Lecture (Jan. 8th, 2019)

The Design Process

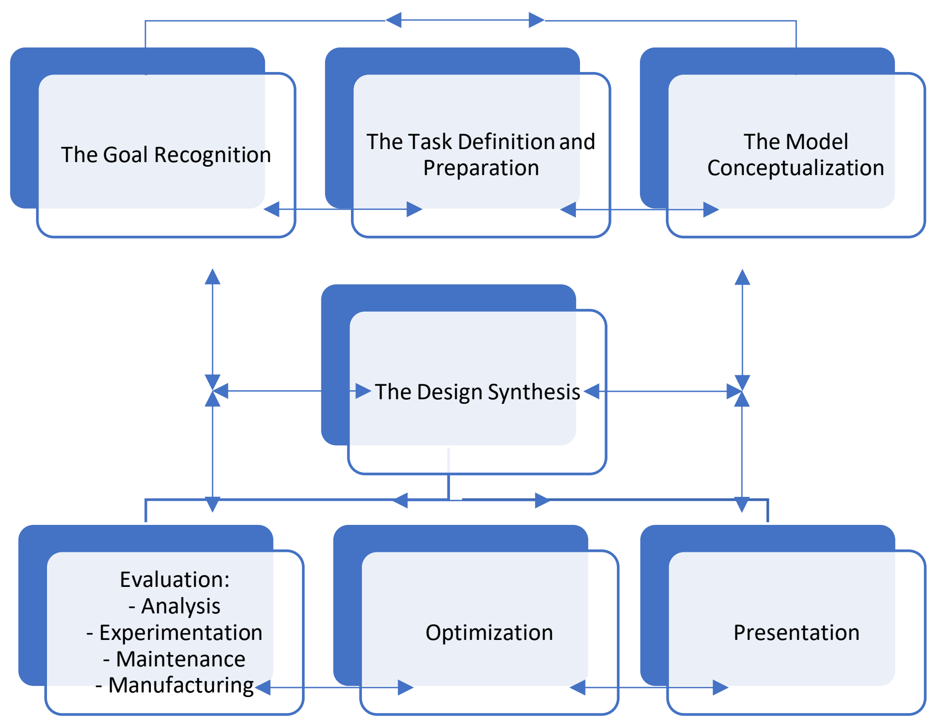
Horizontal Structure of Design

The horizontal structure of design may be considered to involve the following 8 stages:

1. Recognition
2. Definition
3. Preparation
4. Conceptualization
5. Synthesis
6. Evaluation
7. Optimization
8. Presentation

These stages are all inter-related. Reconsideration, re-decision, reformulation, re-examination and re-computation are continuously carried out as needed at any stage of the design.

Design Process Flow Chart



1. Recognition: The need
2. Definition: The problem exactly identified and defined.
3. Preparation: All pertinent information must be collected. Data, formal knowledge, and empirical know-how must be gathered, reviewed, and organized. Where anything is lacking, the gap must be filled in which the proper assumptions, compromises, and sound engineering judgement.

4. Conceptualization: Consideration of alternatives. No alternative arrangement that holds promise of probable success can be omitted.
5. Synthesis: At this stage, details must be conceptualized. Exact mechanisms are synthesized to supply the required motions, dynamic characteristics and time sequences. Components must be selected, material decided upon, fabricating process, etc…
6. Evaluation or Analysis: The designs, models, or possibly analogs must now be analyzed; critical parameters must be checked to see that all is satisfactory. This is performed on all alternatives not previously eliminated. The best design is then selected.
7. Optimization: Such as reliability, economy, weight, and space limitations, and life requirements can be very important factors. The design may thus have to be optimized with respect to the particular criterion.
8. Communication, both verbal and graphic, becomes of prime importance at this stage, because the design has no value until utilized.

Chronological or Vertical Structure of Design

A project must often be carried through the following design and planning phases:

1. Feasibility study: A technical success can easily go bankrupt when a financially sound need is not realized. However, the product must exist at least on paper if its need and feasibility are to be established.
2. Preliminary design: solutions that may have been suggested during the feasibility phase are considered. The surviving alternatives are synthesized sufficiently to reveal their overall features. The specific design concepts are thoroughly evaluated.
3. Detail design: Capacities are exactly sized, dimensions calculated, wear account for, parts detailed, tolerances established, and treatments completely detailed, and clearly described. It is now a producible design.

Lecture (Jan. 10th, 2019)

Decision Tree

Assume that three solutions exist to a primary problem. Should two sub-problems arise in the case of each solution, six choices are generated. If all six result in two new possible paths each, the number of decisions rises to twelve, an so it can continue.

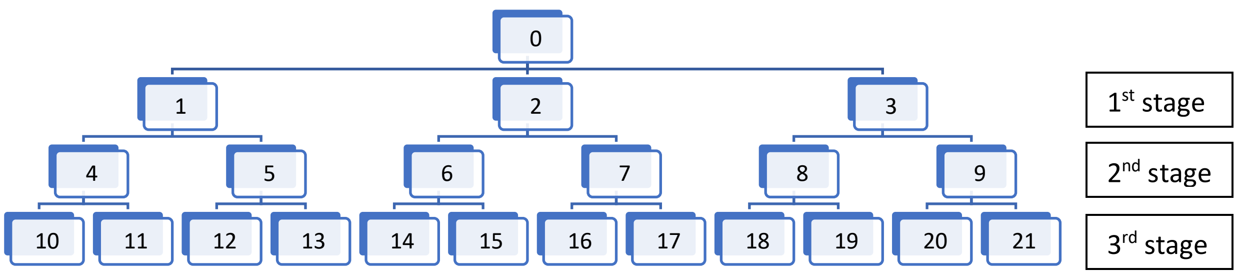
If there are m possibilities at the first stage and $(m \pm i)$ at each other stage, then the total number of required decisions is:

$$1 \cdot m_1(m \pm i)_2(m \pm i)_3 \dots (m \pm i)_n$$

Therefore, for the above example we have:

$$1 \cdot 3(3-1)(3-1) = 1 \cdot 3 \cdot 2 \cdot 2 = 12$$

The tree is shown below:



For the example above, consider the following:
line from 0 to 1 (m_1)
line from 1 to 4 $(m-i)_2$
line from 4 to 10 $(m-i)_3$

If the decision paths could flow from each first stage point to each second stage point, then $(m+i)_2 = (3+3)$ and the number of decision paths becomes:

$$1 \cdot 3(3+3)(3-1) = 36$$

If in addition each second stage point could connect to each third stage point, then:

$$1 \cdot 3(3+3)(3+3) = 108$$

Decision-Making

In making a decision a choice must be made among several possible acts by considering the consequences that may result as each act is applied to the problem at hand. Decisions are made under:

1- Certainty, where the action is known to lead to specific consequences.
2- Risk, when the action leads to specific outcomes that occurs with only known probability.
3- Uncertainty, where the action results in consequences that have unknown probabilities.

Arriving at a conclusion as to which alternative solutions, answer, decision, or design is best in a particular situation is a weighty and not absolutely defined mater. A systematic approach is, nevertheless, much more likely to lead to a more appropriate decision than in mere guessing or even contemplating.

One of the more satisfactory methods of decision-making is based on the Bayesian model which will be demonstrated by the following example:

Example (1.1): A large counterweight is to be made. Two questions needing decision confront the design engineers. Should the counterweight be cast or forged, and should it be made square or round in cross section?

Solution (1.1): The consequence matrix is established as shown below. These consequences are obviously based on a knowledge of the process of casting and forging involved as well as considerations of the relative strengths of different sections.

	Square	Round
Cast	Cheapest	Cheaper
Forged	Faster	Stronger

The desirabilities appear to be as shown in the following matrix. A desirability range other than 0-1 is used to illustrate its validity.

	Square	Round
Cast	2.0	1.0
Forged	0.5	1.5

The probabilities are judged in normalized form:

	Square	Round
Cast	0.4	0.6
Forged	0.5	0.5

Thus, the expected desirabilities are:

Cast: $2 \cdot 0.4 + 1 \cdot 0.6 = 1.4$
Forged: $0.5 \cdot 0.5 + 1.5 \cdot 0.5 = 1.0$

The better choice then seems to be to cast the counterweight. And since there appears a fair chance that the square section will prove cheaper, it is decided upon.

<u>Codes and Standards</u>

Standards: A standard is a set of specifications for parts materials, or processes intended to achieve uniformity, efficiency, and a specified quality.

Code: A code is a set of specifications for the analysis, design, manufacture, and construction of something. The purpose of a code is to achieve a specified degree of safety, efficiency, and performance or quality.

See section 1.6 in your textbook.

Reading assignment: Sections 1.1 to 1.11, and section 1.14
Assignment #1: 1.7, 1.8, 1.9, 1.12, 1.23

Lecture (Jan. 17th, 2019)

Reliability and Probability of Failure

The statistical measure of the probability that a mechanical element will not fail in use is called the reliability of the element and is related to the probability of failure, P_f

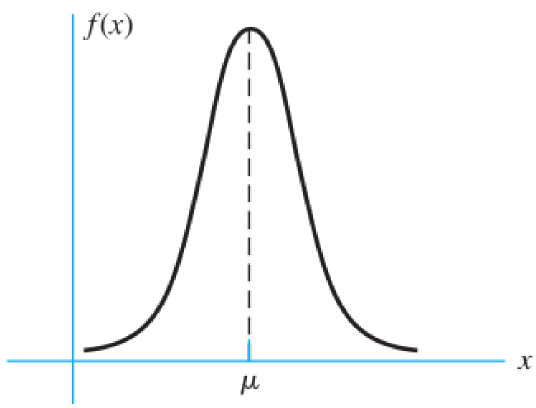
Probability of Failure, P_f

The probability density function (PDF) represents the distribution of events within a given range of values. The Gaussian (normal) distribution and the Weibull distribution are the most important continuous probability distributions for engineering use. The Weibull distribution is used in rolling contact bearing design and will be covered with this topic.

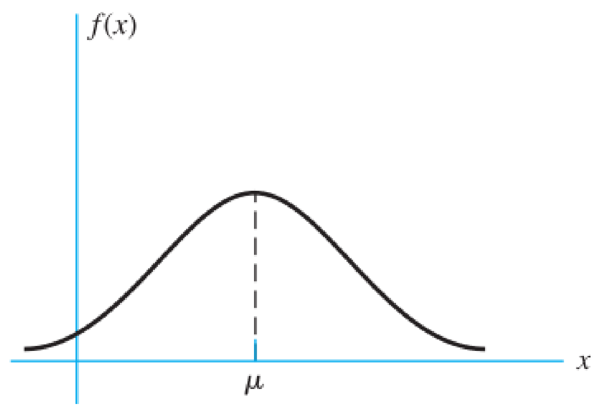
The probability density function (PDF) of the Gaussian (normal) distribution is expressed in terms of its mean, μ_x and its standard deviation $\widehat{\sigma_x}$ as:

$$f(x) = \frac{1}{\widehat{\sigma_x}\ \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{(x-\mu_x)}{\widehat{\sigma_x}}\right)^2\right]$$

And shown graphically below.



a) Small $\widehat{\sigma_x}$

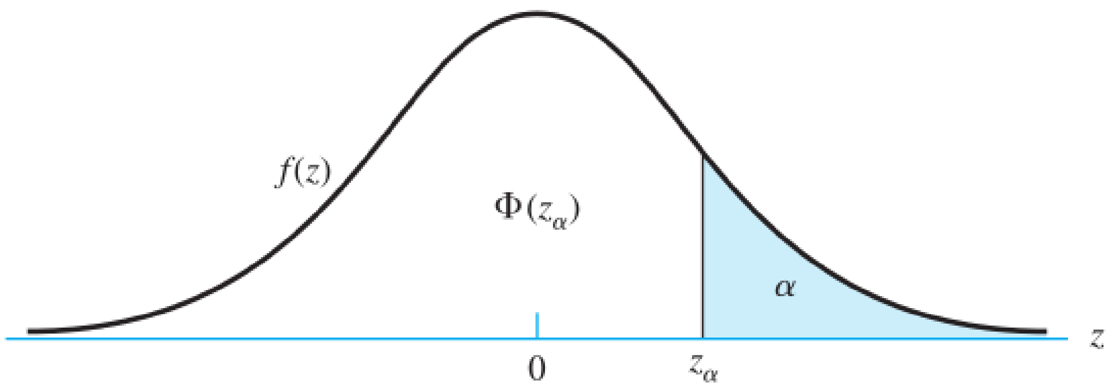


b) Large $\widehat{\sigma_x}$

The obtain the values of p_f the above equation must be integrated. The integrand in tabulated in Table 1-10 where x is placed in dimensionless form using the transform.

$$z = \frac{(x-\mu_x)}{\widehat{\sigma_x}}$$

And ∝ is defined as shown in the figure below.



The variant z has a mean value of zero and a standard deviation of one. In Table A-10, the probability of an observation less than z is $\Phi_{(z)}$ for negative values of z and $1-\Phi_{(z)}$ for positive values of z.

Example:
The lives of parts are often expressed as the number of cycles of operation that a specified percentage of a population will exceed before experiencing failure. The symbol L is used to designate this definition of life. Thus, we can speak of L_{10} life as the number of cycles to failure exceeded by 90 percent of a population of parts. Given a normal distribution model, with a mean of $\bar{L}=122.9$ kilocycles and standard deviation of $S_L=30.3$ kilocycles, estimate the corresponding L_{10} life.

Solution:
$\bar{L}=122.9\ Kcycles\ ;\ S_L=30.3\ Kcycles$

$z_{10}=\dfrac{(x-\mu_x)}{\widehat{\sigma_x}}=\dfrac{x_{10}-122.9}{30.3}$

$or, \quad x_{10}=122.9+30.3\,z_{10}=L_{10}$

$From\ Table\ A-10,\ \ for\ 10\%\ of\ failure,\ \ z_{10}=-1.282$

$\therefore L_{10}=122.9+30.3(-1.282)=84.1\ Kcycles$

Discrete distributions may be approximated by continuous distributions. In an N samples of events, let x_i be the value of an event $(1,2,\ldots,k)$ and f_i the number of times the event x_i occurs within the frequency range. The discrete mean $\bar{x}$ and the standard deviation S_x are thus defined as:

$$\bar{x}=\frac{1}{\mathrm{N}}\sum_{i=1}^{k} f_i x_i$$

$$S_x=\sqrt{\frac{\left(\sum_{i=1}^{k} f_i x_i^2-N\bar{x}^2\right)}{N-1}}$$

Example – Problem 1-14

Determination of the ultimate tensile strength of stainless-steel sheet (17-7 PH, condition TH1050), in sizes from 0.016 to 0.062 in, in 197 tests combined into seven classes were

$S_{ut}\ (kpsi)$	174	182	190	198	206	214	222
f	6	9	44	67	53	12	6

Where S_{ut} is the class mid point and f is the class frequency estimate the mean and the standard deviation.

x	f	fx	fx^2
174	6	1044	181656
182	9	1638	298116
190	44	8360	1588400
198	67	13266	2626668
206	53	10918	2249108
214	12	2568	549552
222	6	1332	295704
Σ	197	39126	7789204

Reliability

$$\bar{x} = \frac{1}{\mathrm{N}}\sum_{i=1}^{k} f_i x_i = \frac{39126}{197} = 198.61\ Kpsi$$

$$S_x = \sqrt{\frac{\left(\sum_{i=1}^{k} f_i x_i^2 - N\bar{x}^2\right)}{N-1}}$$

$$S_x = \left[\frac{7789204 - 197(198.61)^2}{197-1}\right]^{\frac{1}{2}} = 9.68\ Kpsi$$

Reliability Mathematics

Definition: Reliability is defined as the probability that equipment will perform its intended function satisfactorily for the intended time in the intended environment.

If no units are started with, and N_f failure are experienced in a given time t, the reliability R for time t is defined mathematically as (1):

$$R = \frac{N_o - N_f}{N_o} = \frac{N_s}{N_o} = 1 - \frac{N_f}{N_o}$$
$$= 1 - p_f \ \ with \ \ 0 \le R \ \le 1$$

The rate of failure is the time rate of change of reliability or (2):

$$r = \frac{dR}{dt}$$

Substitution (1) in (2) and differentiating:

$$\frac{dR}{dt} = \frac{d}{dt}\left(1 - \frac{N_f}{N_o}\right) = -\frac{1}{N_o}\frac{dN_f}{dt}$$

And the rate at which units fail and/or survive is:

$$\frac{dN_f}{dt} = -N_o\frac{dR}{dt} = \frac{d}{dt}(N_o - N_s) = -\frac{dN_s}{dt}$$

The instantaneous probability of failure per hour, p, can be found by dividing the above rate by the number of units surviving at that instant (3):

$$\rho = \frac{1}{N_s}\frac{dN_f}{dt} = -\frac{N_o}{N_s}\frac{dR}{dt} = -\frac{1}{R}\frac{dR}{dt}$$

$$\therefore \rho\, dt = -\frac{dR}{R}$$

$$and\ \ln R = -\int_0^t \rho\, dt$$

$$But\ at\ t = 0, R = 1\ ;\ and\ R(t) = e^{-\int_0^t \rho\, dt}$$

Combining (2) and (3) to get:

$$r = \frac{dR}{dt} = -\rho R$$

Lecture (Jan. 22nd, 2019)

Figure 9-1

The instantaneous probability of failure ρ, often called hazard rate, varies with time as indicated. This tends to greatly complicate the computation. Fortunately, with little error, for the mid-life period between the rapid burn-in and burn-out intervals, ρ is assumed time invariant ($\bar{\rho}$) as shown in the figure

For this region, the law of reliability simplifies to:

$$R = e^{-\bar{\rho}\int_0^t dt} = e^{-\bar{\rho}\cdot t}$$

The hazard rate is the number of failures occurring per hour for each survived unit. Therefore, its reciprocal σ is the number of survival hours to the next failure or the meantime to failure.

$$\sigma = \frac{1}{\bar{\rho}}$$

and: $$R = e^{-\frac{t}{\sigma}}$$

<u>Reliability of Complex Systems</u>

Systems are made of many components, all interrelated and most of them contributing to the unreliability of the system as a unit. The joint reliability is predicted from individual component probabilities as follows:

Series Systems

Systems consisting of several components so connected and independent that if one part fails the entire system fails, are series systems.

In a series system the reliability is the product of the reliabilities of each component.

$$\therefore R_s = R_1 \cdot R_2 \cdot R_3 \ldots R_n = \prod_1^n R_i$$

$$= e^{-(\bar{\rho}_1 + \bar{\rho}_2 + \bar{\rho}_3 + \cdots + \bar{\rho}_n)t} = e^{-\sum_1^n \bar{\rho}_i t}$$

The unreliability or probability of failure of such a system is:

$$Q_s = 1 - e^{-\sum_1^n \bar{\rho}_i t}$$

Parallel Redundant Systems

When very high system reliabilities are required, duplicate components and even entire duplicate circuits become desirable, so that if the first fails, the second will carry on.

	$R = e_1^{-\overline{\rho_1}t}$	
Begin	$R = e_2^{-\overline{\rho_2}t}$	End
	$R = e_3^{-\overline{\rho_3}t}$	

This parallel reliability is referred to as parallel redundancy, because all units operate simultaneously. The probability that one of the two parallel components will survive is the sum of the probabilities of the three outcomes; neither of components A and B fails, A fails but not B, and B fails but not A.

$$\therefore R_p = e_1^{-\overline{\rho_1}t} + e_2^{-\overline{\rho_2}t} - e_1^{-(\overline{\rho_1}+\overline{\rho_2})t} = R_1 + R_2 - R_1R_2$$

The probability of failure is Q_p:

$$Q_p = \left(1 - e_1^{-\overline{\rho_1}t}\right)\left(1 - e_2^{-\overline{\rho_2}t}\right)$$

The probability of survival for n components is simpler to compute via unreliability:

$$Q_p = Q_{p1} \cdot Q_{p1} \cdot \ldots \cdot Q_{pn} = \prod_{i=1}^{n} Q_i$$

$$\text{and:} \qquad R_p = 1 - Q_p$$

Stand-by Systems

When it is impractical to operate a system with parallel branches, and yet some assurance of continued operation is necessary, stand-by units become advisable. Such a system may be regarded as a simple system with multiple lives.

	Stand-by Unit	
Begin	Primary Unit	End

This obeys a principle known as $Poisson's\ distribution$ which yields:

$$e^{-\bar{\rho}t}\left[1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!} + \cdots + \frac{(\bar{\rho}t)^n}{n!}\right] = 1$$

If one stand-by unit is present, the system reliability would be:

$$R_B = e^{-\bar{\rho}t}(1 + \bar{\rho}t)$$

And with two stand-by units:

$$R_B = e^{-\bar{\rho}t}\left(1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!}\right)$$

And for n stand-by units:

$$R_B = e^{-\bar{\rho}t}\left[1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!} + \cdots + \frac{(\bar{\rho}t)^n}{n!}\right]$$

Example: A series-parallel system is made of components as depicted in the figure. The probabilities for each component, all for the same period of time, is as indicated in the box. Compute the reliability of the system.

Begin	0.96	0.98	0.92	0.95	End
	0.96	0.98	0.92	0.95	

Solution:

The reliability of each series branch (since they are the same) is:

$$R_s = R_1 \cdot R_2 \cdot R_3 \cdot R_4$$
$$R_s = (0.96)(0.98)(0.92)(0.95) = 0.85$$

The reliability of the two parallel branches, and therefore for the system, is

$$R_p = 2R_s - R_1 R_2 = (2)(0.82) - (0.82)(0.82)$$
$$R_p = 0.97$$

Example: Calculate the reliability of a system with two stand-by units if each has a mean life to failure of $100\ hr$, for a period of $10\ hr$. Compute this with the reliability for the system after one stand-by unit is removed; after both stand-by units are removed.

Solution: We have:

$$\bar{\rho} = \frac{1}{m} = \frac{1}{100} = 0.01 = hazard\ rate$$

i) System reliability with two stand-by units:

$$R_B = e^{-\bar{\rho}t}\left(1 + \bar{\rho}t + \frac{(\bar{\rho}t)^2}{2!}\right)$$

$$R_B = e^{-(0.01)(10)}\left[1 + (0.01)(10) + \frac{\left((0.01)(10)\right)^2}{2!}\right] = 0.9998453$$

ii) System reliability with one stand-by unit:

$$R_B = e^{-(0.01)(10)}[1 + (0.01)(10)] = 0.9953211$$

iii) System reliability with no stand-by:

$$R_B = e^{-(0.01)(10)} = 0.9048374$$

<u>Relating Design Factor to Reliability</u>

Stress and strength are statistical in nature. In the probability density functions for stress σ and strength S shown in the figure below, the mean values of stress and strength are $\bar{\sigma} = \mu_\sigma$ and $\bar{S} = \mu_s$ respectively.

