of presenting engineering fatigue data is by means of the $S - N\ curve.$





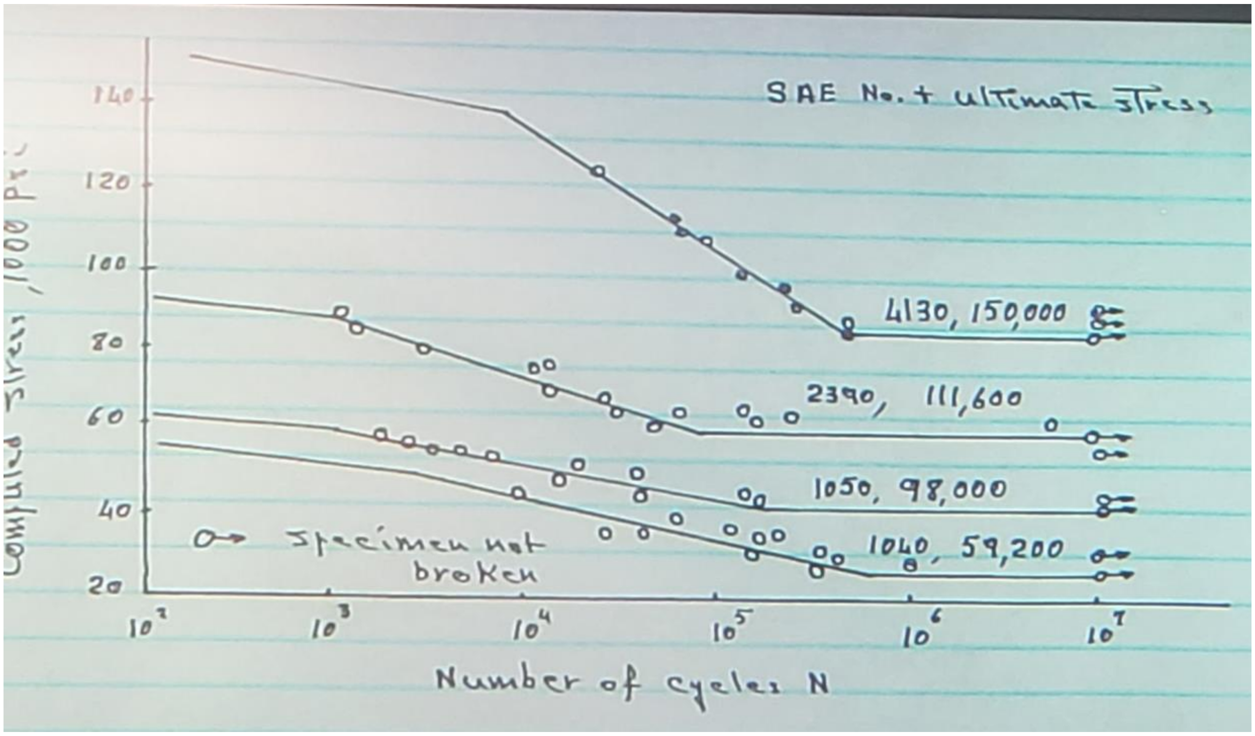
Fatigue-test results for endurance-limit determination.





Failure as a function of mean stress s_m and alternating stress s_r.

In this particular graph, if the $SAE\ No.$ or the ultimate strength is known, and if N is known (number of cycles). The fatigue or endurance limit of the material can be found. However, the above data is for zero mean stress $S_m = 0$. To solve for cases where $S_m \neq 0$, first $S_A - N\ Cures$ are plotted as shown. Then for a given $N = N_1$ the $S_a = S_m$ curve is plotted.