$\therefore The\ average\ designfactor\ \bar{n}_d$:

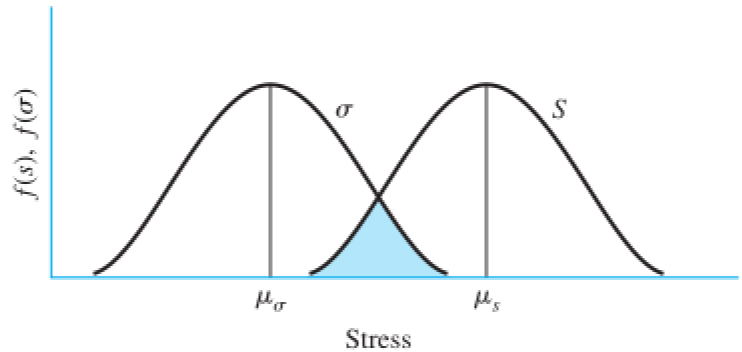
$$\bar{n}_d = \frac{\mu_s}{\mu_\sigma}$$

The margin of safety m at any value of σ and s is:

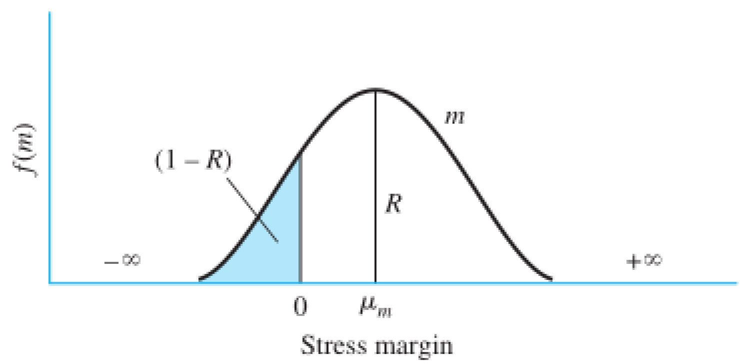
$$m = S - \sigma$$

The average margin of safety $\overline{m}$ is:

$$\overline{m} = \mu_s - \mu_\sigma$$



(a)



Lecture (Jan. 24th, 2019)

For the overlap in Fig. (a) $\sigma > S$ and the margin of safety is negative. The reliability that a part will perform without failure, R, is the area of the margin of safety distribution (Fig. (b)) for $m > 0$.

Reliability is then the probability that $m > 0$.

$$R = p\ (S > \sigma) = p\ (\{S - \sigma\} > 0) = p\ (m > 0)$$

Noting that for normal distributions:

$$\mu_m = \mu_s - \mu_\sigma$$

And:

$$\hat{\sigma}_m = (\hat{\sigma}_s^2 + \hat{\sigma}_\sigma^2)^{\frac{1}{2}}$$

We write:

$$Z_m = \frac{(m - \mu_m)}{\hat{\sigma}_m}$$

To find the probability that $m > 0$ we substitute $m = 0$ in Z_m.

$$Z_m = \frac{(0 - \mu_m)}{\hat{\sigma}_m} = \frac{-\mu_m}{\hat{\sigma}_m} = -\frac{(\mu_s - \mu_\sigma)}{(\hat{\sigma}_s^2 + \hat{\sigma}_\sigma^2)^{\frac{1}{2}}}$$

Dividing all terms by μ_σ:

$$Z_m = -\frac{\left(\frac{\mu_s}{\mu_\sigma} - 1\right)}{\left(\frac{\hat{\sigma}_s^2}{\mu_\sigma} + \frac{\hat{\sigma}_\sigma^2}{\mu_\sigma}\right)^{\frac{1}{2}}} = -\frac{\bar{n}_d - 1}{\left(\frac{\hat{\sigma}_s^2}{\mu_\sigma}\frac{\mu_s}{\mu_2} + \frac{\hat{\sigma}_\sigma^2}{\mu_\sigma}\right)^{\frac{1}{2}}} = -\frac{\bar{n}_d - 1}{\left(\bar{n}_d^2\frac{\hat{\sigma}_s^2}{\mu_s^2} + \frac{\hat{\sigma}_\sigma^2}{\mu_\sigma}\right)^{\frac{1}{2}}}$$

Introducing the terms $C_s = \sigma_s/\mu_s$ and $C_\sigma = \sigma_\sigma/\mu_\sigma$

$$Z_m = -\frac{\bar{n}_d - 1}{(\bar{n}_d^2 C_s^2 + C_\sigma^2)^{\frac{1}{2}}}$$

Solving for n_d:

$$n_d = \frac{1 + [1 - (1 - Z^2 C_s^2)(1 - Z^2 C_\sigma^2)]^{\frac{1}{2}}}{1 - Z^2 C_s^2} \quad where\ R > 0.5$$

$$n_d = \frac{1 - [1 - (1 - Z^2 C_s^2)(1 - Z^2 C_\sigma^2)]^{\frac{1}{2}}}{1 - Z^2 C_s^2} \quad where\ R \leq 0.5$$

Where Z refers to Z_m.

Important note: Comparing Fig(b) to Tab. A-10

$$R = 1 - \Phi_{(z)} \quad z \leq 0$$

$$R = \Phi_{(z)} \qquad z > 0$$

<u>Optimizing by Differentiation</u>

When all functional constrains can be involved in a single criterion function, the parameters are readily optimized. The derivative of the criterion function with respect to each parameter is set to zero.

The n equations are then solved simultaneously for the optimum parametric values. Of course, these must be established consistent with any regional limitations that may apply.

If the criterion is expressible in terms of a single significant parameter, the mathematical problem reduces to finding where the stope is zero.

Example: A rectangular tank with its base twice as long as wide is to have a volume of $12\ ft^2$. Determine the most economical dimensions, if the bottom sheet material costs $20¢/ft^2$ and the sides $10¢/ft^2$.

Solution: Cost of bottom = $a\ \times 2a\ \times 2a = 40a^2¢$

And the four sides cost = $2 \times a \times b \times 10 + 2 \times 2ab \times 10 = 60ab¢$

The total cost = $C = 40a^2 + 60ab$

$$V = 2a \times a \times b = 12 \quad \text{or} \quad b = 6/a^2$$

$$c = 40a^2 + \frac{360}{a}$$

$$\frac{dc}{da} = 80a - \frac{360}{a^2} = 0$$

From which $a = 1.65'$ and $b = 2.2'$

And the most economical tank is:

$$1.65\ ft\ \times 3.30\ ft\ \times 2.20\ ft$$

The cost is:

$$C = 40a^2 + \frac{360}{a} = 40(1.65)^2 + \frac{360}{1.65}$$

$$C \cong 3.27$$

<u>Optimization by Dual Variables</u>

This method consists of the replacement of the generalized function by a dual problem that results in the simultaneous solutions of a system of linear equation. If the number of unknowns exceeds the number of equations that can be written, the method will not yield a solution. The dual problem derives from a particular treatment of arithmetic and geometric mean expressions.

If $a_1 + a_2 + a_3 + \cdots + a_n = 1$, the expression concerned with the weighted arithmetic mean is:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

And that for the weighted geometric mean is:

$$(x_1^{a1})(x_2^{a2})(x_3^{a3}) \dots (x_n^{an})$$

The inequality (the arithmetic average is greater than the geometric) becomes an equality only if all of the x-terms are, in addition to the above, equal. The inequality is best written in the following form. If $a_ix_i = u_i$, then

$$u_1 + u_2 + \cdots + u_n \geq \left(\frac{u_1}{a_1}\right)^{a1}\left(\frac{u_2}{a_2}\right)^{a2} \dots \left(\frac{u_n}{a_n}\right)^{an}$$

Example: Four hundred (400) yd^3 of sand must be ferried across a river. The sand is to be shipped across in open containers of length L, width W, and height h. The bottom and sides of the container cost $\$10/yd^2$ and the ends $\$20/yd^2$. Each round trip on the ferry costs $10¢$. The containers are assumed to have no salvage value after the transfer. Minimize the transfer cost.

Solution: The total transportation cost is:

$$\begin{aligned} &C = (\$10)(bottom) \\ &+(\$10)(side)(2\ sides) \\ &+(\$20)(end)(two\ ends) \\ &+(total\ volume\ of\ sand/volume\ of\ container)(cost\ of\ trip) \end{aligned}$$

$$C = 10lw + 20lh + 40wh + \left(\frac{400}{lwh}\right)\left(\frac{10}{100}\right)$$

The dual function is:

$$C(a) = \left(\frac{10lw}{a_1}\right)^{a_1}\left(\frac{20lh}{a_2}\right)^{a_2}\left(\frac{40wh}{a_3}\right)^{a_3}\left(\frac{40}{a_4lwh}\right)^{a_4}$$

To satisfy a minimum C, the dual variables must conform to:

$$(lw)^{a_1}(lh)^{a_2}(wh)^{a3}\left(\frac{1}{lwh}\right)^{a_n} = 1$$

And eliminating the variables with the dual function becomes:

$$C(a) = \left(\frac{10}{a_1}\right)^{a_1}\left(\frac{20}{a_2}\right)^{a_2}\left(\frac{40}{a_3}\right)^{a_3}\left(\frac{40}{a_4}\right)^{a_4}$$

In order that the dual variables relation be satisfied, the sum of the exponents for each variable must equal zero. Thus,

$$\begin{aligned} &For\ l, \quad a_1 + a_2 + 0 - a_4 = 0 \\ &For\ w, \quad a_1 + 0 + a_3 - a_4 = 0 \\ &For\ h, \quad 0 + a_2 + a_3 - a_4 = 0 \end{aligned}$$

And $a_1 + a_2 + a_3 + a_4 = 1$ for the inequality to become and equality.

Solving simultaneously,

$a_1 = 1/5$
$a_2 = 1/5$
$a_3 = 1/5$
$a_4 = 2/5$

And $C(a) = \left(\frac{10}{1/5}\right)^{1/5} \left(\frac{20}{1/5}\right)^{1/5} \left(\frac{40}{1/5}\right)^{1/5} \left(\frac{40}{2/5}\right)^{2/5}$

The minimum cost is $100

To obtain the design parameters, observe that the exponent values yield the proportionate cost of each contributing cost.

Thus for:

$$\begin{aligned} &a_1: \quad 10lw = \frac{1/5}{5/5} \times 100 = 20 \\ &a_2: \quad 20lh = \frac{1/5}{5/5} \times 100 = 20 \\ &a_3: \quad 40wh = \frac{1/5}{5/5} \times 100 = 20 \\ &a_4: \quad \frac{40}{lwh} = \frac{2/5}{5/5} \times 100 = 40 \end{aligned}$$

Solving the above, the optimum parameter values are:

$$l = 2yd \quad ; \quad w = 1\ yd \quad ; \quad h = 1/2\ yd$$

Lecture (Jan. 29th, 2019)

Minimization by Method of Lagrange Multipliers

Given a function

$$U(x_1, x_2, \dots, x_n)$$

We wish to find its minimum value & co-ordinates of $x_1, x_2 ---$ etc. at this minimum. Let the function be subject to the constraints

$$\varphi_1(x_1, x_2 --- x_n) = 0$$

$$\varphi_2(x_1, x_2 --- x_n) = 0$$

Procedure:

(1) Form a new function

$$U(x_1, x_2, \dots, x_n) = U(x_1, x_2, \dots, x_n) + \lambda_1\varphi_1(x_1, x_2, \dots, x_n) + \lambda_2\varphi_2(x_1, x_2, --- x_n) \dots\ etc.$$

Let there be N variable & M constraints. We now treat the λ's as variables and write the $(m+n)$ equations.

$$\frac{\partial U^*}{\partial x_1} = 0$$

$$\frac{\partial U^*}{\partial x_2} = 0$$

$$etc.$$

$$\frac{\partial U^*}{\partial \lambda_1} = 0$$

Final step: solve the set of algebraic equations (2) for the variables $x_1, x_2, --- x_n$. The function $U(x_1, x_2)$ will be minimum at this point.

NOTE: The method of Lagrange Multipliers is generally good for handling problems where constraints on the variables exist.

Assignment #2: 3.4, 3.5, 3.6, 3.9, 3.10, 3.14

Midterm is February 12th, unless we can find a room, and then it will be on the Saturday.

Load and Stress Analysis

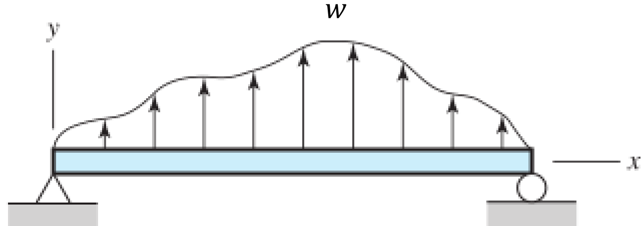
Beams

1. Loads Distributed Along a Line

Let's assume that the function w describing a particular distributed load is known. The graph of w is called the loading curve. The force acting on an element dx of the line is wdx.

The total force F is:

$$F = \int_L w\,dx$$



The moment about the origin due to the force exerted on the element dx is $xwdx$, so the total moment about the origin due to the distributed load is:

$$M = \int_L xw\,dx$$

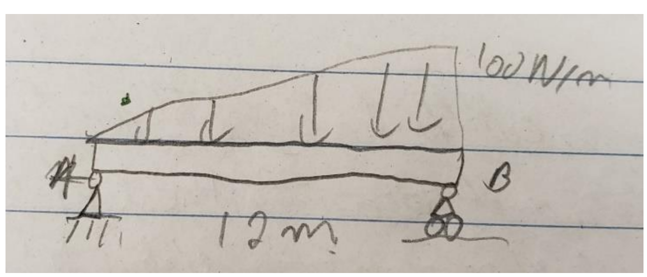
Or:

$$M = \bar{x}F = \int_L xw\,dx$$

Where F is the equivalent load if placed at the position:

$$\bar{x} = \frac{\int_L xw\,dx}{\int_L w\,dx}$$

Example: The beam is subjected to a triangular distributed load whose value at B is $100\ N/m$. Determine the reactions at A and B.



First method:

$$w = \frac{100}{12}x \quad (N/m)$$

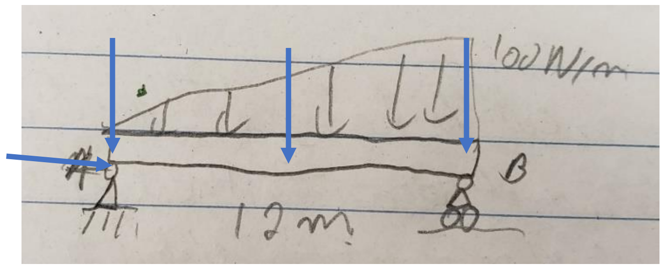
The total load is:

$$F = \int_L w\,dx = \int_0^{12} \left(\frac{100}{12}\right) x\,dx = 600\ N$$

The clockwise moment about A due to the load is:

$$M_A = \int_L xw\,dx = \int_0^{12} \left(\frac{100}{12}\right) x^2\,dx = 4800\ N \cdot m$$

From the equilibrium equations:

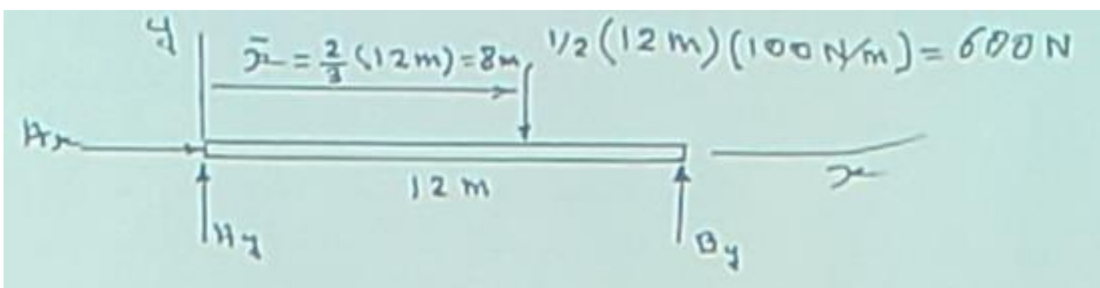


$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = A_y + B_y - 600 = 0$$

$$-\sum M_A = 12B_y - 4800 = 0$$

$$\therefore A_x = 0 \quad ; \quad A_y = 200\ N \quad ; \quad B_y = 400\ N$$

Second Method



$$F = \left(\frac{1}{2}\right) \cdot (12\ m) \cdot \left(100\,\frac{N}{m}\right) = 600\ N$$

$$\bar{x} = \left(\frac{2}{3}\right) \cdot (12\ m) = 8\ m$$

$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = A_y + B_y - 600 = 0$$

$$-\sum M_A = 12B - 8 \cdot 600 = 0$$

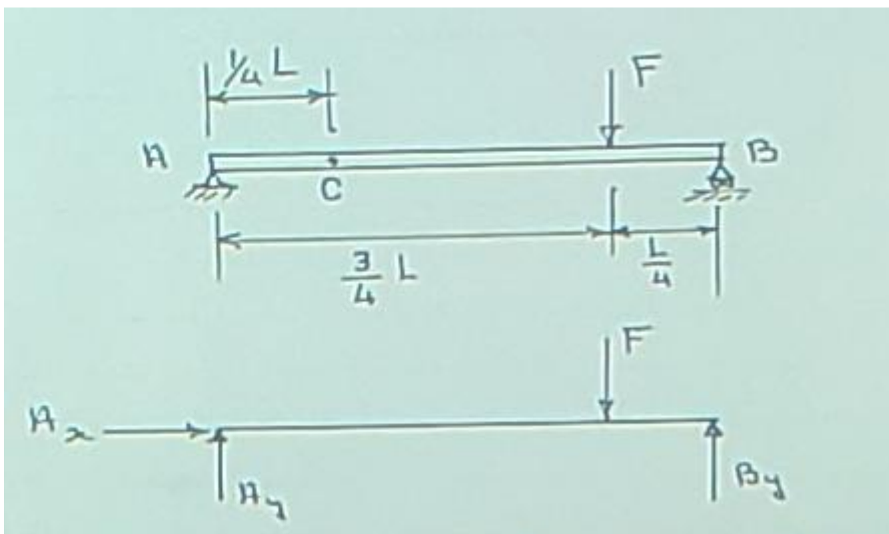
$$\therefore A_x = 0 \quad ; \quad A_y = 200\ N \quad ; \quad B_y = 400\ N$$

2. Internal Forces and Moments in Beams

Determining the internal forces and moment at a particular cross section of a bream typically involves three steps:

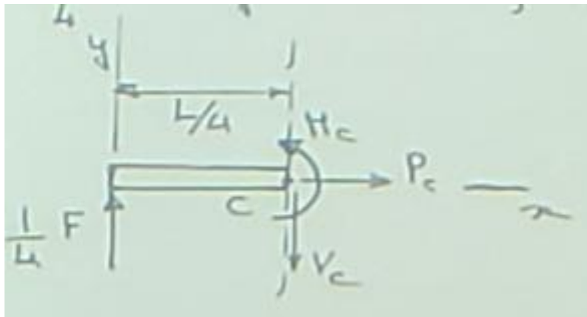
1. Determine the external forces and moments – Draw free-body diagram of the beam and determine the reactions at tits supports. If the beam is a member of a structure, you must first analyze the structure.
2. Draw the free-body diagram of part of the beam – cut the beam at the point at which you want to determine the internal forces and moment and draw the free-body diagram of one of the resulting parts. You can choose the part with the simplest free-body diagram, if your cut divides a distributed load, don't represent the distributed load by an equivalent force until after you have obtained your free-body diagram.
3. Apply the equilibrium equations – use the equilibrium equations to determine the axial force P, the shear force V, and the bending moment M.

Example: Determine the internal forces and moment at C.



$$\sum F_x = A_x = 0 \quad ; \quad \sum M_A = LB_y - F\left(\frac{3}{4}L\right) = 0 \quad ; \quad B_y = \left(\frac{3}{4}\right)F$$

$$\sum F_y = \left(\frac{3}{4}\right)F + A_y - F = 0 \quad ; \quad A_y = \left(\frac{1}{4}\right)F$$

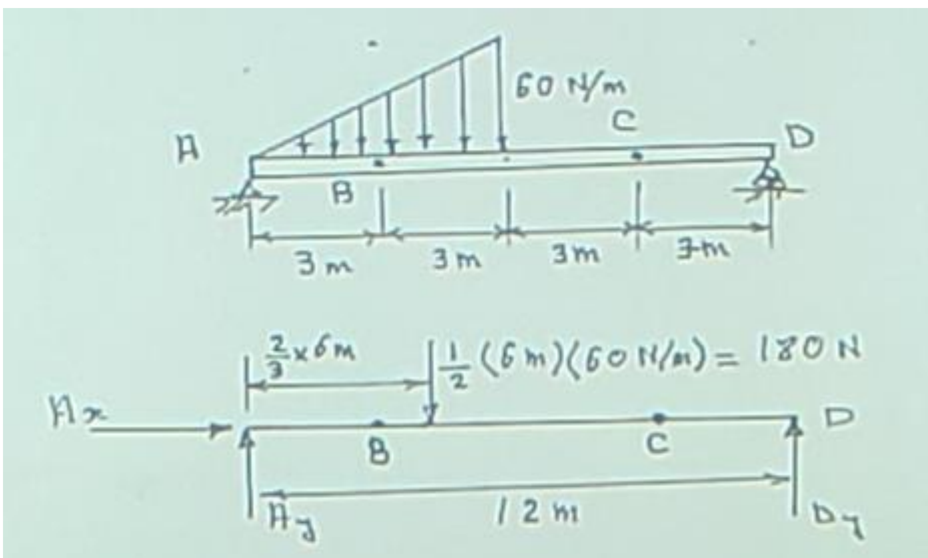


$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = \left(\frac{1}{4}\right)F - V_c = 0 \quad ; \quad V_c = \left(\frac{1}{4}\right)F$$

$$\sum M_C = M_c - \left(\frac{1}{4}\right)\left(\frac{F}{4}\right) = 0 \quad ; \quad M_c = \left(\frac{1}{16}\right)FL$$

$$P_c = 0 \quad ; \quad V_c = \left(\frac{1}{4}\right)F \quad ; \quad M_c = \left(\frac{1}{16}\right)FL$$

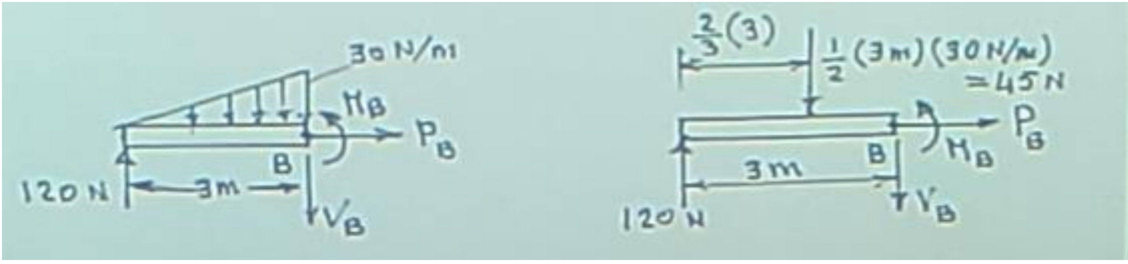
Example: Determine the internal forces and moment at (a) and B and (b) at C



$$\sum F_x = A_x = 0 \quad ; \quad \sum F_y = A_y + D_y - 180 = 0$$

$$\sum M_A = 12D_y - 4(180) = 0$$

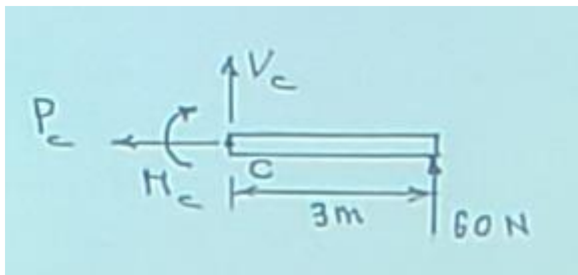
$$A_x = 0 \quad ; \quad A_y = 120\ N \quad ; \quad D_y = 60\ N$$



$$\sum F_x = P_B = 0 \quad ; \quad \sum F_y = 120 - 45 - V_B = 0$$

$$\sum M_B = M_B + (1)(45) - (3)(120) = 0$$

$$P_B = 0 \quad ; \quad V_B = 75\ N \quad ; \quad M_B = 315\ N \cdot m$$



$$\sum F_x = -P_C = 0$$

$$\sum F_y = V_c + 60 = 0$$

$$\sum M_C = -M_C + (3)(60) = 0$$

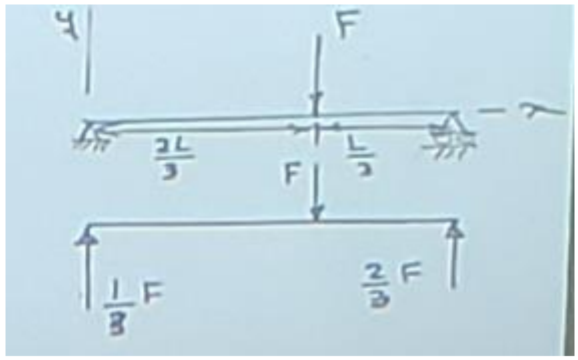
Solving we obtain:

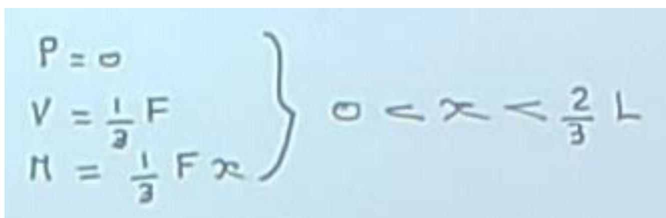
$$P_C = 0 \quad ; \quad V_C = -60 \quad ; \quad M_C = 180\ N \cdot m$$

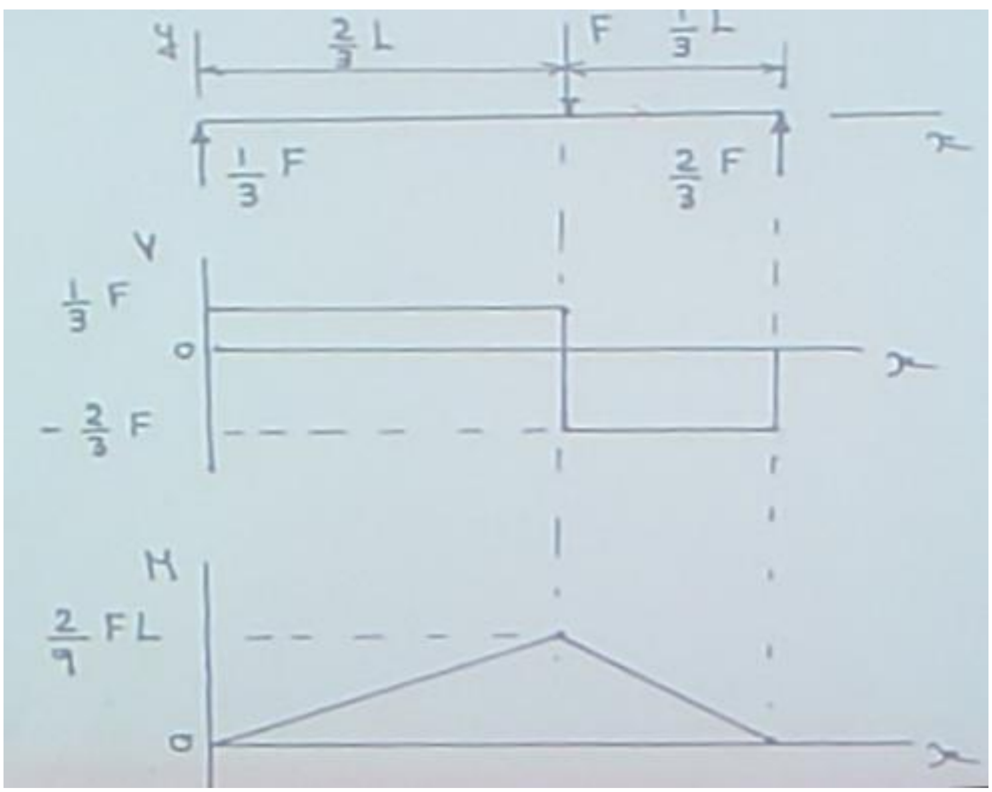
Lecture (Jan. 31st, 2019)

3. Shear Force and Bending Moment Diagrams

The shear force and bending moment diagrams are simply the graphs of V and M respectively, as functions of x. They show the changes in the shear force and bending moment that occur along the beam's length as well as their maximum and minimum values.







3. <u>Relations between Distributed Load, Shear Force, and Bending Moment</u>

$$\frac{dv}{dx} = -w$$

$$\frac{dM}{dx} = v$$

If we define $q(x)$ as the load intensity with units of force per unit length and is positive in the positive y −direction. Then,

$$q = -w = \frac{dv}{dx} = \frac{d^2M}{dx^2}$$

And:

$$V = \int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q\,dx$$

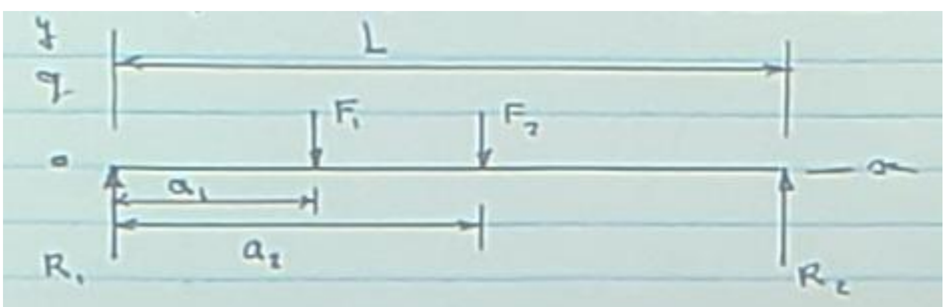
And:

$$M = \int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V\,dx$$

Singularity Functions

Singularity functions are shown in Table 3.1 constitute an easy means of integrating across discontinuities. Consequently, they are used to write general expressions for shear force and bending moment in beams in the presence of concentrated forces and moments.

Example – Derive expressions for the loading, shear force, and bending moment of the beams as shown.



Solution

$$q = R_1 < x >^{-1} - F_1 < x - a_1 >^{-1} - F_2 < x - a_2 >^{-1} + R_2 < x - L >^{-1}$$

Have:

$$\int_{V_A}^{V_B} dV = \int_{x_A}^{x_B} q\,dx = V_B - V_a$$

And:

$$V = 0 \; at \; x = -\infty$$

$$\therefore V = \int_{-\infty}^{x} q \, dx = R_1 < x >^0 - F_1 < x - a_1 >^0 - F_2 < x - a_2 >^0 + R_2 < x - L >^0$$

Also have:

$$V = 0 \; at \; x > L$$

$$\therefore R_1 - F_1 - F_2 + R_2 = 0 \; - \; Equation \; (1)$$

The bending moment:

$$\int_{M_A}^{M_B} dM = \int_{x_A}^{x_B} V \, dx = M_B - M_A$$

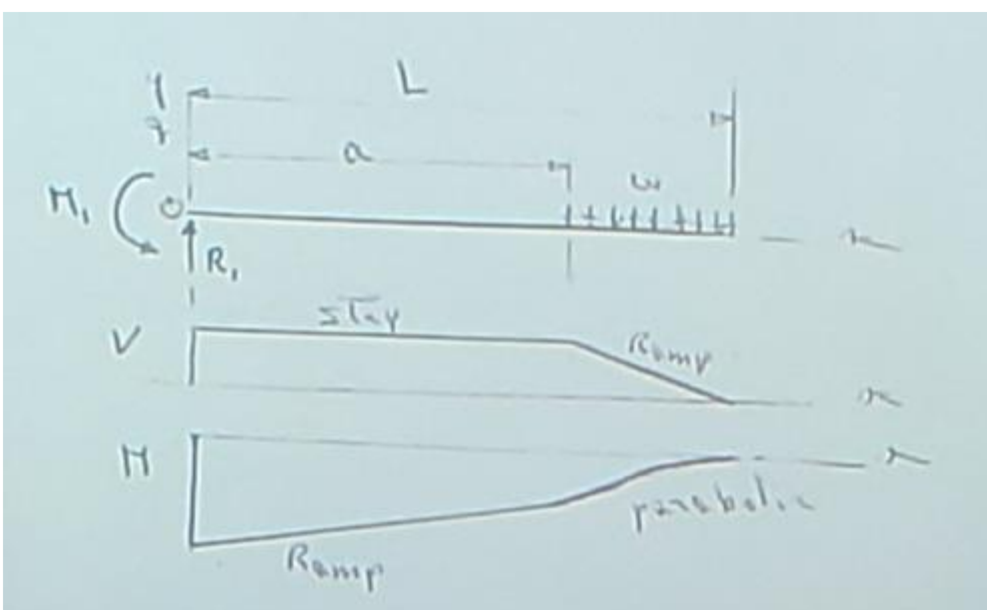
$$\therefore M = \int_{-\infty}^{x} V \, dx = R_1 < x >^1 - F_1 < x - a_1 >^1 - F_2 < x - a_2 >^1 + R_2 < x - L >^1$$

And:

$$R_1 L - F_1(L - a_1) - F_2(L - a_2) = 0 \; - Equation \; (2)$$

$$(1) \; and \; (2) \; are \; solved \; for \; R_1 \; and \; R_2$$

Example – The figure shows the loading diagram for a beam cantilevered at 0 and having a uniform load w acting on the position $a \leq x \leq L$. Derive the shear force and moment sections. M_1 and R_1 are the support reactions.



Solution – The loading function is:

$$q = -M_1 < x >^{-2} + R_1 < x >^{-1} - w < x - a >^0$$

First integration to obtain V:

$$V = \int_{-\infty}^{x} q\, dx = -M, x >^{-1} + R_1 < x >^0 - \frac{w}{2} < x - a >^1$$

Second Integration:

$$M = \int_{-\infty}^{x} V\, dx = -M_1 < x >^0 + R_1 < x >^{-1} - \frac{w}{2} < x - a >^2$$

For x slightly larger than L:

$$V = M = 0$$

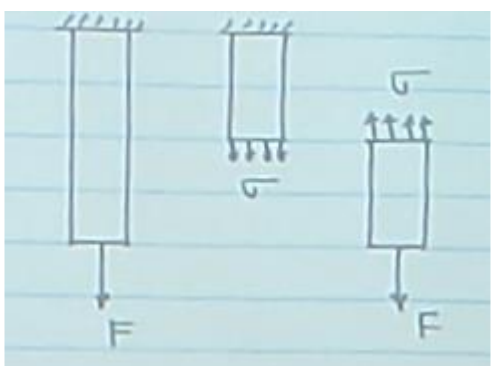
$$\therefore V_{x>L} = -M_1(0) + R_1 - w(L - a) = 0 \quad - \quad Equation\ (1)$$

And:

$$M_{x>L} = -M_1 + R_1 L - \frac{w}{2}(L - a)^2 = 0 \ - \ Equation\ (2)$$

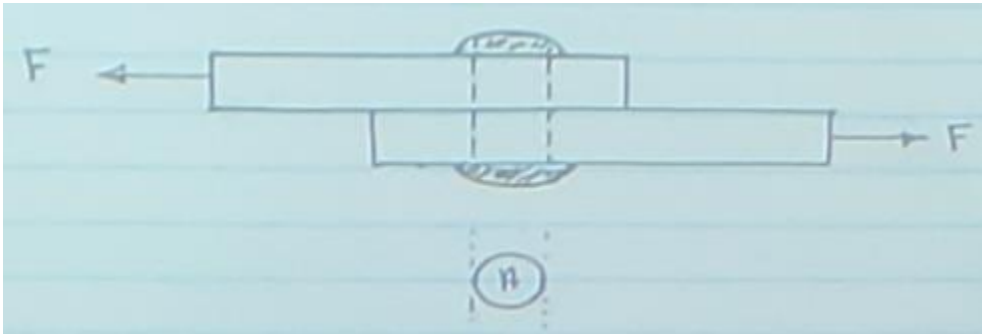
- *Pure Tension or Compression*

$$\sigma = \frac{F}{A}$$



- *Pure Shear Stress*

$$\tau = \frac{F}{A}$$



Elastic Strain

- *Strain*

$$\varepsilon = \frac{\delta}{L}$$

From Hooke's Law:

$$\sigma = E\varepsilon$$

$$\tau = G\gamma$$

Where:

$$E - modulus\ of\ elasticity$$
$$\varepsilon - strain$$
$$G - shear\ modulus$$
$$\gamma - shear\ strain$$

And:

$$\therefore\ \frac{F}{A} = E\frac{\delta}{L}$$

And:

$$\delta = \frac{FL}{AE}$$

- Poisson's Ratio μ or v

$$\mu = -\frac{lateral\ strain}{axial\ strain}$$

The three elastic constants are related by:

$$E = 2G(1 + \mu)$$

Stress-Strain Relations

- *Uniaxial Stress*

$$\varepsilon_1 = \frac{\sigma_1}{E} \quad ; \quad \varepsilon_2 = -\mu \cdot \varepsilon_1 \quad ; \quad \varepsilon_3 = -\mu \cdot \varepsilon_1$$

- *Biaxial Stress*

$$\varepsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E}$$

$$\varepsilon_3 = \frac{-\mu\sigma_1}{E} - \frac{\mu\sigma_2}{E}$$

Solving for σ_1 and σ_2:

$$\sigma_1 =$$

$$\sigma_2 =$$

- *Triaxial Stress*

$$\varepsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$\varepsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Or:

$$\sigma_1 = \frac{Ee_1(1-\mu) + \mu E(\varepsilon_2 + \varepsilon_3)}{1 - \mu - 2\mu^2}$$

$$\sigma_2 = \frac{Ee_2(1-\mu) + \mu E(\varepsilon_1 + \varepsilon_3)}{1 - \mu - 2\mu^2}$$

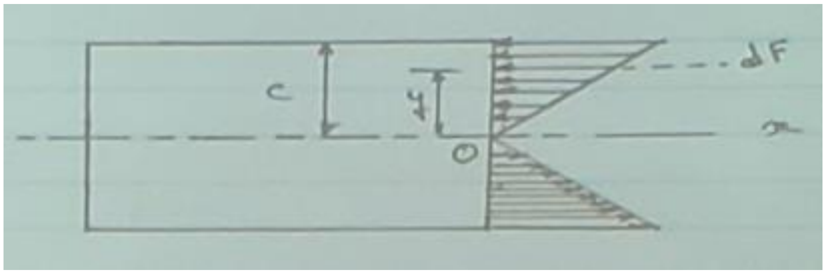
$$\sigma_3 = \frac{Ee_3(1-\mu) + \mu E(\varepsilon_1 + \varepsilon_2)}{1 - \mu - 2\mu^2}$$

Lecture (Feb. 5th, 2019)

Midterm - ATAC-1003, 1:00pm on Saturday
Will cover material from Weeks 1-4 only

Normal Stress in Bending

It can be shown that the neutral axis and centroidal axis in straight beams are the same.



Moment of dF about O is:

$$dM = dF\, y$$

And:

$$M = \int^{A} y\, \sigma\, dA$$

But:

$$\frac{\sigma}{y} = \frac{\sigma_c}{c} \quad ; \quad \sigma = \frac{\sigma_c}{c}$$

And:

$$M = \int^{A} y^2\, \frac{\sigma_c}{c}\, dA$$

$$= \frac{\sigma_c}{c} \int^{A} y^2\, dA$$

$$= \frac{\sigma}{y} \int^{A} y^2\, dA$$

But:

$$\int^{A} y^2 \cdot dA = I$$

And:

$$M = \frac{\sigma}{y} \cdot I$$

And:

$$\sigma = \frac{My}{I}$$

$\sigma =$ Stress at distance y from the neutral axis
$I =$ Moment of inertia of the cross-section about the neutral axis
$M =$ Applied bending moment

$$\sigma_{t,max} = \frac{Mc_t}{I} \quad ; \quad \sigma_{c,max} = \frac{Mc_c}{I}$$

For symmetric beam sections:

$$c_t = c_c = c$$

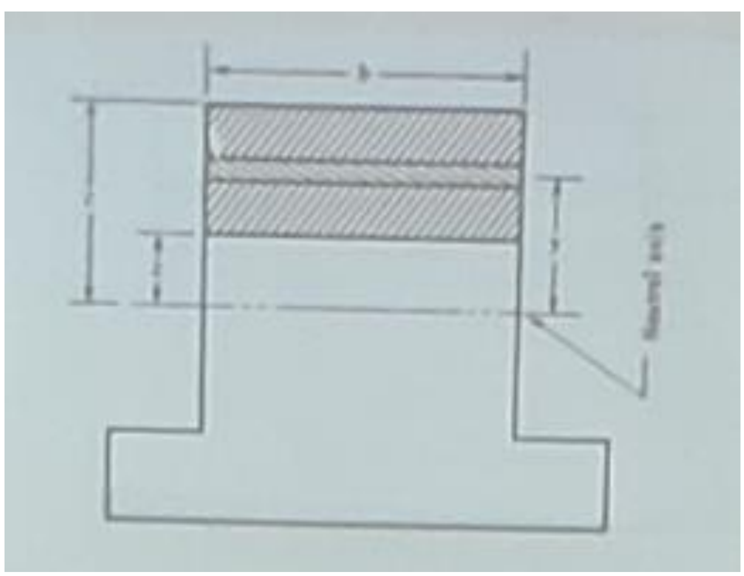
$$|\sigma_t| = |\sigma_c| = \frac{Mc}{I}$$

<u>Shear Stress Due to Bending</u>

In addition to normal stresses induced by bending of a beam, transverse shearing stresses are induced between he elements or fibers, provided the bending moment varies along the length of the beam. According to the strength-of-materials methods.

$$\tau = \frac{V}{Ib}\int_z^c y\,dA = \frac{VQ}{Ib}$$

$\tau =$ Shear Stress
$I =$ Moment of inertia of the cross-section
$b =$ beam width at the section
$Q = \int_z^c y\,dA =$ moment of area of the element about the neutral axis
$V =$ shearing force at the section
$z =$ location where shear stress is of interest



For rectangular cross-section:

$$\tau_{max} = \frac{3V}{A}$$

For solid circular cross section:

$$\tau_{max} = \frac{4V}{3A}$$

For thin-walled circular tube:

$$\tau_{max} = \frac{2V}{A}$$

$A =$ cross sectional area

Two-Plane Bending

When bending occurs in both xy and xz planes of cross sections with one or two planes of symmetry, the bending stresses are given by:

$$\sigma_x = \frac{M_z\, y}{I_z} + \frac{M_y\, z}{I_y}$$

The maximum bending stress for a solid circular section in this case is:

$$\sigma_{max} = \frac{M\, c}{I} = \frac{\left(M_y^2 + M_z^2\right)^{\frac{1}{2}}\left(\frac{d}{2}\right)}{\pi\frac{d^4}{64}}$$

$$\sigma_{max} = \frac{32}{\pi d^3}\left(M_y^2 + M_z^2\right)^{\frac{1}{2}}$$

Torsion of Circular Shafts

Torsional moments induce shear stresses on cross-sections normal to the axis of bars and shaft.

For circular shafts:

$$\tau = \frac{T\, r}{J}$$

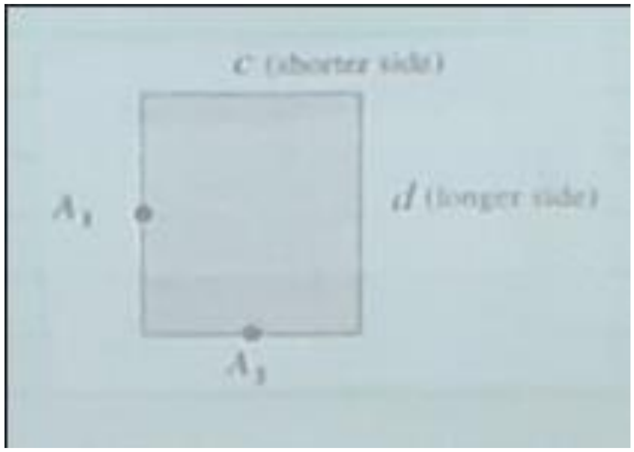
$\tau =$ Induced shear stress
$r =$ distance from the center of the shaft to the point of stress
$J =$ polar moment of inertia

For a solid circular shaft:

$$\tau_{max} = \frac{T\, d}{2\, J} = \frac{T\, d}{2\,\pi\frac{d^4}{32}} = \frac{16\, T}{\pi\, d^3}$$

Torsion of Rectangular Bars

The general equations for stress and deformation in rectangular bars may be written in the following form.



For point A_1:

$$\tau = \frac{T}{\propto_1 \ b\, c^2}$$

For point A_2:

$$\tau = \frac{T}{\propto_2 \ b\, c^2}$$

For angular deformation (radians per inch of length):

$$\theta_1 = \frac{T}{\beta\, G\, b\, c^3}$$

$T =$ Torque
$b = d =$ breadth of section (width)
$t = c =$ thickness of section
$\propto_i =$ coefficient from the table below
$\beta =$ coefficient from the table below
$G =$ shear modulus

The maximum shear stress on the cross-section occurs at the center A_1 of the long side and is found by using $\propto_1$

TABLE 3-3 CONSTANT FOR TORSION OF RECTANGULAR BARS

b/c	1.00	1.20	1.50	1.75	2.00	2.50	3.00	4.00	5.00	6.00	8.00	10.00	∞
α_1	0.208	0.219	0.231	0.239	0.246	0.258	0.267	0.282	0.291	0.299	0.307	0.312	0.333
α_2	0.208	0.235	0.269	0.291	0.309	0.336	0.355	0.378	0.392	0.402	0.414	0.421	...
β	0.1406	0.166	0.196	0.214	0.229	0.249	0.263	0.281	0.291	0.299	0.307	0.312	0.333

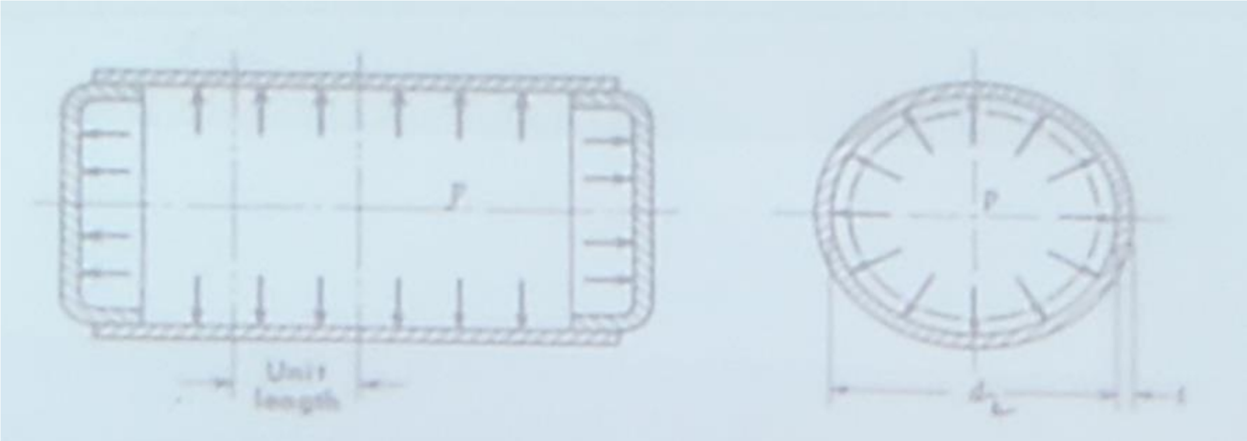
The following approximate formula for the maximum torsional stress in a rectangular section was given by Timoshenko and McCullough

$$\tau_{max} = \frac{T}{b\,t^2}\left(3 + 1.8\frac{t}{b}\right)$$

Pressure Cylinder

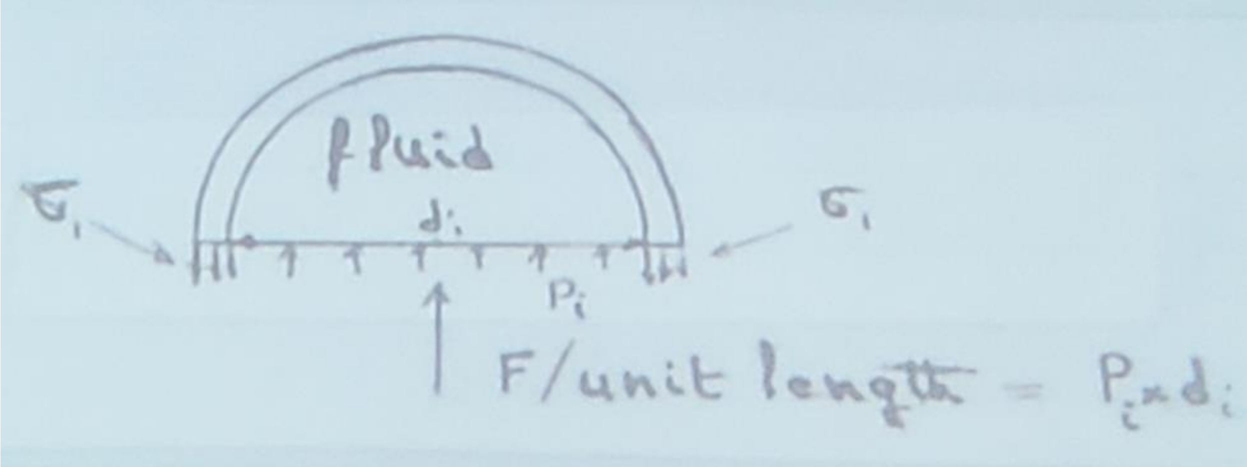
Thin walled cylinders $d/t \gg 10$:

Neglecting the effects of curvature of the cylinder wall, and assuming tensile stresses are uniformly distributed over the section of the wall.



Cylinder with internal pressure.

$2\,\sigma_1\,t = P_i\,d_i$



The average tangential stress is:

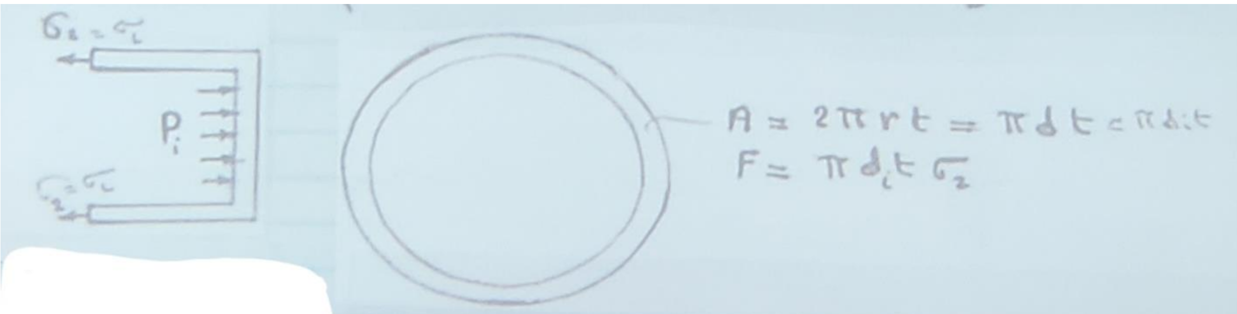
$$\sigma_1 = \sigma_t = \frac{P_i d_i}{2t}$$

$\sigma_1 = \sigma_t =$ tangential stress or hoop stress

And an approximation to the maximum tangential stress is:

$$\sigma_{t,max} = \frac{P_i(d_i + t)}{2t}$$

The longitudinal tensile stress or σ_z is:



$$F = p_i A \quad ; \quad A = \frac{\pi d_i^2}{4}$$

$$F = p_i \frac{\pi d_i^2}{4}$$

$$\therefore\ \pi\, d_i\, t\, \sigma_2 = \frac{\pi\, d_i^2\, p_i}{4}$$

$$\sigma_2 = \sigma_L = \frac{p_i\, d_i}{4\, t}$$

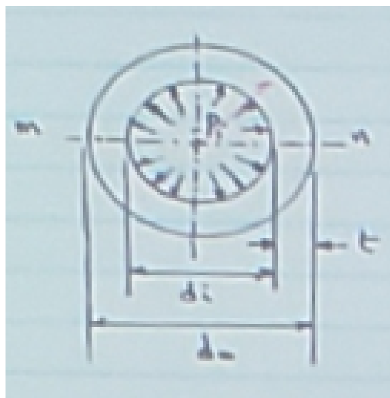
σ_1 = circumferential, or hoop, or tangential stress
σ_2 = longitudinal stress
d_i = internal diameter
t = wall thickness
p = internal pressure

Lecture (Feb. 12th, 2019)

$$\pi d_i t \sigma_2 = \frac{\pi d_i^2 p_i}{4}$$

Where:
σ_1 = circumferential (hoop) stress, tangential stress
σ_2 = longitudinal stress
d_i = internal diameter
t = wall thickness
p = internal pressure

Thick-walled Cylinders with Internal and External Pressures



$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{(p_i - p_o) r_o^2 r_i^2}{r^2 (r_o^2 - r_i^2)}$$

$$\sigma_t = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{(p_i - p_o) r_o^2 r_i^2}{r^2 (r_o^2 - r_i^2)}$$

Thick Walled Cylinders with Internal Pressure only

Tangential Stress:

$$\sigma_t = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

Radial Stress:

$$\sigma_r = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$

Axial Stress:

$$\sigma_t = \frac{p_i r_i^2}{r_o^2 - r_i^2}$$

$$\sigma_{t,max} = \frac{p_i(r_o^2 - r_i^2)}{r_o^2 - r_i^2} \quad (At\, r = r_i)$$

$$\sigma_{r,max} = -p_i \quad (At\, r = r_i)$$

23.2 Metal Fits

- Basic size is the exact theoretical size. Limiting variations begin from the basic dimension.

- The nominal size of a part is the designation used for the purpose of general identification.

- Limits are the stated maximum and minimum permissible dimensions.

- Tolerance is the total permissible variation in size. (- The difference between the two limits)

Example: A $1.500 \pm 0.010\ in$ shaft is a shaft that has a basic size of $1 - 1/2\ in$, (in this case the basic size is also the nominal size), in diameter and a tolerance of $0.020\ in$.

- Unilateral tolerance is when one of the limits is the basic size

Example: $1.500^{+0.000}_{-0.010}$

Unilateral tolerances are usually used in specifying fits for interchangeable parts.

- Bilateral tolerance is when variation is permitted in both directions from the basic size.

Example: $1.500\ \pm 0.010$

- Natural tolerance is equal to plus and minus three standard deviations from the mean. For normal distributions, 99.73% of production is within natural tolerance limits.
- Clearance is used when the internal member of two mating parts is smaller than the external member.
  a – diametral clearance is the measured difference in the two diameters.
  b – radial clearance is the difference in the two radii.
- Interference is when the internal member is larger than the external member.

23.3 Force Fits and Shrink Fit

In a force-fit assembly, the pressure between the parts depends on the amount of interference.

If the radial interference is δ, the contact pressure at the interference radius R is:

$$p = \frac{\delta}{R\left[\frac{1}{E_o}\left(\frac{r_o^2 + R^2}{r_o^2 - R^2} + v_o\right) + \frac{1}{E_i}\left(\frac{R^2 + r_i^2}{R^2 - r_i^2} - v_i\right)\right]}$$

If the members are of the same material, then:

$$p = \frac{E\delta}{2R^3}\left[\frac{(r_o^2 - R^2)(R^2 - r_i^2)}{r_o^2 - r_i^2}\right]$$

If the mating parts are of the same material and $r_i = 0$ (hub and solid shaft):

$$p = \frac{E\delta}{2R}\left[1 - \frac{R^2}{r_o^2}\right]$$

The maximum tangential and radial stresses at the inside surface of the external member are:

$$(\sigma_t)_{max} = p\frac{r_o^2 + R^2}{r_o^2 - R^2} = \frac{p\left(1 + \frac{R^2}{r_o^2}\right)}{\left(1 - \frac{R^2}{r_o^2}\right)} \quad ; \quad (\sigma_p)_{max} = -p$$

Substituting for p:

$$(\sigma_t)_{max} = \frac{E\delta}{2R}\left[1 + \frac{R^2}{r_o^2}\right]$$

$$(\sigma_r)_{max} = \frac{-E\delta}{2R}\left[1 - \frac{R^2}{r_o^2}\right]$$

The maximum shearing stress is:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_t - \sigma_r}{2}\right)^2} = \frac{E\delta}{2R}$$

For brittle material, the maximum normal stress should not exceed the ultimate tensile strength of the material.

$$\frac{S_{ult}}{n_d} = \frac{E\delta}{2R}\left(1 + \frac{R^2}{r_o^2}\right)$$

For ductile material, based on the maximum shear theory.

$$\frac{S_{yp}}{n_d} = \frac{E\delta}{R}$$

Where
n_d = design factor
S_{ult} = ultimate tensile strength, psi
S_{yp} =yield strength, psi

$$(\sigma_r)_{max} = -\frac{E\delta}{2R}\left(1 - \frac{R^2}{r_o^2}\right)$$

The maximum shearing stress is:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_t - \sigma_r}{2}\right)^2} = \frac{E\delta}{2R}$$

For brittle material, the maximum normal stress should not exceed the ultimate tensile strength of the material.

$$\frac{S_{ult}}{n_d} = \frac{E\delta}{}$$

23-4 Force Fits – Steel Shaft & Cast-iron Hub

$$p = \frac{E_c \delta \left[1 - \left(\frac{d_i^2}{d_o^2}\right)\right]}{d_i \left[1.53 + 0.47 \left(\frac{d_i^2}{d_o^2}\right)\right]}$$

Where, E_c = modulus of elasticity of cast iron. And:

$$\frac{S_{ult}}{f_s} = \frac{E_c \delta \left[1 - \left(\frac{d_i^2}{d_o^2}\right)\right]}{d_i \left[1.53 + 0.47 \left(\frac{d_i^2}{d_o^2}\right)\right]}$$

23-5 Holding ability of Force and Shrink Fits

$$T = \frac{f p \pi d_i^2 L}{2}$$

Where:
$T =$ Transmitted torque, $lb - in$
$p =$ contact pressure, psi
$d_i =$ Diameter, in
$L =$ length of hub, in
$f =$ coefficient of friction (usually from 0.1 to 0.05)
$\delta =$ diametral interference, in

-Thermal-stresses and strains

When the temperature of an unrestrained body is uniformly increased, the body expands, and the normal strain is:

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha(\Delta T)$$

Where:
$\alpha =$ coefficient of thermal expansion (Table 3.3)
$\Delta T =$ temperature change in degrees

If a straight bar is restrained at the ends, the compressive stress is:

$$\sigma = \varepsilon E = \alpha(\Delta T)E$$

If a uniform plate is restrained at the edges

$$\sigma = \frac{\alpha(\Delta T)E}{1 - v}$$

Although referred to as thermal stresses, the above are not thermal stresses, but arise from the edge restrains. A thermal stress is one which arises because of the existence of a temperature gradient in a body.

23-6 Assembly of Shrink Fits

- The minimum change in temperature for assembly is:

$$\Delta T = \frac{\delta}{\alpha d_i}$$

Where:
$\delta =$ diametral interference, in
$\alpha =$ coefficient of expansion, $in\ per\ in\ per\ °F$
$\Delta T =$ exchange in temperature, $°F$

- The force required to press the parts together is:

$$F = 2\pi r_i L p f$$

$$\pi d_i L p f$$

Lecture (Feb. 14th, 2019)

Curved Beams in Bending

The neutral axis and the centroidal axis of a curbed beam do not coincide, and the stress distribution is not linear.

The location of the neutral axis with respect to the center of curvature is given by:

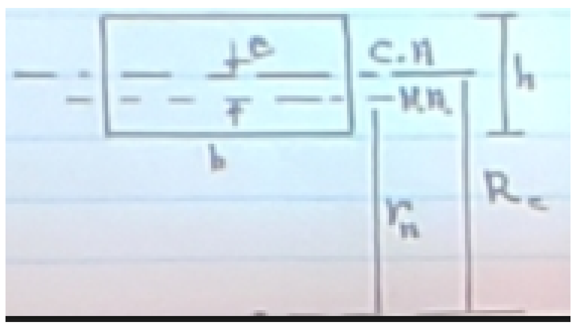
$$r_n = \frac{A}{\int \frac{dA}{r}}$$

For rectangular section:

$$R_c = r_i + \frac{h}{2}$$

And:

$$r_n = \frac{A}{\int \frac{dA}{r}} = \frac{bh}{\int_{r_i}^{r_o} \frac{b}{r}\, dr} = \frac{h}{\ln\left(\frac{r_o}{r_i}\right)}$$



For solid round section:

$$R_c = r_i + \frac{d}{2}$$

And:

$$r_n = \frac{d^2}{4\left(2R_c - \sqrt{4R_c^2 - d^2}\right)}$$

The stress distribution is given by:

$$\sigma = \frac{My}{Ae(r_n - y)}$$

(Where e is the distance between the neutral axis and the centroidal axis.)

At the inner fiber:

$$\sigma_i = \frac{Mc_i}{Aer_i}$$

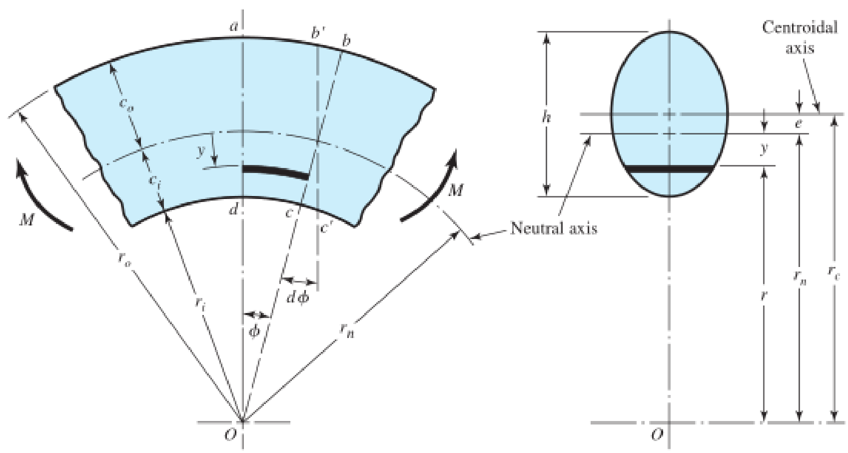
At the outer fiber:

$$\sigma_o = \frac{Mc_o}{Aer_o}$$

(For other cross-section shapes refer to Table 3.4)

Figure 3–34

Note that y is positive in the direction toward the center of curvature, point O.



r_o = radius of outer fiber

r_i = radius of inner fiber

h = depth of section

c_o = distance from neutral axis to outer fiber

c_i = distance from neutral axis to inner fiber

r_n = radius of neutral axis

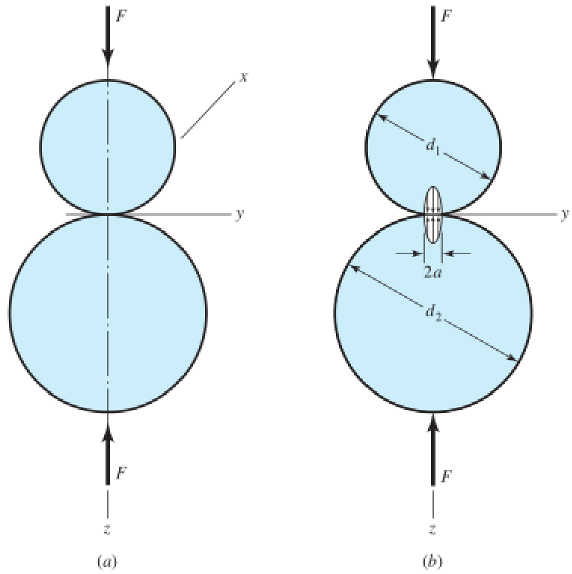
r_c = radius of centroidal axis

$e = r_c - r_n$, distance from centroidal axis to neutral axis

M = bending moment; positive M decreases curvature

Figure 3–36

(*a*) Two spheres held in contact by force *F*; (*b*) contact stress has a hemispherical distribution across contact zone of diameter 2*a*.



Hertz Contact Stresses

When two solid spheres are passed together with a force F, the radius of the circular contact area is:

$$a = \sqrt[3]{\frac{3F}{8}\frac{[(1-v_1^2)/E] + [(1-v_2^2)/E_2]}{(1/d_1)+(1/d_2)}}$$

Where:

a = radius of the circular area of contact
d_1 = diameter of sphere 1
d_2 = diameter of sphere 2
E_1 = modulus of elasticity of sphere 1
E_2 = modulus of elasticity of sphere 2
v_1 = Poisson's radio of sphere 1
v_2 = Poisson's ratio of sphere 2
F = applied force

The maximum pressure at the centre of the contact area is:

$$p_{max} = \frac{3F}{2\pi a^2}$$

The above equations are also valid for the case of a sphere and a plane surface on a sphere and an internal spherical surface. For a plane surface use $d = \infty$, and for internal surfaces the diameter is expressed as a negative quantity.

Plane: $d = \infty$

Internal Spherical Surface: $d < 0$

The maximum stress occur on the $z - axis$ which is the axis of application of external force: (These are principal stresses)

$$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = -p_{max}\left[\left(1 - \left|\frac{z}{a}\right|\tan^{-1}\frac{1}{|z/a|}\right)(1+\nu) - \frac{1}{2\left(1 + \frac{z^2}{a^2}\right)}\right]$$

$$\sigma_3 = \sigma_z = \frac{-p_{max}}{1 + \frac{z^2}{a^2}}$$

In using the above equations, the value of Poisson's ratio used must be that of the sphere under consideration.

Also:

$$\tau_{xz} = \tau_{yz} = \frac{\sigma_x - \sigma_z}{2} = \frac{\sigma_y - \sigma_z}{2} = \tau_{max}$$

Since:

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

In the case of two contacting cylinders of length l and diameter, d_1 and d_2, the area of contact is a rectangle of width $2b$ where:

$$b = \sqrt{\frac{2F}{\pi l}\frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{1/d_1 + 1/d_2}}$$

And the maximum pressure is:

$$p_{max} = \frac{2F}{\pi b l}$$

The above equations are also applicable for a cylinder and a plane surface as well as for a cylinder and an internal cylindrical surface where:

For a plane surface: $d = \infty$

For cylindrical surface: $d < 0$

The stress state on the $z - axis$ given by the following:

$$\sigma_x = -2\nu p_{\max}\left(\sqrt{1 + \frac{z^2}{b^2}} - \left|\frac{z}{b}\right|\right)$$

$$\sigma_y = -p_{\max}\left(\frac{1 + 2\frac{z^2}{b^2}}{\sqrt{1 + \frac{z^2}{b^2}}} - 2\left|\frac{z}{b}\right|\right)$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{\sqrt{1 + z^2/b^2}}$$

Note that:

For $0 \leq z \leq 0.436b$: $\sigma_1 = \sigma_x$; $\tau_{max} = (\sigma_1 - \sigma_3)/2$

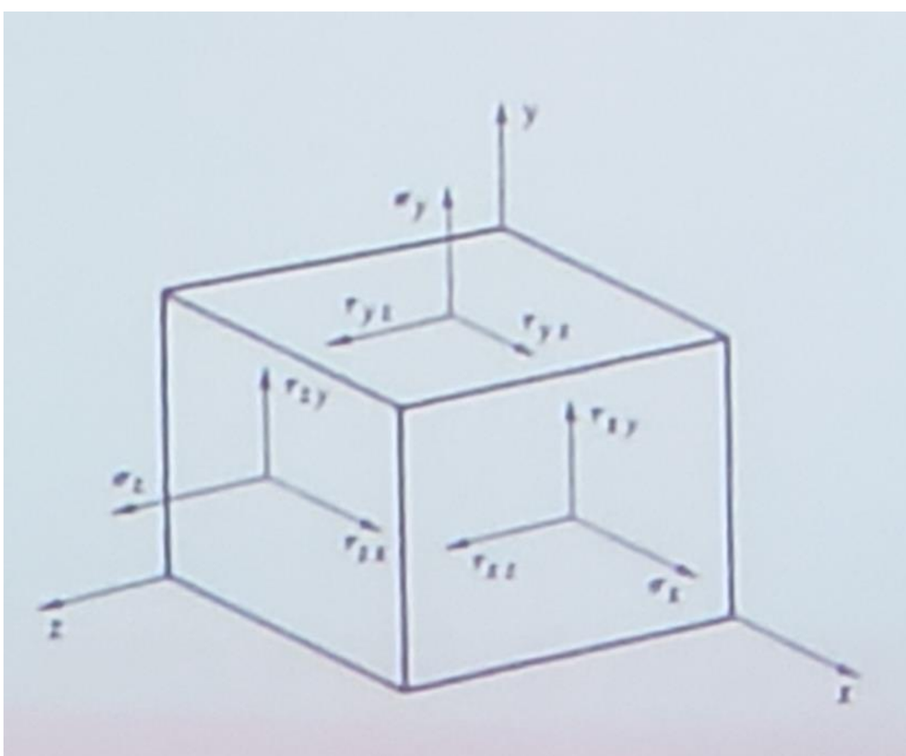
And:

For $z \geq 0.436b$: $\sigma_1 = \sigma_y$; $\tau_{max} = (\sigma_1 - \sigma_3)/2$

Also note that τ_{xy} here is not the largest of the three shears for all values of z/b, but is max for $z/b = 0.786$ and is the larger at this point.

<u>6.2 - Determination of Principal Stresses</u>

Whatever the aspect of the stress at a joint may be, it can always be expressed in terms of normal stresses and shear stresses.