Lecture (Mar. 14th, 2019)

From the $S_n - S_m$ curve as shown in the figure, the following empirical relation was found:

$$S_a = S_e\left[1 - \left(\frac{S_m}{S_{ult}}\right)^P\right]$$

(TODO – Picture)

For Gerber Curve: $P = 2$
For Goodman Line: $P = 1$

When design is based on S_{yp} (yield strength) the Soderberg law is followed.

$$S_m = S_e\left(1 - \frac{S_m}{S_{yp}}\right)$$

And when a factor of safety is required

$$\sigma_a = \frac{S_e}{n_d}\left(1 - \frac{\sigma_m}{S_{yp}}n_d\right)$$

Where:

$$\sigma_a = \frac{S_a}{n_d} \quad ; \quad \sigma_m = \frac{S_m}{n_d}$$

Or:

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{yp}} = \frac{1}{n_d}$$

Where:
S_e = endurance strength for $S_m = 0$
S_{yp} = yield strength
S_{ult} = minimum ultimate tensile strength
n_d = design factor

Using $Goodman$ line:

$$\frac{\sigma_a}{S_e} = \frac{\sigma_m}{S_{ult}} = \frac{1}{n_d}$$

Using $Gerber$ line:

$$\frac{n_d\sigma_a}{S_e} = \left(\frac{n_d\sigma_m}{S_{ult}}\right)^2 = 1$$

Using $ASME$-elliptic line:

$$\left(\frac{n_d\sigma_a}{S_e}\right)^2 + \left(\frac{n_d\sigma_m}{S_{ult}}\right)^2 = 1$$

Using $Langer$ first-cycle-yielding:

$$\sigma_a + \sigma_m = \frac{S_{yp}}{n_d} 7$$

<u>Endurance Limit</u>

Based on a large number of actual test data from several sources, Charles R Mischke, in his paper "Prediction of Stochastic Endurance Strength,"

Trans. Of ASME, J. Vibration Acoustics Stress and Reliability in Design, Vol. 109, No.1, pp 113-122, January 1987, concluded that endurance limit can eb related to tensile strength.

For Steels:

$$S'_e = \begin{cases} 0.504\, S_{ut} \quad ; \quad S_{ut} \leq 200\ kpsi\ (1400\ MPa) \\ 100\ kpsi \quad ; \quad S_{ut} > 200\ kpsi \\ 700\ MPa \quad ; \quad S_{ut} > 1400\ MPa \end{cases}$$

Where:
S_{ut} = minimum tensile strength
S_e = endurance limit
S'_e = endurance limit of the rotating beam specimen

<u>Fatigue Strength</u>

Recall that:

$$\frac{\Delta\varepsilon_e}{2} = \frac{\sigma'_F}{E}(2N)^b$$

Where:
$\Delta\varepsilon_e$ = elastic strain range
σ'_F = true stress corresponding to fracture in one reversal
b = fatigue strength exponent
N = number of reversals or expected life
E = modulus of elasticity

Defining the specimen fatigue strength at a specific number of cycles as:

$$\left(S'_f\right)_N = \frac{E\Delta\varepsilon_e}{2}$$

Then,

$$\left(S'_f\right)_N = \frac{E\Delta\varepsilon_e}{2} = \sigma'_F(2N)^b$$

At 10^3 cycles:

$$\left(S'_f\right)_{10^3} = \sigma'_F(2 * 10^3)^6 = fS_{ut}$$

Where:

$$f = \frac{\sigma_F'}{S_{ut}} = (2 * 10^3)^b$$

See $Table\ A - 23$ for reliable value of σ_F' for selected steels. Or use $\sigma_F' = \sigma_o \varepsilon_m$, with $\varepsilon = \varepsilon_F'$ is known. Otherwise, you may use the SAE approximation for steels with $H_B \leq 500$ given as:

$$\sigma_F' = S_{ut} + 50\ kpsi \quad ; \quad \sigma_F' = S_{ut} + 345\ MPa$$

Substituting the endurance strength S_e' and corresponding cycles N_e and solving for b:

$$b = -\frac{\log(\sigma_F'/S_e')}{\log(2N_e)}$$

With values of σ_F' and b known for $70 \leq S_{ut} \leq 200\ kpsi$, Figure 6-18 is plotted where the graph is used to find approcimate values of f for various values of S_{ut} between 70 and 200 kpsi.