Where:

$\sigma_x, \sigma_y, \sigma_z$; are normal stresses

$\tau_{yx} = \tau_{xy}$
$\tau_{yz} = \tau_{zy}$
$\tau_{zx} = \tau_{xz}$; are shear stresses

Two-dimensional Stress

Consider a section of this element:

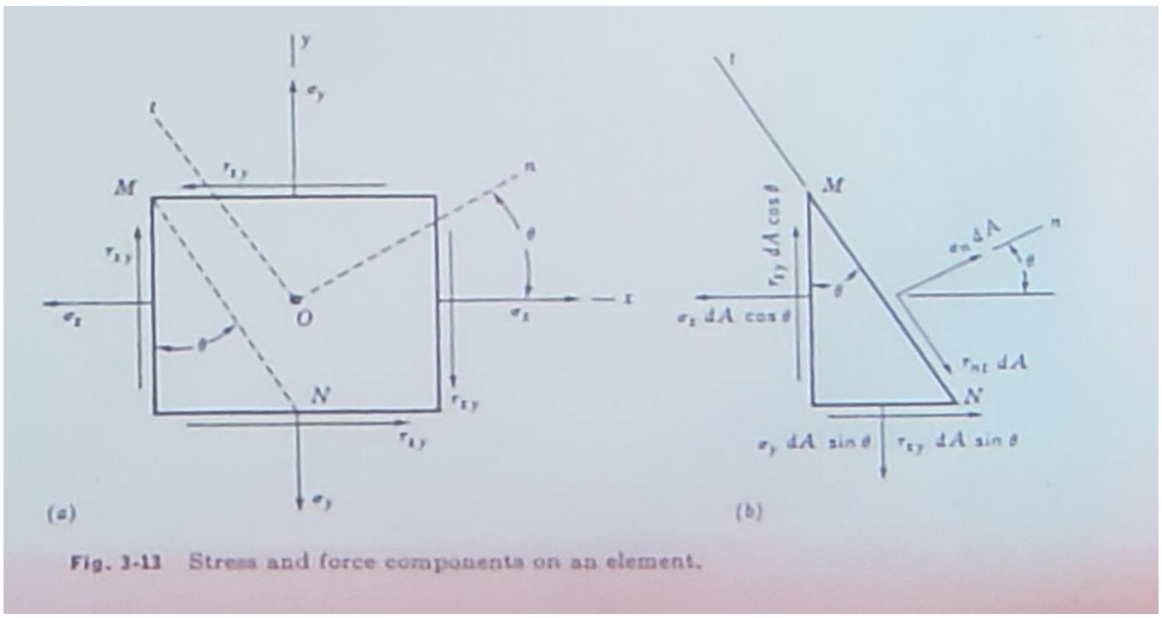


Fig. 1-13 Stress and force components on an element.

$$\sum F_n = 0$$
$$\sigma_n\, dA - \sigma_x\, cos\theta\, dA - \sigma_y\, sin\theta\, dA\, sin\theta + \tau_{xy}\, cos\theta\, dA\, sin\theta + \tau_{xy}\, sin\theta\, dA\, cos\theta = 0$$
$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - 2\tau_{xy} sin\theta cos\theta$$
$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau_{xy} 2sin\theta$$
$$\{2sin\theta cos\theta = sin2\theta\}$$
$$Replacing \cos^2\theta = \left(\frac{1}{2}\right)(1 + cos2\theta)$$

$\sum F_t = 0$ leads to

$$\tau_{nt} = (\sigma_x - \sigma_y) sin\theta cos\theta + \tau_{xy}(\cos^2\theta + \sin^2\theta)$$
$$\tau_{nt} = \frac{\sigma_x - \sigma_y}{2} sin2\theta + \tau_{xy} cos2\theta$$

The direction of the principal stresses (maximum and minimum values) is found by differentiating σ_n with respect to θ, setting the values to zero and solving for θ. The result is:

$$tan2\theta_{1,2} = -\frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

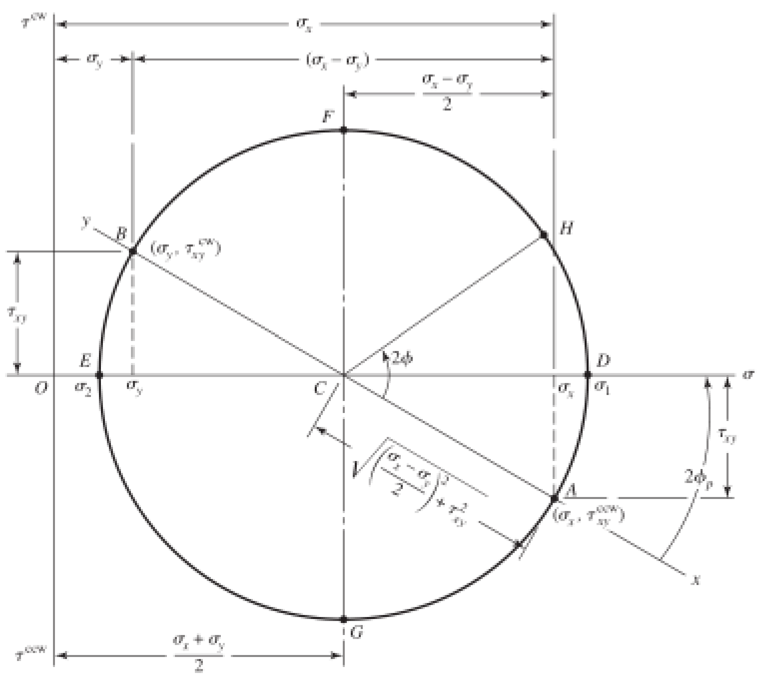
Substituting in the expression of σ_n to find:

$$\sigma_{1,2} = \frac{\sigma_x - \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{1,2} = 0$$

6.3 - Mohr's Circle

The above results can be represented graphically by a diagram known as "Mohr's Circle" as shown:



Lecture (Feb. 26th, 2019)

General Three-Dimensional Stresses

With the exception of contact stress, most maximum stress states occur under plane (2-D) stress conditions.

In the presence of 3-D stresses, the three principal stresses are found by solving the following equilibrium equation:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \quad (3\text{–}15)$$

The root of which are:

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

The three principal shear stresses are:

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \quad (3\text{–}16)$$

Hooke's Law

$$F = k\delta$$

Or:

$$\delta = \frac{F}{k}$$

Where:
F = The applied force
k = Spring rate (elastic zone)
δ = Resulting spring deflection

We also have:

$$\sigma = \frac{F}{A} \; or \; \frac{P}{A} \quad (1)$$

$$\varepsilon = \frac{\delta}{L} \quad (2)$$

$$\sigma = \varepsilon E \; or \; \varepsilon = \frac{\sigma}{E} \quad (3)$$

Substituting (1) and (3) into (2):

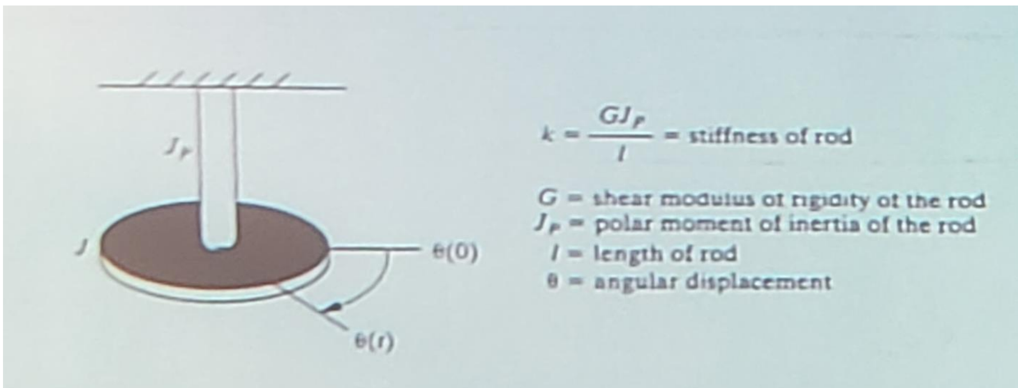
$$\delta = \frac{Fl}{AE} = \frac{Pl}{AE} = \frac{F}{k}$$

Or:

$$k = \frac{AE}{l}$$

Where:
$A =$ cross-section
$E =$ modulus of elasticity
$l =$ length

Torsion



Hooke's Law:

$$T = k_t \theta$$

Or:

$$\theta = \frac{T}{k_t}$$

$$\tau = \frac{Tr}{J_p} \quad (1)$$

$$\gamma = \frac{r\theta}{l} \quad (2)$$

$$\tau = \gamma G \quad or \quad \gamma = \frac{\tau}{G} \quad (3)$$

Substituting (1) and (3) into (2):

$$\frac{Tr}{J_p G} = \frac{r\theta}{l}$$

Or:

$$\theta = \frac{TL}{J_p G} = \frac{T}{k_t}$$

$$k_t = \frac{J_p G}{L}$$

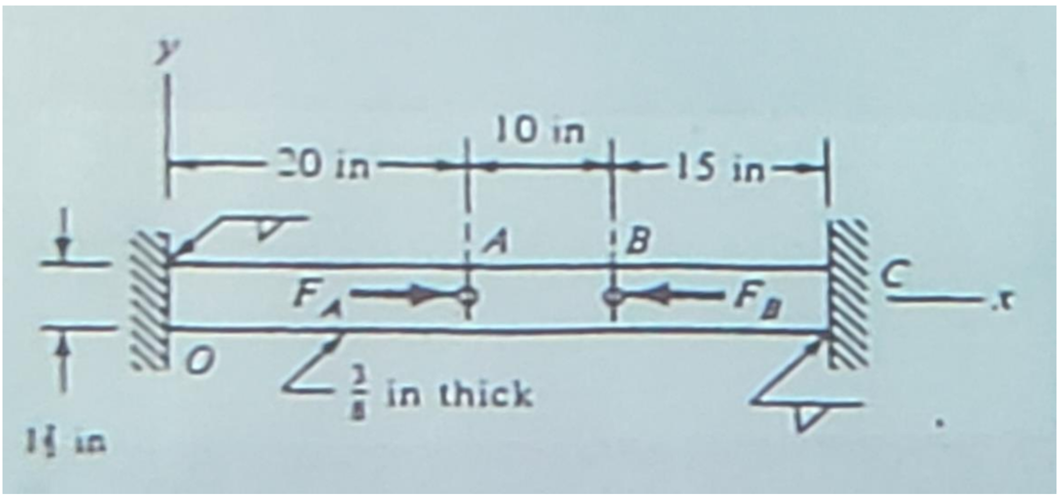
Statically Indeterminate Problems

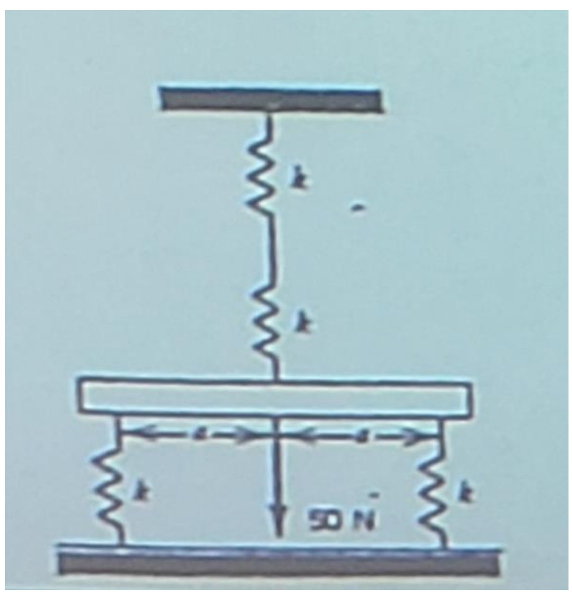
Statically indeterminate systems are characterized by the presence of more supports or members than the minimum required for the equilibrium of the structure. For such situations, the deformations of the parts must be taken into consideration.

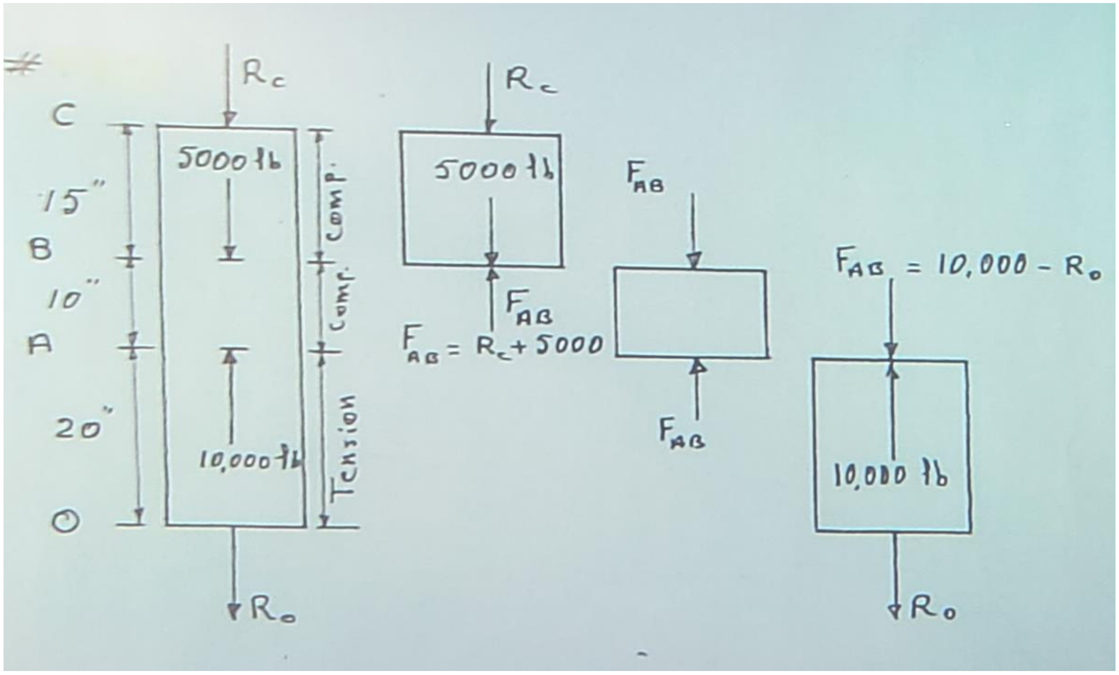
Example:

a- The figure shows a $\frac{3}{8}$ x $1\frac{1}{2}$ in rectangular steel bar welded to fixed supports as each end. The bar is axially loaded by the forces $F_A = 10{,}000\ lb$ and $F_a = 5{,}000\ lb$ acting on pins at A and B. Assuming that the bar will not buckle laterally, find the reactions at the fixed supports.

b- A very stiff horizonal bar, supported by four identical spring as shown is subjected to a center load of $50\ N$. What load is applied to each spring?







$$\therefore R_c + 5{,}000 = 10{,}000 - R_o$$

$$\therefore R_c + R_o = 5000 \quad (1)$$

Also:

$$\delta_{OA} = \delta_{AB} + \delta_{BC}$$

Or:

$$\frac{20R_o}{AE} = \frac{10(R_c + 5000)}{AE} + \frac{15R_c}{AE}$$

$$20R_o = 50{,}000 + 25R_c$$

From (1): $R_o = 5000 - R_c$

Substituting in (2): $20(5000 - R_c) = 50{,}000 + 25R_c$

$$or\ R_c = 1111.11\ lb$$

$$and\ R_o = 5000 - 1111.111 = 3888.888\ lb$$

$$R_c \cong 1111\ lb$$
$$R_o \cong 3889\ lb$$

Upper springs each deflect only half as much as lower springs, hence carry only half the load.

Let $L =$ load carried by each lower spring.

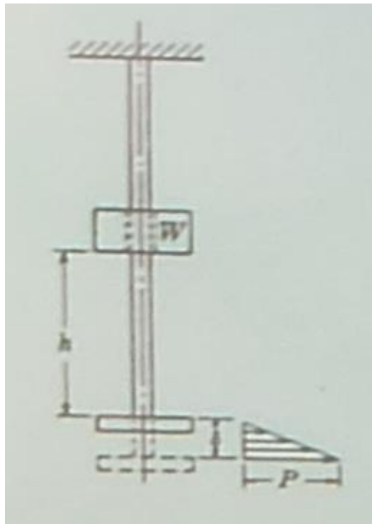
$$2L + \frac{L}{2} = 50\ N$$

$$\therefore L = 20\ N$$

Lower springs carry $20\ N$ each.

Upper springs carry $10\ N$ each.

<u>Stresses Due to Shock and Impact Loading</u>



Where:
W = falling weight, lb
h = height of free fall, in
δ = deflection, in
P = impact load, lb
$C = P/\delta$ = lb/in of deflection

Energy balance:

$$\frac{1}{2}P\delta = W(h+\delta)$$

$$P = 2\frac{W}{\delta}(h+\delta)$$

$$\frac{P}{W} = 2\left(\frac{h}{\delta}+1\right)$$

But:

$$\delta = \frac{P}{c}$$

$$\therefore \frac{P}{W} = 2\left(\frac{hC}{P}+1\right)$$

$$P^2 = 2W(hC+P)$$

$$P^2 - 2WP - 2WhC = 0$$

$$P = \frac{2W \pm \sqrt{4W^2 + 8WhC}}{2}$$

$$P = W\left(1 + \sqrt{1 + \frac{2hC}{W}}\right)$$

$$\frac{P}{W} = 1 + \sqrt{1 + \frac{2hC}{W}}$$

For a bar in Tension:

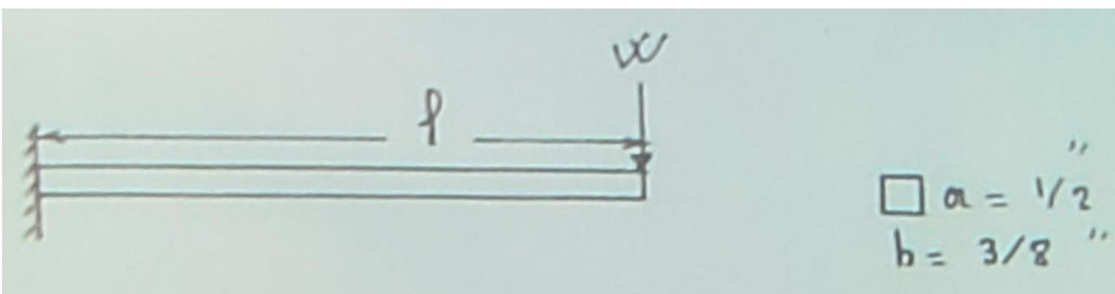
$$\delta = \frac{PL}{AE}$$

$$\therefore C = \frac{P}{\delta} = \frac{P}{PL/AE} = \frac{AE}{L}$$

Special case: if the load is applied instantaneously without velocity of approach then:

$$P = 2W$$

Example 9:



Where:
$W = 8\ lb$
$a = 1/2\ in$
$b = 3/8\ in$
$l = 12\ in$

Find the max bending stress in the beam

$a -$ load applied gradually
$b -$ load dropped from a distance of $3/16$ in

Solution:

$$S_{max} = \frac{M}{I/c}$$

$$\frac{I}{c} = \frac{ba^2}{6} = \frac{\left(\frac{3}{8}\right)(0.5)^2}{6} = 0.0156\ in^3$$

a) $M = (8)(12) = 96\ lb - in$

$$\sigma_{max} = \frac{96}{0.0156} \cong 6154\ psi$$

b) Impact stress

$$\delta = \frac{Pl^3}{3EI}$$

$$I = \frac{ba^3}{12} = \frac{\left(\frac{3}{8}\right)\left(\frac{1}{2}\right)^3}{12} = 0.0039\ in^4$$

$$C = \frac{P}{\delta} = \frac{3EI}{L^3} = \frac{(3)(30)(10^6)(0.0039)}{(12)^3} = 204\ lb/in$$

$$P = W\left(1 + \sqrt{1 + \frac{2hC}{W}}\right) = (8)\left(1 + \sqrt{1 + \frac{(2)(204)(3/16)}{(8)}}\right)$$

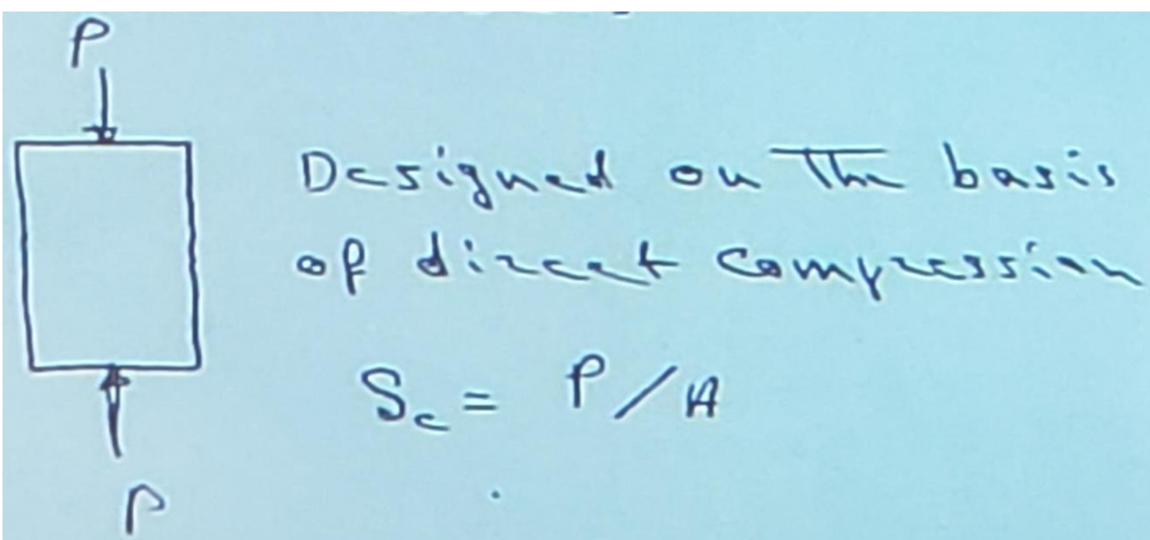
$$P = 34.64\ lb$$

$$\sigma_{max} = \frac{M}{I/C} = \frac{(34.64)(12)}{(0.0156)} = 26646\ psi$$

Lecture (Feb. 28th, 2019)

3.7 <u>Instability Considerations</u>

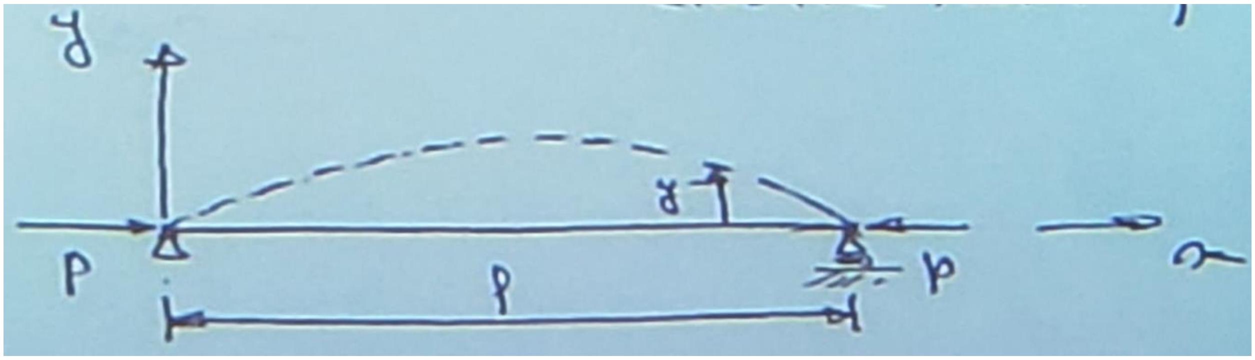
- Short compressive members



- <u>Euler Analysis</u> (Slender columns, i.e. long with small cross-section)

$$\frac{B}{\rho^2} > 2$$

Elastic instability; deformation within the elastic limit



$$M = -P\, y(x)$$

But from strength of material

$$M = EI \frac{d^2 y}{dx^2} = EI\, y''(x)$$

$$\therefore\ EI\, y''(x) + P\, y(x) = 0 \quad D.E.$$

Let

$\alpha^2 = \frac{P}{EI}$

Then

$y''(x) + \alpha^2\, y(x) = 0$

Solution

$y(x) = A \sin(\alpha x + P)$

B.C. $y = 0 \ \ at \ \ x = 0$

and $y = 0 \ \ at \ \ x = \rho$

$$\therefore For\ x = 0 \quad 0 = A \sin(0 + \beta) \quad \therefore P = 0$$

$$For\ x = \rho \quad 0 = A \sin \alpha\rho$$

$\therefore sin\alpha\rho = 0 \quad since\ A \neq 0$

$But\ \alpha^2 = \frac{P}{EI}$

$\therefore (\alpha\rho)^2 = n^2\pi^2 = \frac{P}{EI}\rho 2$

$P = \frac{n^{2\pi^2 EI}}{\rho^2}$

$P_{cont} = \frac{2\pi^2 EI}{\rho^2}$

Solving for other boundary conditions it can be proven that:

$F_{crit} = \frac{n\pi^2 EI}{\rho^2}$

Where:
$n =$ end fixity coefficient (Figure 4-18, page 196, Table 4-2 Page 199)

Introducing the quantity

$$B = \frac{S_y \rho^2}{n\pi^2 E}$$

And replacing I by $\rho^2 A$

Then:

$$F_{crit} = \frac{n\pi^2 AE}{(L/\rho^2)} = \frac{S_y A \rho^2}{B}$$

- J.B. Johnson Formula (less slender formula)

$$\frac{B}{\rho^2} < 2$$

(plastic instability; allowable stresses exceeded)

$\therefore Experimental\ results$

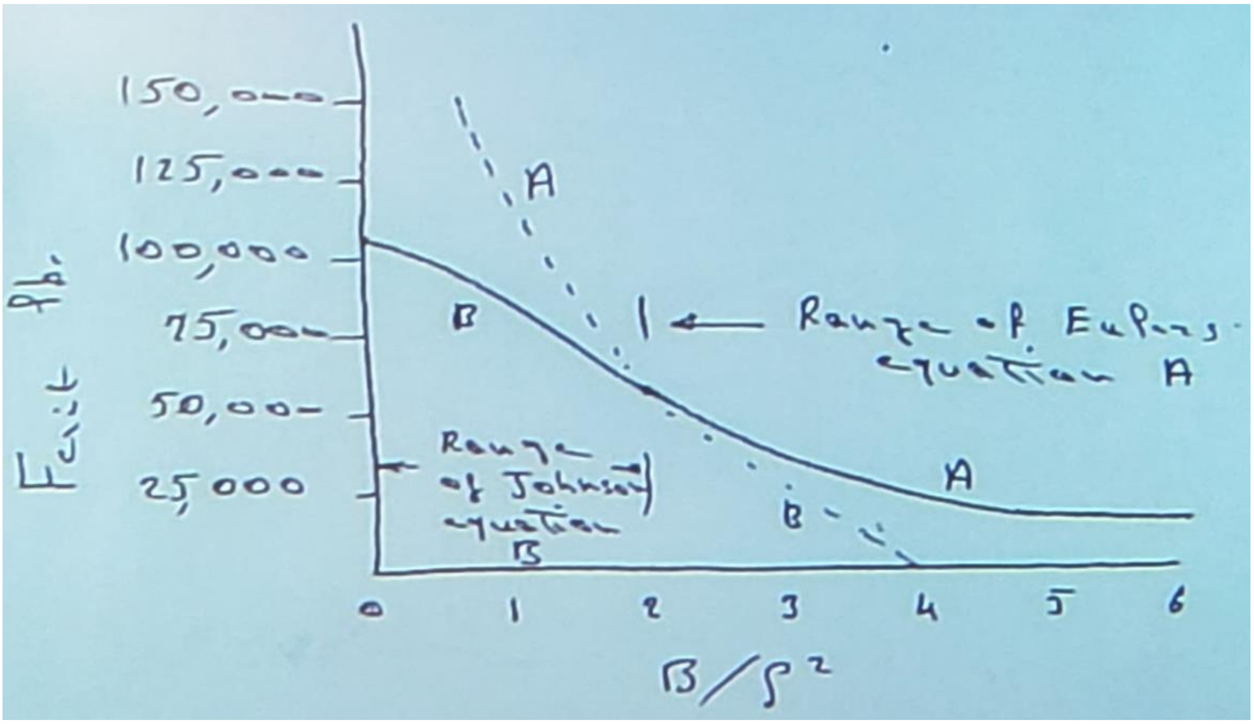
$F_{crit} = AS_y \left(1 - \frac{S_y \rho^2}{4n\pi^2 E \rho^2}\right)$

introducing B:

$F_{crit} = AS_y \left(1 - \frac{B}{4\rho^2}\right)$

If F_{crit} (Euler) equated to F_{crit} (Johnson) we find:

$$\frac{B}{\rho^2} = 2$$



$\therefore if\ \dfrac{B}{\rho^2} < 2$ ***Use Johnson***

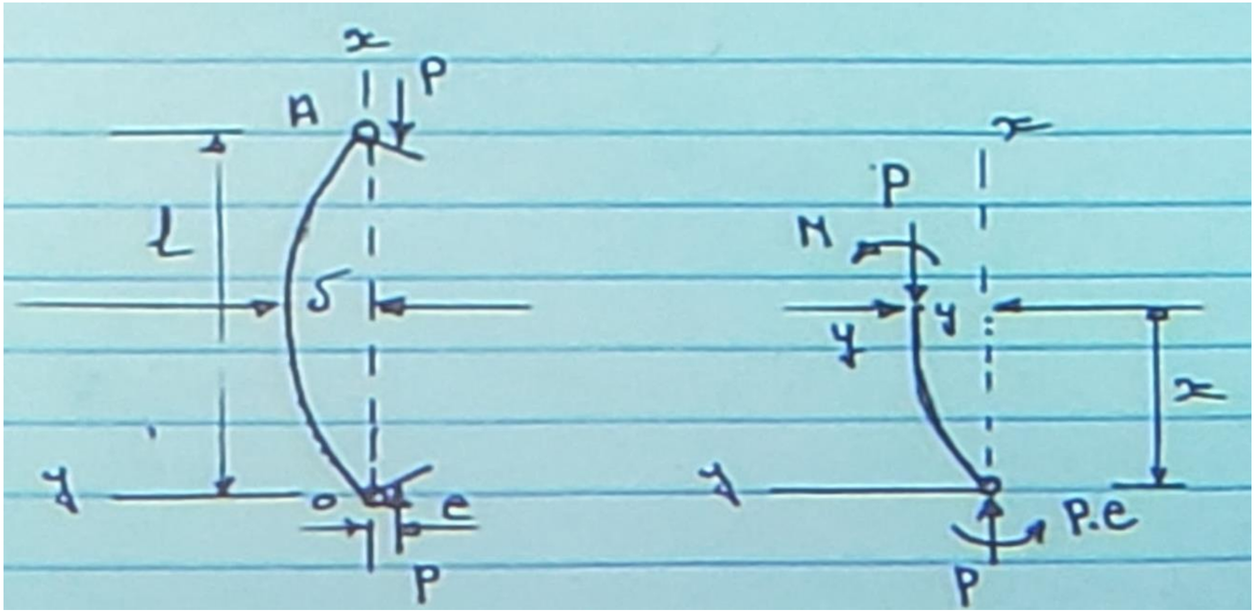
(most machine members are in this range)

$\therefore if\ \dfrac{B}{\rho^2} > 2$ ***Use Euler***

Start with Johnson. Find B/ρ^2 – if $< 2\ O.K.$ if not go to Euler

Where:
F_{crit} = critical load causing failure, lb
A = cross-sectional area, in^2
I = moment of inertia of area, in^4
L = length of column, in
ρ = least radius of gyration of cross-section, in
n = end-fixity coefficient, see Figure 4-18 and Table 4-2
E = modulus of elasticity, psi
S_y = yield point of material, psi
$B = S_y L^2 / n\pi^2 E$

Columns with Eccentric Loading



IT can be shown that at $x = l/2$, the deflection:

$$\delta = y_{\frac{l}{2}} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}\right) - 1\right]$$

The maximum bending moment occurs at $\rho/2$ and is:

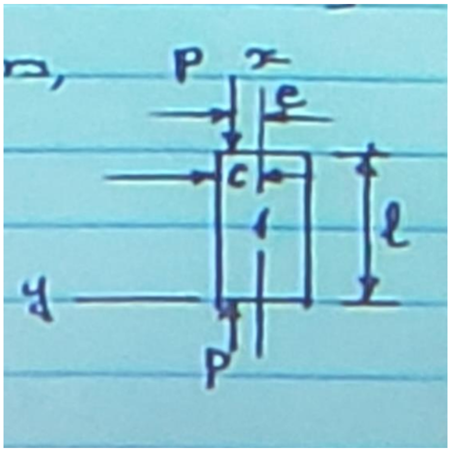
$$M_{max} = P(e + \delta) = P \cdot e \cdot \sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

The maximum compressive stress:

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{A\rho^2} = \frac{P}{A}\left[1 + \frac{ec}{\rho^2} sec \frac{l}{2\rho}\sqrt{\frac{P}{EA}}\right]$$

In the case of short compressive members, valid if $\left(\frac{l}{\rho}\right) \leq 0.282 \ \left(\frac{AE}{P}\right)^{\frac{1}{2}}$

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{P \cdot e \cdot c \cdot A}{I \cdot A} = \frac{P}{A}\left(1 + \frac{ec}{\rho^2}\right)$$

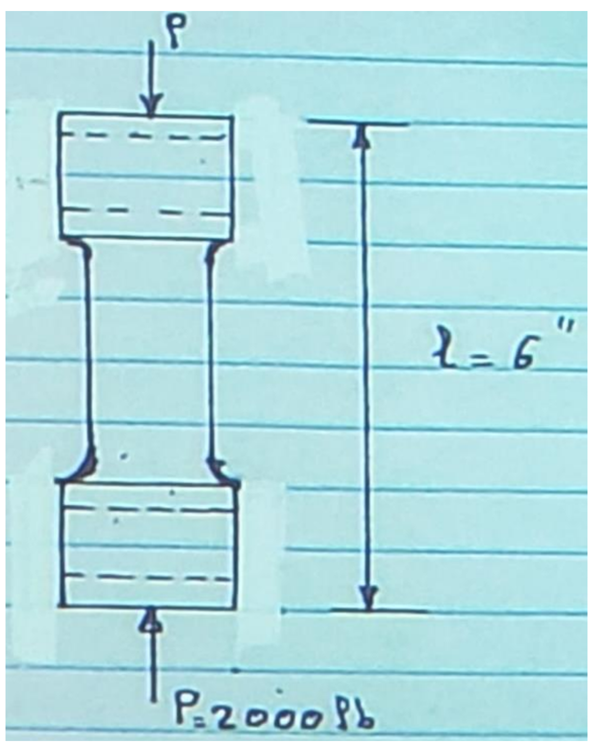


Where:

$$\rho^2 = \frac{I}{A}$$

Example:
A strut of circular cross-section
Material is SAE10HR Steel
Diameter of loading pin 0.5”
Allowable bearing pressure at pin = $10{,}000\ psi$
$l = 6"$
$P = 2{,}000\ lb$

Using a design factor of 1.5 determine the dimensions for the strut



Solution:
From Table A-20, $S_y = 42{,}000\ psi$
From Table A-23, $E = 29 \cdot 10^6\ psi$
From Table 4-2, $n = 1$

$$B = \frac{S_y l^2}{n\pi^2 E} = \frac{(42 \cdot 10^3)(6^2)}{(1)\pi^2(29 \cdot 10^6)} = 0.00528$$

$$F_{crit} = n_d P = (1.5)(2{,}000) = 3{,}000\ lb$$

Starting with Johnson's equation, assuming $B/\rho^2 < 2$:

$$F_{crit} = AS_y\left(1 - \frac{B}{4\rho^2}\right) \quad ; \quad A = \frac{\pi d^2}{4} \quad ; \quad \rho = \frac{d}{4}$$

Substituting and solving for d^2:

$$d^2 = \left(\frac{4F_{crit}}{\pi S_y}\right) + 4B$$

$$d^2 = \frac{(4)(3{,}000)}{(\pi)(42{,}000)} + (4)(0.00528) = 0.112\ in^2$$

$$d = 0.335\ in$$

Using standard $3/8\ in$ we check for B/ρ^2

$$\frac{B}{\rho^2} = \frac{16B}{d^2} = \frac{(16)(0.00528)}{\left(\frac{3}{8}\right)^2} = 0.601$$

$$\therefore \frac{B}{\rho^2} < 2; \quad Johnson's\ Formula\ Justified$$

For the eye, $S = P/td$

$$10{,}000 = 2{,}000/(t \cdot 0.5)$$

$$t = \frac{0.2}{0.5} = 0.4\ in$$

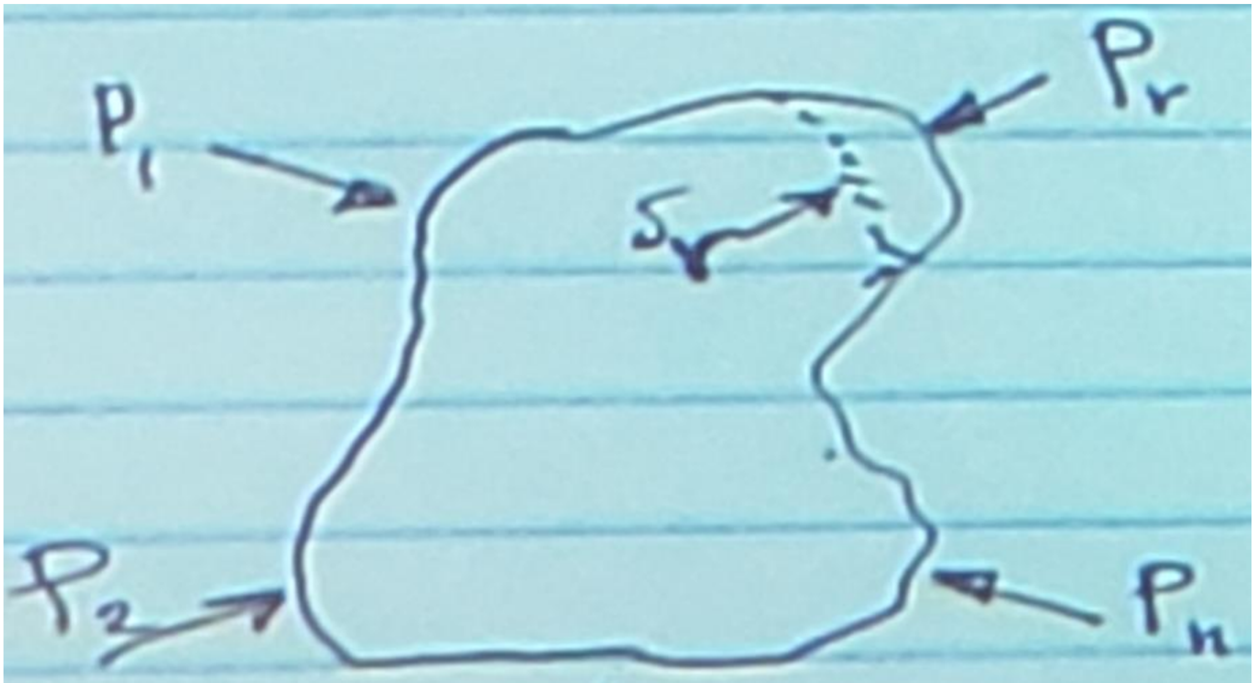
Use $1/2\ in$ to allow for machining of the faces of the eye.

Beam Deflection

- Castigliano's Theorem

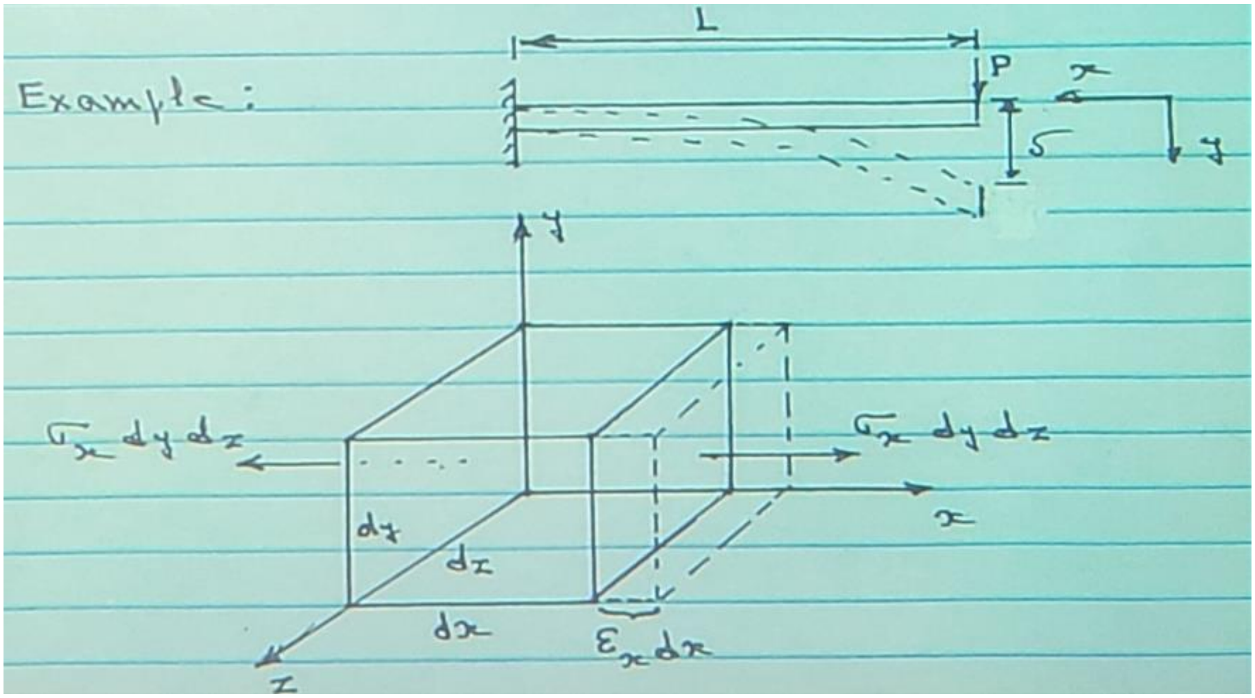
The displacement corresponding to any force of a system of forces acting on an elastic body can be determined by taking the partial derivative of the elastic strain energy with respect to that force.

$$\frac{\delta u}{\delta P_r} = \delta_r$$



P_1
P_r
S_r
P_2
P_n

Lecture (Mar. 5th, 2019)



$$Du = \left(\frac{1}{2}\right)(\sigma_x\, dy\, dx)(\varepsilon_x\, dx)$$

$$Du = \left(\frac{1}{2}\right)(\sigma_x \varepsilon_x)(dx\, dy\, dz)$$

The strain energy per unit volume is:

$$du = \left(\frac{1}{2}\right)\sigma_x \varepsilon_x (dx\, dy\, dz)/(dx\, dy\, dz)$$

$$du = \left(\frac{1}{2}\right)\sigma_x \varepsilon_x = \frac{\sigma_x^2}{2E} \quad ; \quad Note\ that\ \varepsilon_x = \frac{\sigma_x}{E}$$

And:

$$U = \int^{v} du \quad = \quad \left(\frac{1}{2}\right)\int_L \int_A \frac{\sigma_x^2}{E}\, dA\, dx$$

$$= \quad \left(\frac{1}{2}\right)\int_L \int_A \left(\frac{1}{E}\right)\left(\frac{My}{I}\right)^2 dA\, dx$$

$$U = \left(\frac{1}{2}\right)\int_L \frac{1}{E}\frac{M^2}{I^2}\, dx \int_A y^2\, dA = \left(\frac{1}{2}\right)\int_0^L \frac{M^2}{EI}\, dx$$

From Castegliano's Theorem:

$$\delta = \frac{\partial u}{\partial P}$$

$$\delta = \frac{\partial u}{\partial P} = \left(\frac{1}{2}\right)\int_L \frac{\partial}{\partial P}\left(\frac{M^2}{EI}\right)dx$$

$$= \int_L M\frac{\partial M/\partial P}{EI}dx$$

$$\delta = \int_0^L \frac{(-Px)(-x)}{EI}dx = \frac{PL^3}{3EI}$$

The deflection is always in the direction of the force.

If an applied force does not exist at the pint where the deflection is to be determined, then a fictitious force Q must be applied. After the strain energy equation has been differentiated with respect to Q, the force Q is set equal to zero. The resulting expression is the displacement at the point of application of Q and Is in the same direction as Q was assumed to be acting.

The following provided strain energy expressions for various types of loading:

Tension and Compression:

$$U = \frac{F^2L}{2AE}$$

Torsion:

$$U = \frac{T^2L}{2AG}$$

Direct Shear:

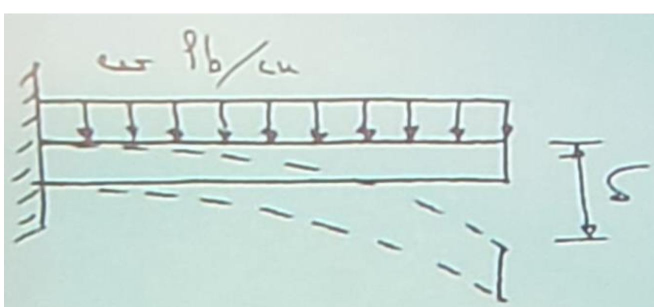
$$U = \frac{F^2L}{2AG}$$

Bending:

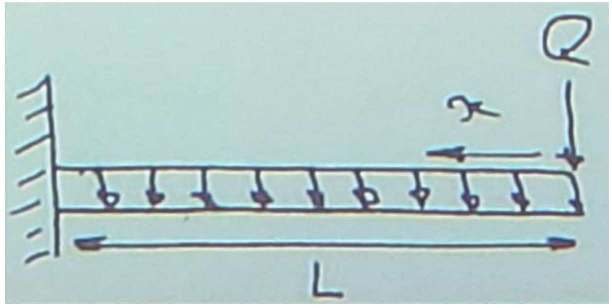
$$U = \int \frac{M^2}{2EI}dx$$

Example:

Determine the end deflection of a uniformly loaded cantilever beam.