For actual mechanical component, we may write:

$$S_f = aN^b$$

Where is can be shown that for $10^3 \leq N \leq 10^6$

$$a = \frac{(f\ S_{ut})^2}{S_e}$$

$$b = -\frac{1}{3}\log\left(\frac{f\ S_{ut}}{S_e}\right)$$

If a completely reversed stress σ_{rev} is given, then:

$$N = \left(\frac{\sigma_{rev}}{a}\right)^{\frac{1}{b}}$$

For low-cycle, $1 \leq N \leq 10^3$ cycles:

$$S_f \geq S_{ut} N^{(\log f)/3}$$

For problems in the finite life range, $10^3 \leq N \leq 10^6$, stresses σ_m and σ_a are transformed into an equivalent completely reversing stress σ_R as follows:

For Goodman:

$$\sigma_R = \frac{\sigma_a S_{ut}}{S_{ut} - \sigma_m} = \sigma_{rev}$$

For Gerber:

$$\sigma_R = \frac{\sigma_m}{1 - \left(\frac{\sigma_m}{S_{ut}}\right)^2} = \sigma_{rev}$$

Reading assignment:
Example 6-2

Example: A bar of steel has the minimum properties $S_e = 40\ kpsi$, $S_y = 60\ kpsi$, and$S_{ut} = 80\ kpsi$. The bar is subjected to a steady torsional stress of $15\ kpsi$ and an alternating bending stress of $25\ kpsi$. Find the factor of safety guarding against a static failure, and either the factor of safety guarding against a fatigue failure or the expected life of the part. For static failure use the $Distorsion - Energy\ Theory$ (DE). For fatigue analysis use:

a) Modified Goodman criterion
b) Gerber criterion
c) ASME-elliptic criterion

Solution:

Given:
$S_e = 40\ kpsi$
$S_y = 60\ kpsi$
$S_{ut} = 80\ kpsi$
$\sigma_a = 25\ kpsi$
$\sigma_m = \tau_a = 0$
$\tau_m = 15\ kpsi$

Using the Distortion Energy Theorem for the alternating, mid-range, and maximum stresses, the $von - Mises$ stresses are:

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

Here $\sigma_y = 0$

$$\therefore \sigma' = \left(\sigma_x^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

And:

$$\sigma_a' = (\sigma_a^2 + 3\tau_a^2)^{\frac{1}{2}} = [(25)^2 + (3)(0)^2]^{\frac{1}{2}} = 25{,}000\ kpsi$$

$$\sigma_m' = (\sigma_m^2 + 3\tau_m^2)^{\frac{1}{2}} = [(0)^2 + (3)(15)^2]^{\frac{1}{2}} = 25.98\ kpsi$$

$$\sigma_{max}' = (\sigma_{max}' + 3\tau_{max}^2)^{\frac{1}{2}} = [(\sigma_a + \sigma_m)^2 + 3(\tau_a + \tau_m)^2]^{\frac{1}{2}}$$
$$= [(25)^2 + (3)(15)^2]^{\frac{1}{2}} = 36.06\ kpsi$$

$$n_y = \frac{S_y}{\sigma_{max}'} = \frac{60}{36.06} = 1.66$$

a) Modified Goodman:

$$n_f = \frac{1}{\left(\frac{\sigma_a}{S_e}\right) + \left(\frac{\sigma_m}{S_{ut}}\right)}$$

$$n_f = \frac{1}{\left(\frac{25}{40}\right) + \left(\frac{35.98}{80}\right)} = 1.05$$

b) Gerber:

$$\frac{n\sigma_a}{S_e} + \left(\frac{n\sigma_m}{S_{ut}}\right)^2 = 1$$

Or:

$$n_f = \left(\frac{1}{2}\right)\left(\frac{S_{ut}}{\sigma_m}\right)^2 \frac{\sigma_a}{S_e}\left[-1 + \sqrt{1 + \left(\frac{2\sigma_m S_e}{S_{ut}\sigma_a}\right)^2}\right]$$

$$n_f = 1.31$$

c) ASME Elliptic:

$$n_f = \sqrt{\frac{1}{\left(\frac{\sigma_a}{S_e}\right)^2 + \left(\frac{\sigma_m}{S_y}\right)^2}}$$

$$n_f = 1.32$$