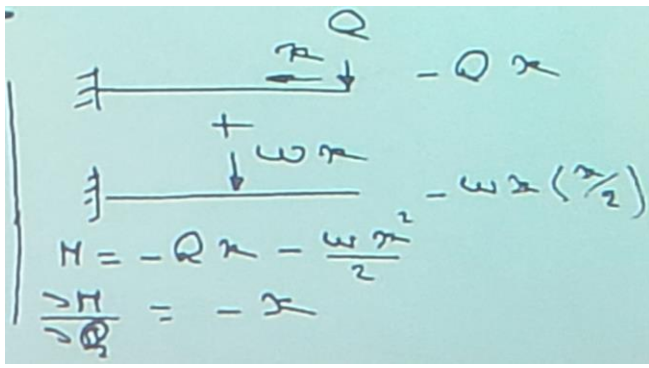


Assume a fictitious force Q acting as shown at the point where the deflection is required.



$$M = -Qx - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial Q} = -x$$



$$\delta = \frac{\partial u}{\partial Q} = \int_0^L M \frac{(-Px)(-x)}{EI} dx = \frac{wL^4}{8EI}$$

Alternative solution:

Recall that if y is the beam deflection function and q is the load per unit length, then:

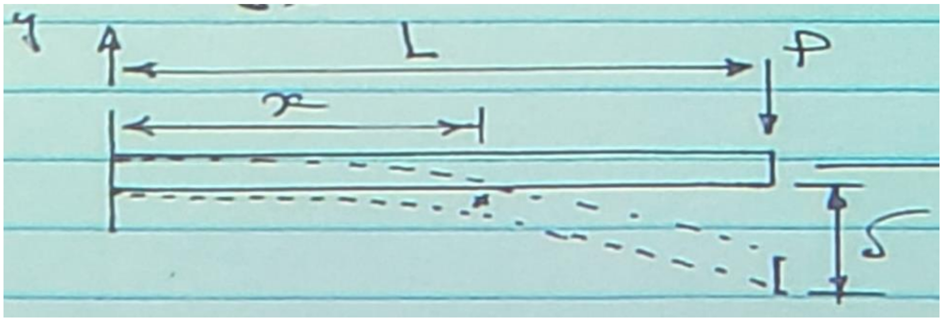
$$y = f(x)$$

$$\theta = \frac{dy}{dx} = slope$$

$$M = \frac{d^2y}{dx^2} EI$$

$$V = \frac{d^3y}{dx^3} EI$$

$$q = \frac{d^4y}{dx^4} EI$$



$$M = \frac{EI\, d^2y}{dx^2} = -P(L - x)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{EI}[P(L - x)]$$

$$\frac{dy}{dx} = -\frac{P}{EI}\int (L - x)\, dx = -\frac{P}{EI}\left(Lx - \frac{x^2}{2} + C_1\right)$$

$$y = -\frac{P}{EI}\int \left(Lx - \frac{x^2}{2} + C_1\right) dx$$

$$y = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_1 x + C_2\right)$$

BC: $at\ x = 0;\quad y' = 0 \therefore C_1 = 0$

$y = 0 \therefore C_2 = 0$

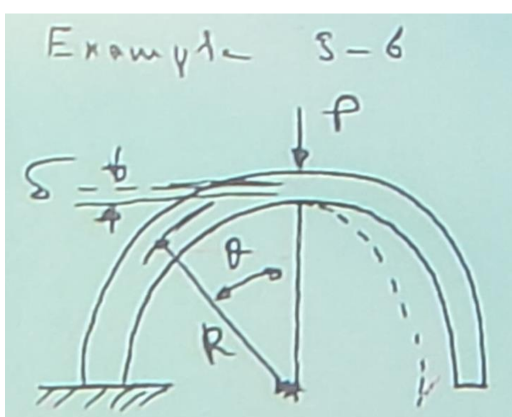
And:

$$y = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

At $x = L$;

$$y = \delta = -\frac{P}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{PL^3}{3EI} = \frac{PL^3}{3EI} \downarrow$$

Deflection by use of singularity functions:



$$For\ 0 \le x \le L \quad ; \quad \frac{EI\, d^4 y}{dx^4} = q = -P < x - L >^{-1}$$

$$V = -P < x - L >^{0} + C_1$$

$$M = -P < x - L >^{1} + C_1 x + C_2$$

$$EI\,\theta = -\frac{P}{2} < x - L >^{2} + \frac{C_1}{2} x^2 + C_2 x + C_3$$

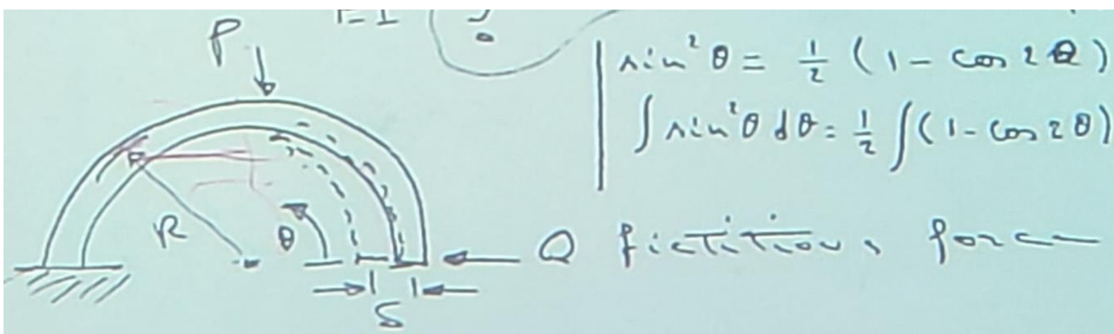
$$EI\,y = -\frac{P}{6} < x - L >^{3} + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$$

$BC:$	$At\ x = 0;$	$EI\,\theta = EI\,y = 0$	$\therefore C_3 = C_4 = 0$
	$At\ x = 0;$	$V = R_1$	$\therefore C_1 = R_1$
	$at\ x = L;$	$M = 0$	$\therefore C_2 = -R_1 L$

$$\therefore EI\,y = -\frac{P}{6} < x - L >^{3} +$$

Beam Deflection by Superposition

The results of many simple load cases and boundary conditions have been solved and tabulated. A limited number of these cases is presented in Table A-9. The effect of a combined loading on a structure can be obtained by adding the effects of each individual loading algebraically.



$$M = -PR\sin\theta$$

$$\frac{\partial M}{\partial P} = -R\sin\theta$$

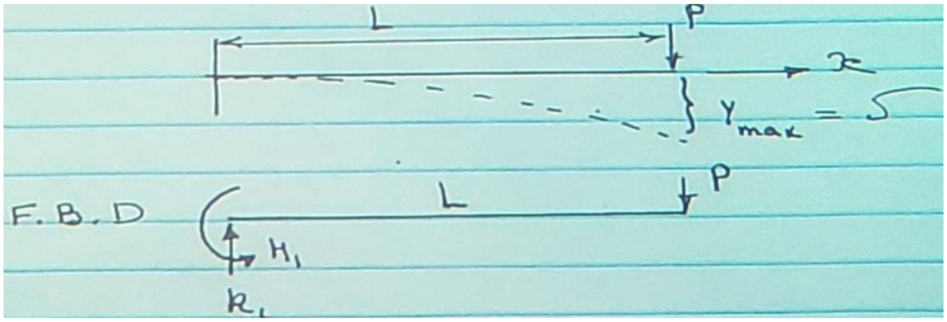
$$\delta = \frac{\partial u}{\partial P} = \int_0^{\frac{\pi}{2}} \frac{M\left(\frac{\partial M}{\partial P}\right)}{EI} R\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(-PR\sin\theta)(-R\sin\theta)}{EI} R\,d\theta$$

$$= \frac{PR^3}{EI} \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{\pi P R^3}{4EI}$$

$$\sin^2\theta = \left(\frac{1}{2}\right)(1-\cos 2\theta)$$

$$\int \sin^2\theta \; d\theta = \left(\frac{1}{2}\right)\int (1-cos2\theta)d\theta = \left(\frac{1}{2}\right)\theta - \left(\frac{1}{4}\right)sin2\theta + C$$



$$M = QRsin\theta \qquad\qquad for \quad 0 \le \theta \le \pi/2$$

$$M = QRsin\theta + PRsin\left(\theta - \frac{\pi}{2}\right) \qquad\qquad \frac{\pi}{2} \le \theta \le \pi$$

$$= QRsin\theta - PRcos\theta$$

$$\frac{\partial M}{\partial Q} = Rsin\theta \qquad\qquad for \quad 0 \le \theta \le \pi$$

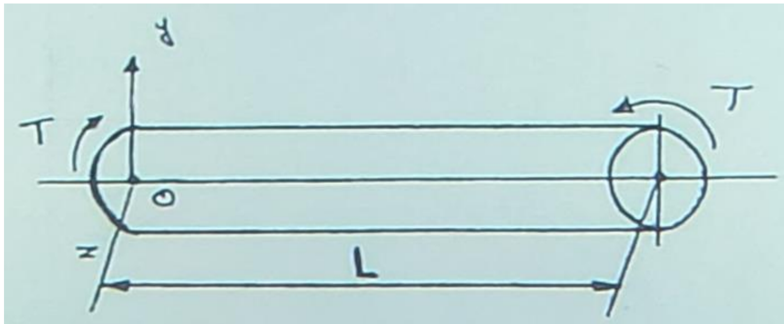
$$\delta_q = \frac{\partial u}{\partial Q} = \int_0^{\frac{\pi}{2}} \frac{(QRsin\theta)(Rsin\theta)}{EI} R\,d\theta + \int_{\pi/2}^{\pi} \frac{\pi(QRsin\theta - PRcos\theta)(Rsin\theta)}{EI} R\,d\theta$$

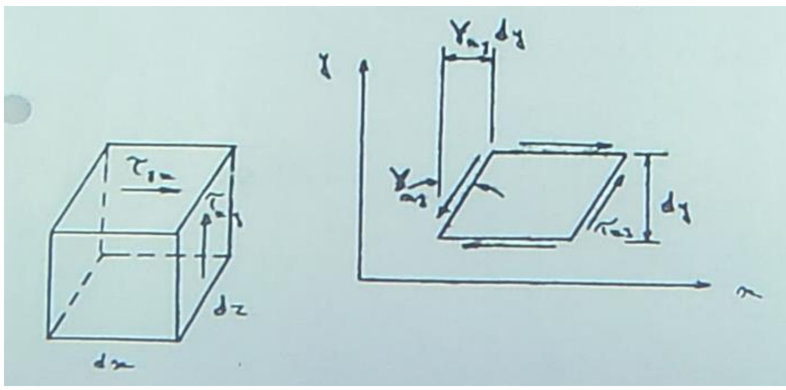
$$= -\frac{PR^3}{EI}\int_{\frac{\pi}{2}}^{\pi} sin\theta\; cos\theta\; d\theta = -\frac{PR^3\sin^2\theta}{2EI}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\delta_q = \frac{PR^3}{2EI}$$

Lecture (Mar. 7th, 2019)

Castigliano's Theorem may also be employed to calculate the angle of twist in members subject to torsion.





$$dU_{shear} = \left(\frac{1}{2}\right)\tau_{xy}\, dx\, dz \cdot \gamma_{xy}\, dy = \left(\frac{1}{2}\right)\tau_{xy}\, dv\, \gamma_{avg}$$

But $\gamma_{avg} = \tau/G$

$$\therefore du = \frac{\tau^2}{2g}\, dv$$

And:

$$U = \int_L \int_A \frac{\tau^2}{2G}\, dx\, dA$$

But:

$$\tau = \frac{Tr}{J}$$

$$\therefore U = \int_L \int_A \frac{T^2\tau^2}{2GJ^2}\, dx\, dA$$

$$U = \int_L \frac{T^2}{2GJ^2}\, dx = \int_A r^2\, dA = \int_0^L \frac{T^2}{2JG}\, dx$$

Where:
$\tau =$ shear stress, psi
$\gamma =$shear strain, in/in
$A =$ cross-sectional area, in^2
$T =$ torque, $in - lb$
$J =$polar moment of inertia, in^4

$$(for\ circular\ shaft)\ =\ \int_A r^2\, dA$$

$$\theta = \frac{\partial U}{\partial T} = \int_0^L \frac{2T}{2JG}\, dx = \int_0^L \frac{T\, dx}{JG}$$

If the torque is uniform along the length of the shaft:

$$\theta = \frac{TL}{JG} = total\ angle\ of\ twist$$

The rotation of a section of a beam at a particular section is found to be:

$$\theta = \frac{\partial U}{\partial C} = \int_0^L \frac{M(\partial M/\partial C)}{EI}\, dx \qquad (C = couple)$$

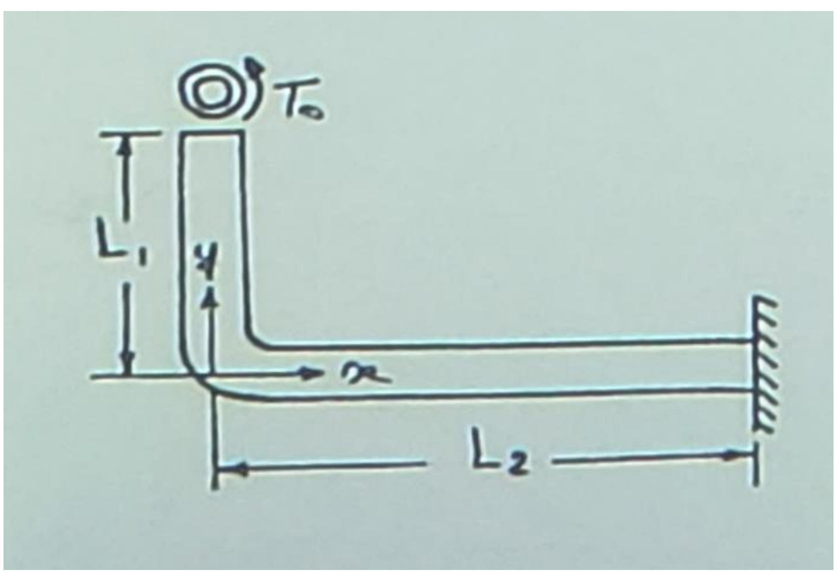
Where C is the couple at the section of interest.

Case of pure bending where $M = C$ throughout the length of the beam:

$$\theta = ML/EI$$

Where E and I are assumed to be constant

Example: Determine the rotation θ of the free end of a tube in the plane of a torque T_o; see Fig. Both portions of the tube lie in the same plane. Neglect the effect of deflection of the radius of the quarter load.



Length L_1 of pipe is subjected to torque T_o
Length L_2 of pipe is subjected to bending moment T_o (about $y - axis$)

$$\therefore U_1 = \int_0^{L_1} \frac{T^2}{2JG}\, dy \quad ; \quad U_2 = \int_0^{L_2} \frac{M^2}{2EI}\, dx$$

Where:
$M = T = T_o$

$$U = U_1 + U_2 = \int_0^{L_1} \frac{T_o^2}{2JG}\, dy + \int_0^{L_2} \frac{T_o^2}{2EI}\, dx$$

And:

$$\theta = \frac{\partial U}{\partial T_o} = \int_0^{L_1} \frac{T_o}{JG}\, dy + \int_0^{L_2} \frac{T_o}{EI}\, dx = \frac{T_o L_1}{JG} + \frac{T_o L_2}{EI}$$

<u>Failure Preventions = Static Loading</u>

Stress Concentration: Stress concentration is a localized effect that may be caused by a surface scratch, variation in material properties, localized high pressure points, or abrupt changes of section.

The stress at a point in a member influenced by one or more of these causes is, in general, grater than the nominal stress determined by elementary strength of materials.

The definition of geometric or theoretical stress concentration factor for normal stress (k_t) and shear stress (k_{ts}) is given by:

$$\sigma_{max} = k_t \sigma_{nom} \quad ; \quad \tau_{max} = k_{ts} \tau_{nom}$$

Table A.15 provides charts for the theoretical stress concentration factors for several load conditions and geometry.

Material	Static Load	Cyclic Load
Brittle	Serious	Very Serious
Ductile	Not Serious	Serious

Failure Theories: The generally accepted failure theories for ductile materials (yield criteria) are:
- Maximum Shear Stress theory (MSS)
- Distortion Energy theory (DE)
- Ductile Coulomb-Mohr theory (DCM)

And for brittle materials (fracture criteria) are:
- Maximum normal stress theory (MNS)
- Brittle Coulomb-Mohr Theory (BCM)
- Modified Mohr Theory (MM)

Maximum Shear-Stress Theory (For Ductile Materials)

This theory assumes that failure occurs for a combined stress condition when the maximum shear stress equals the value of a critical shear stress produced in an element subjected to simple tension, which is:

$$(S_s)_{yp} = \frac{S_{yp}}{2}$$

For $3D$ stressed, the maximum shear stress is given by one of the following, whichever is largest:

$$\frac{(\sigma_1 - \sigma_2)}{2} \quad ; \quad \frac{(\sigma_2 - \sigma_3)}{2} \quad ; \quad \frac{(\sigma_3 - \sigma_1)}{2}$$

Or:

$$\frac{S_{yp}}{2} = \begin{cases} \dfrac{(\sigma_1 - \sigma_2)}{2} \\ \dfrac{(\sigma_2 - \sigma_3)}{2} \\ \dfrac{(\sigma_3 - \sigma_1)}{2} \end{cases} \quad or\ S_{yp} = \begin{cases} \sigma_1 - \sigma_2 \\ \sigma_2 - \sigma_3 \\ \sigma_3 - \sigma_1 \end{cases}$$

For $2D$ stresses, $\sigma_3 = 0$, then:

If σ_1 and σ_2 are of **opposite** sign:
$\sigma_1 - \sigma_2 = \pm S_{yp}$
[or $n_d = S_{yp}/(\sigma_1 - \sigma_2)$]

If σ_1 and σ_2 are of the **same** sign:
$\sigma_1 = \pm S_{yp}$ if $|\sigma_1| > |\sigma_2|$
[or $n_d = S_{yp}/\sigma_1$]

$\sigma_2 = \pm S_{yp}$ if $|\sigma_2| > |\sigma_1|$
[or $n_d = S_{yp}/\sigma_2$]

Distortion-Energy Theorem (For Ductile Materials)

This theory assumes that yielding will occur when the strain energy of distortion per unit volume equals the strain energy of distortion per unit volume for a specimen in uniaxial tension or compression strained to the yield stress. This energy is found to be for the body under $3D$ stress.

$$U_s = \frac{(1+v)}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

For the specimen,

$$U_s = \frac{(1+v)}{6E}(2S_{yp}^2)$$

$$\therefore (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2S_{yp}^2$$

Lecture (Mar. 12th, 2019)

$$U_s = \frac{1+v}{6E}\left(2S_{yp}^2\right)$$

$$\therefore (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2S_{yp}^2$$

Introducing the design factor n_d we have,

$$\sigma' = \sigma_{eq} = \frac{S_{yp}}{n_d}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{\frac{1}{2}}$$

Where:

$$\sigma' = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{\frac{1}{2}}$$

IS known at the $Von\ Mises\ Stress$

For plane stress, $\sigma_3 = 0$,

$$\sigma' = (\sigma_1 + \sigma_2 + \sigma_1\sigma_2)^{\frac{1}{2}}$$

Which is the equation of an eclipse.

Note that in the case of pure shear, $\sigma_1 = -\sigma_2$ or $3\sigma_1^2 = S_{yp}^2$, and $\sigma_1 = 0.577S_{yp}$ while the maximum shear stress theory assumes $\sigma_1 = 0.5S_{yp}$

In terms of the rectangular stress components we can write σ' as

$$\sigma' = \frac{1}{\sqrt{2}}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_x - \sigma_z)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{\frac{1}{2}}$$

And for $2-D$ stress:

$$\sigma' = \left(\sigma_x^2 - \sigma_{x_y} + \sigma_y^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

<u>Coulomb-Mohr Theory (For Ductile Materials)</u>

This theory can be used to predict failure for materials whose strength in tension and compression are not equal. If states that:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

Where either yield strength or ultimate strength can be used.

Incorporating the design factor n_d:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n_d}$$

For plane stress, if the two nonzero principal stresses are $\sigma_A \geq \sigma_B$ then:

If $\sigma_A \geq \sigma_B \geq 0$, then $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$

$$\therefore \sigma_A = \frac{S_t}{n_d}$$

If $\sigma_A \geq 0 \geq \sigma_B$, then $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$

$$\therefore \frac{\sigma_A}{S_t} - \frac{S_t}{S_c} = \frac{1}{n_d}$$

If $0 \geq \sigma_A \geq \sigma_B$, then $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$

$$\therefore \sigma_B = -\frac{S_c}{n_d}$$

Note that for pure shear τ, $\sigma_1 = -\sigma_3 = \tau$

The torsional yield strength occurs when $\tau_{max} = S_{sy}$

Substituting into:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

We get:

$$\frac{S_{sy}}{S_{yt}} + \frac{S_{sy}}{S_{yc}} = 1$$

$$S_{sy}S_{yc} + S_{sy}S_{yt} = S_{yt}S_{yc}$$

$$S_{sy} = \frac{S_{yt}S_{yc}}{S_{yt} + S_{yc}}$$

Reading Assignment:
Example 5-1
Example 5-2

Maximum-Normal-Stress Theory (For Brittle Materials)

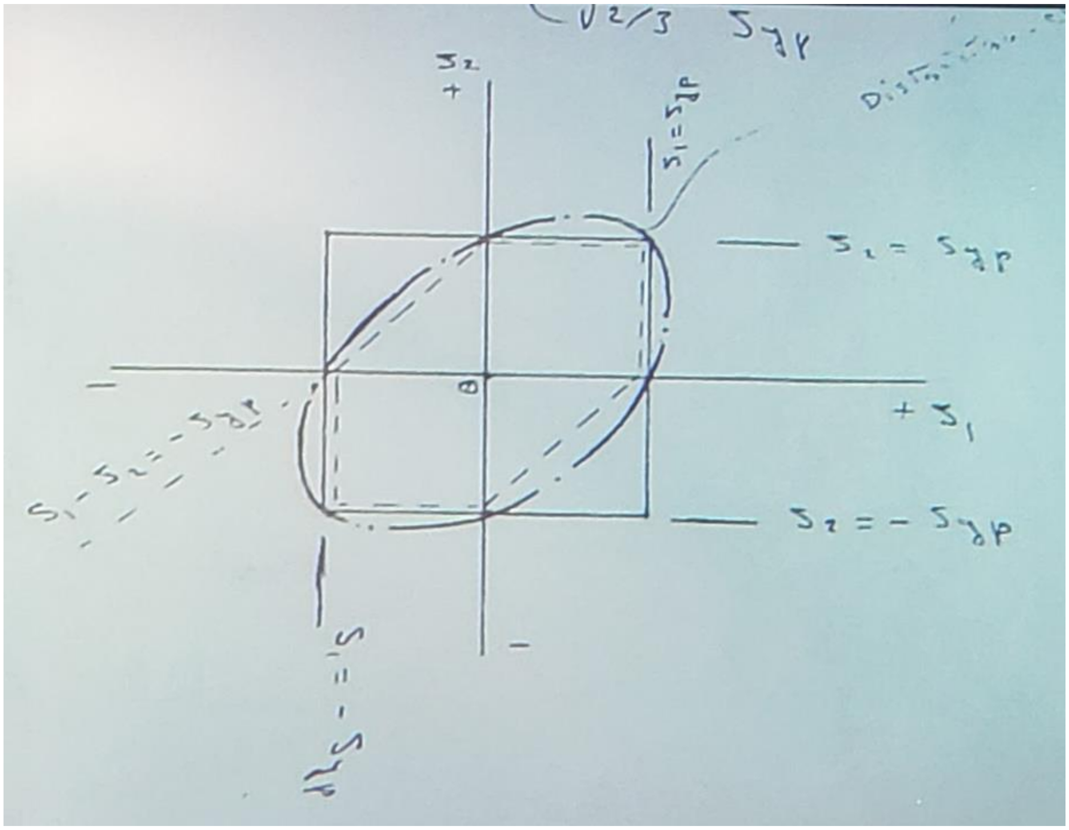
According to this theory, failure occurs at a point in a body when one of the principal stresses at that point equals the critical stress for that material.

If:

$$|\sigma_1| > |\sigma_2| > |\sigma_3|$$

Then:

$$\sigma_1 = \frac{S_{ut}}{n_d}$$



<u>Brittle Coulomb-Mohr Theory:</u>

$$\sigma_A = \frac{S_{ut}}{n_d} \quad ; \quad For\ \sigma_A \geq \sigma_B \geq 0$$

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n_d} \quad ; \quad For\ \sigma_A \geq 0 \geq \sigma_B$$

$$\sigma_B = -\frac{S_{uc}}{n_d} \quad ; \quad For\ 0 \geq \ \sigma_A \geq \sigma_B$$

Modified Coulomb-Mohr Theory:

$$\sigma_A = \frac{S_{ut}}{n_d}$$

$$For\ \sigma_A \geq \sigma_B \geq 0\ and\ \sigma_A \geq 0 \geq \sigma_B\ and\ \left|\frac{\sigma_B}{\sigma_A}\right| \leq 1$$

$$\frac{\sigma_A(S_{uc} - S_{ut})}{S_{uc}S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n_d}$$

$$For\ \sigma_A \geq 0 \geq \sigma_B\ and\ \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

$$\sigma_B = -\frac{S_{uc}}{n_d}$$

$$For\ 0 \geq \sigma_A \geq \sigma_B\ and\ \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

Reading Assignment:
Example 5-3
Example 5-4
Example 5-5

<u>Fatigue Failure (Variable Loading)</u>

Reading Assignment:
Sections 6.1 to 6.6

Fluctuating Stresses

Although most fluctuating stresses in machinery are sinusoidal in nature due to rotating elements, some irregular patterns do occur. However, regardless of its shape, if a pattern exhibits a single maximum and a single minimum force, its shape is not important, but the peaks are important. Let F_{max} be the largest force and F_{min} be the smallest force. Then a steady component, F_m, and an alternating component, F_a, can be constructed.

$$F_m = \frac{F_{max} + F_{min}}{2} \quad ; \quad F_a = \left|\frac{F_{max} - F_{min}}{2}\right|$$

(TODO – Picture)

Where:
σ_{min} = minimum stress
σ_{max} = maximum stress
σ_a = stress amplitude = $(\sigma_{max} - \sigma_{min})/2$
σ_m = mean stress or midrange stress = $(\sigma_{max} + \sigma_{min})/2$
σ_r = stress range = $2\sigma_a$
σ_s = steady, or state stress

- Key Factors in Fatigue Failure

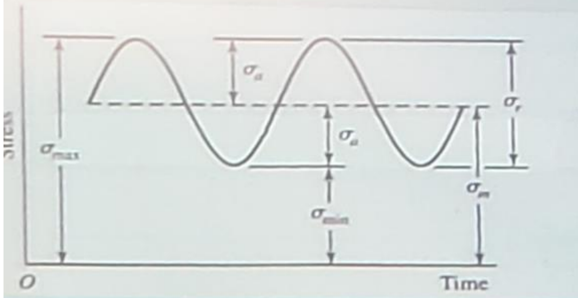
1 – A maximum stress of sufficient magnitude

2 – An applied stress fluctuation of large enough magnitude

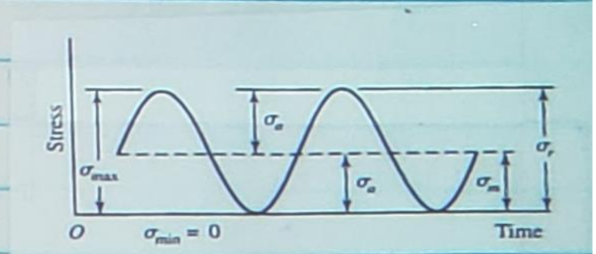
3 – A sufficient number of cycles of the applied stress

Fatigue design procedure

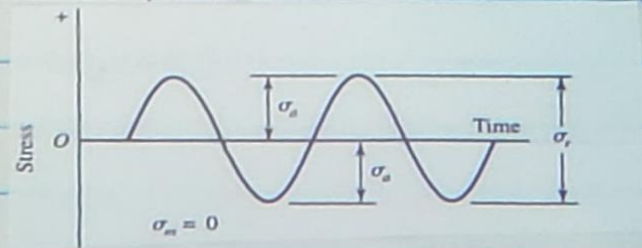
One of the most common methods of presenting engineering fatigue data is by means of the $S - N\ curve.$



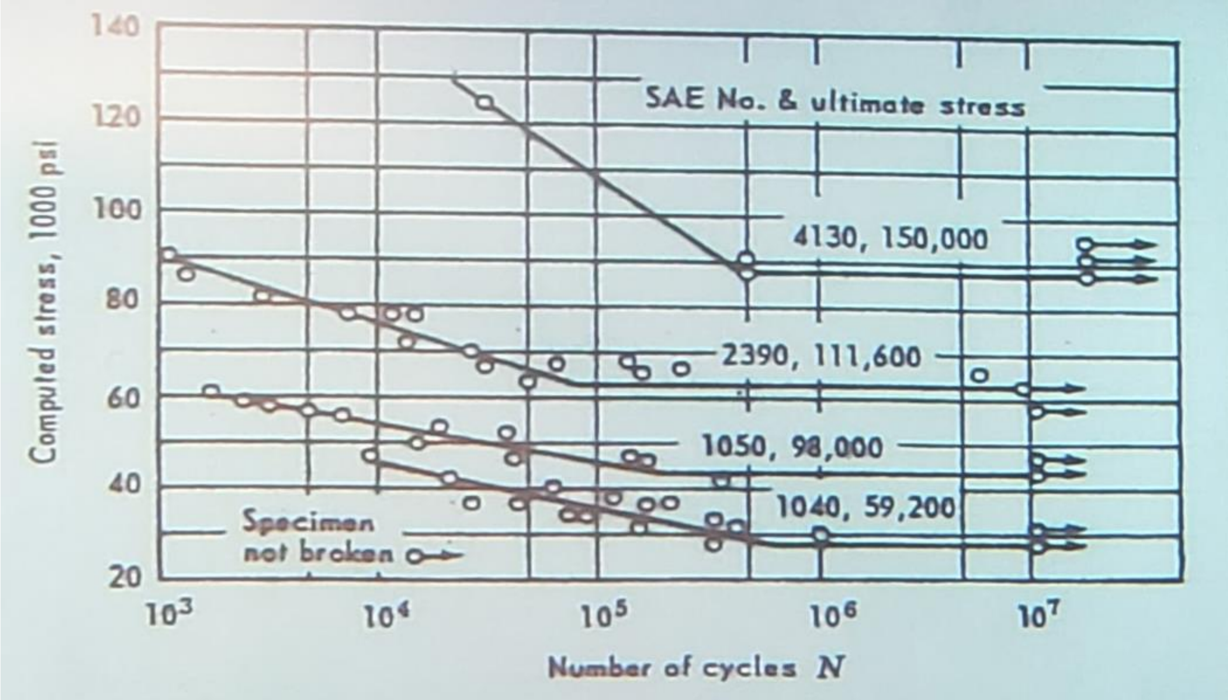
Sinusoidal



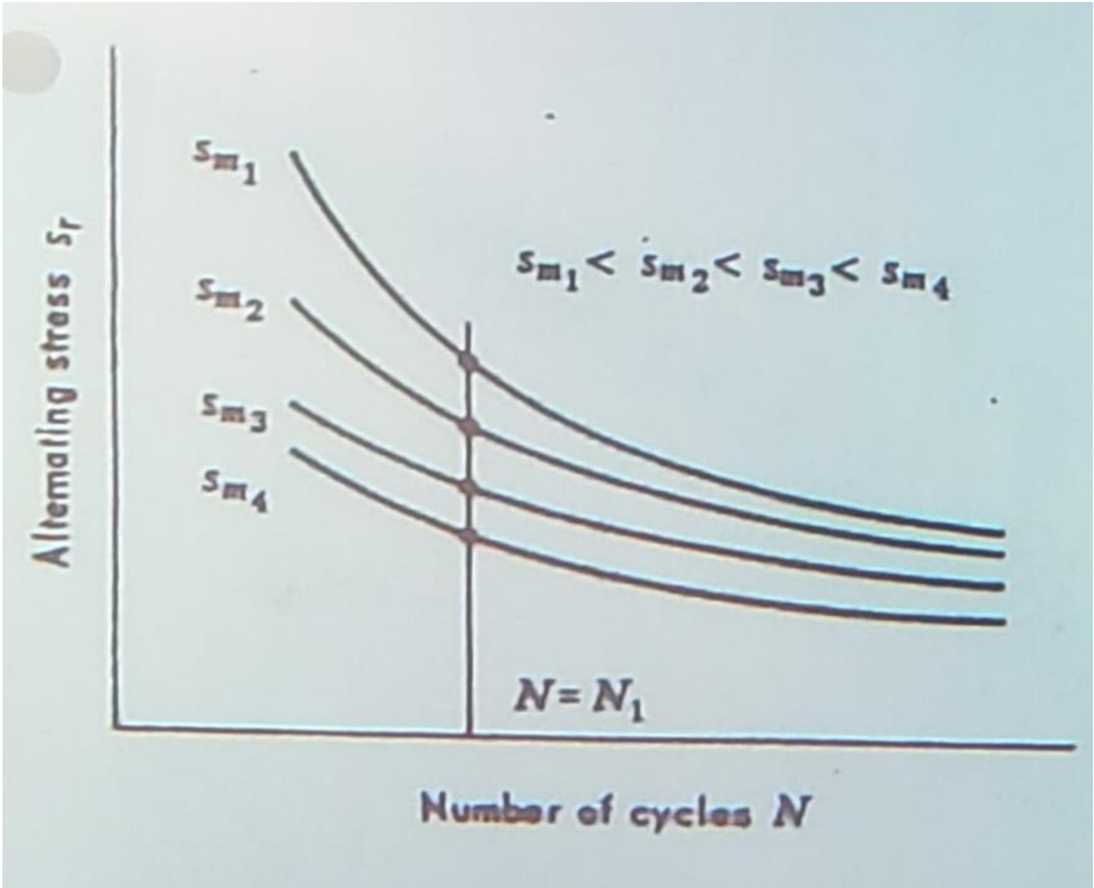
repeated

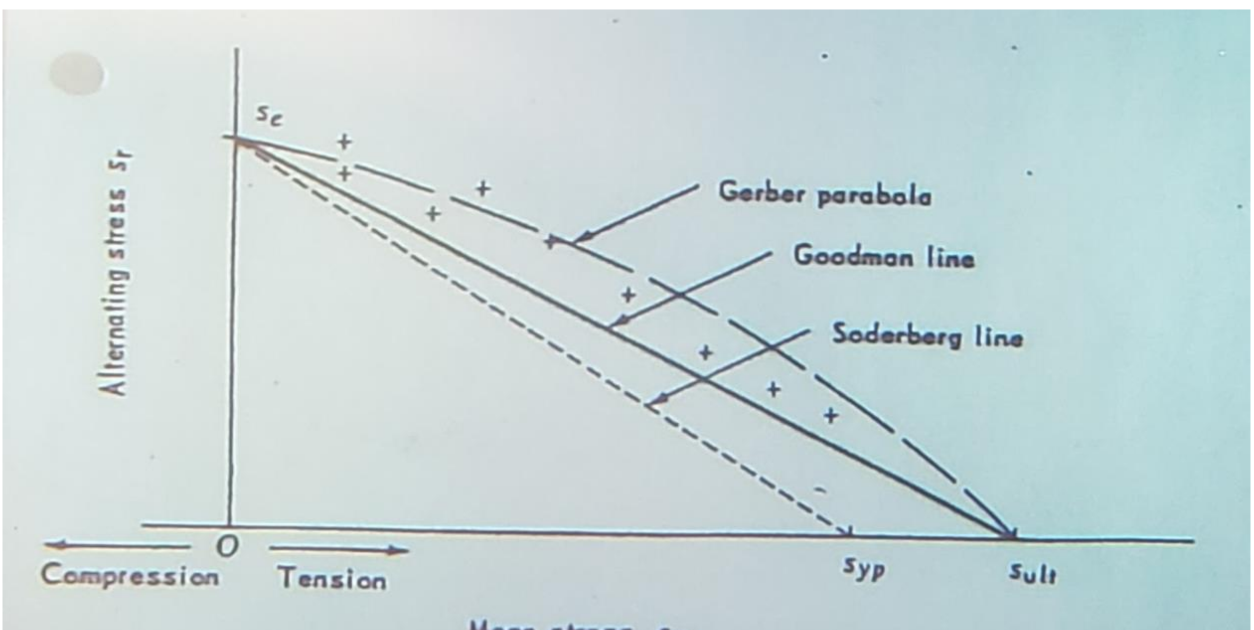


Completely reversed



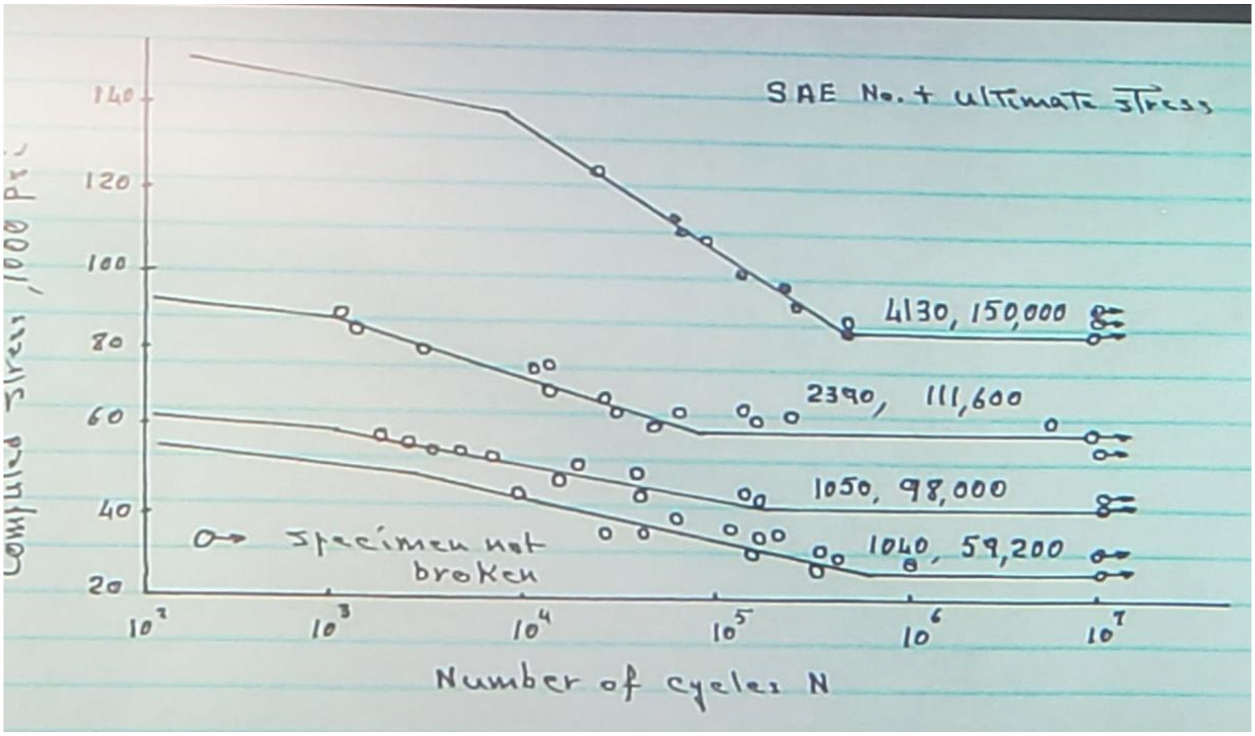
Fatigue-test results for endurance-limit determination.





Failure as a function of mean stress s_m and alternating stress s_r.

In this particular graph, if the $SAE\ No.$ or the ultimate strength is known, and if N is known (number of cycles). The fatigue or endurance limit of the material can be found. However, the above data is for zero mean stress $S_m = 0$. To solve for cases where $S_m \neq 0$, first $S_A - N\ Cures$ are plotted as shown. Then for a given $N = N_1$ the $S_a = S_m$ curve is plotted.



Lecture (Mar. 14th, 2019)

From the $S_n - S_m$ curve as shown in the figure, the following empirical relation was found:

$$S_a = S_e\left[1 - \left(\frac{S_m}{S_{ult}}\right)^P\right]$$

(TODO – Picture)

For Gerber Curve: $P = 2$
For Goodman Line: $P = 1$

When design is based on S_{yp} (yield strength) the Soderberg law is followed.

$$S_m = S_e\left(1 - \frac{S_m}{S_{yp}}\right)$$

And when a factor of safety is required

$$\sigma_a = \frac{S_e}{n_d}\left(1 - \frac{\sigma_m}{S_{yp}}n_d\right)$$

Where:

$$\sigma_a = \frac{S_a}{n_d} \quad ; \quad \sigma_m = \frac{S_m}{n_d}$$

Or:

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{yp}} = \frac{1}{n_d}$$

Where:
S_e = endurance strength for $S_m = 0$
S_{yp} = yield strength
S_{ult} = minimum ultimate tensile strength
n_d = design factor

Using $Goodman$ line:

$$\frac{\sigma_a}{S_e} = \frac{\sigma_m}{S_{ult}} = \frac{1}{n_d}$$

Using $Gerber$ line:

$$\frac{n_d\sigma_a}{S_e} = \left(\frac{n_d\sigma_m}{S_{ult}}\right)^2 = 1$$

Using $ASME$-elliptic line:

$$\left(\frac{n_d\sigma_a}{S_e}\right)^2 + \left(\frac{n_d\sigma_m}{S_{ult}}\right)^2 = 1$$

Using $Langer$ first-cycle-yielding:

$$\sigma_a + \sigma_m = \frac{S_{yp}}{n_d} 7$$

<u>Endurance Limit</u>

Based on a large number of actual test data from several sources, Charles R Mischke, in his paper "Prediction of Stochastic Endurance Strength,"

Trans. Of ASME, J. Vibration Acoustics Stress and Reliability in Design, Vol. 109, No.1, pp 113-122, January 1987, concluded that endurance limit can eb related to tensile strength.

For Steels:

$$S'_e = \begin{cases} 0.504\, S_{ut} \quad ; \quad S_{ut} \leq 200\ kpsi\ (1400\ MPa) \\ 100\ kpsi \quad ; \quad S_{ut} > \ 200\ kpsi \\ 700\ MPa \quad ; \quad S_{ut} > \ 1400\ MPa \end{cases}$$

Where:
S_{ut} = minimum tensile strength
S_e = endurance limit
S'_e = endurance limit of the rotating beam specimen

<u>Fatigue Strength</u>

Recall that:

$$\frac{\Delta\varepsilon_e}{2} = \frac{\sigma'_F}{E}(2N)^b$$

Where:
$\Delta\varepsilon_e$ = elastic strain range
σ'_F = true stress corresponding to fracture in one reversal
b = fatigue strength exponent
N = number of reversals or expected life
E = modulus of elasticity

Defining the specimen fatigue strength at a specific number of cycles as:

$$\left(S'_f\right)_N = \frac{E\Delta\varepsilon_e}{2}$$

Then,

$$\left(S'_f\right)_N = \frac{E\Delta\varepsilon_e}{2} = \sigma'_F(2N)^b$$

At 10^3 cycles:

$$\left(S'_f\right)_{10^3} = \sigma'_F(2 * 10^3)^6 = fS_{ut}$$

Where:

$$f = \frac{\sigma'_F}{S_{ut}} = (2 * 10^3)^b$$

See $Table\ A - 23$ for reliable value of σ'_F for selected steels. Or use $\sigma'_F = \sigma_o \varepsilon_m$, with $\varepsilon = \varepsilon'_F$ is known. Otherwise, you may use the SAE approximation for steels with $H_B \leq 500$ given as:

$$\sigma'_F = S_{ut} + 50\ kpsi \quad ; \quad \sigma'_F = S_{ut} + 345\ MPa$$

Substituting the endurance strength S'_e and corresponding cycles N_e and solving for b:

$$b = -\frac{\log(\sigma'_F / S'_e)}{\log(2N_e)}$$

With values of σ'_F and b known for $70 \leq S_{ut} \leq 200\ kpsi$, Figure 6-18 is plotted where the graph is used to find approcimate values of f for various values of S_{ut} between 70 and 200 kpsi.

For actual mechanical component, we may write:

$$S_f = aN^b$$

Where is can be shown that for $10^3 \leq N \leq 10^6$

$$a = \frac{(f\ S_{ut})^2}{S_e}$$

$$b = -\frac{1}{3}\log\left(\frac{f\ S_{ut}}{S_e}\right)$$

If a completely reversed stress σ_{rev} is given, then:

$$N = \left(\frac{\sigma_{rev}}{a}\right)^{\frac{1}{b}}$$

For low-cycle, $1 \leq N \leq 10^3$ cycles:

$$S_f \geq S_{ut} N^{(\log f)/3}$$

For problems in the finite life range, $10^3 \leq N \leq 10^6$, stresses σ_m and σ_a are transformed into an equivalent completely reversing stress σ_R as follows:

For Goodman:

$$\sigma_R = \frac{\sigma_a S_{ut}}{S_{ut} - \sigma_m} = \sigma_{rev}$$

For Gerber:

$$\sigma_R = \frac{\sigma_m}{1 - \left(\frac{\sigma_m}{S_{ut}}\right)^2} = \sigma_{rev}$$

Reading assignment:
Example 6-2

Example: A bar of steel has the minimum properties $S_e = 40\ kpsi$, $S_y = 60\ kpsi$, and$S_{ut} = 80\ kpsi$. The bar is subjected to a steady torsional stress of $15\ kpsi$ and an alternating bending stress of $25\ kpsi$. Find the factor of safety guarding against a static failure, and either the factor of safety guarding against a fatigue failure or the expected life of the part. For static failure use the $Distorsion - Energy\ Theory$ (DE). For fatigue analysis use:

a) Modified Goodman criterion
b) Gerber criterion
c) ASME-elliptic criterion

Solution:

Given:
$S_e = 40\ kpsi$
$S_y = 60\ kpsi$
$S_{ut} = 80\ kpsi$
$\sigma_a = 25\ kpsi$
$\sigma_m = \tau_a = 0$
$\tau_m = 15\ kpsi$

Using the Distortion Energy Theorem for the alternating, mid-range, and maximum stresses, the $von - Mises$ stresses are:

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

Here $\sigma_y = 0$

$$\therefore \sigma' = \left(\sigma_x^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

And:

$$\sigma_a' = (\sigma_a^2 + 3\tau_a^2)^{\frac{1}{2}} = [(25)^2 + (3)(0)^2]^{\frac{1}{2}} = 25{,}000\ kpsi$$

$$\sigma_m' = (\sigma_m^2 + 3\tau_m^2)^{\frac{1}{2}} = [(0)^2 + (3)(15)^2]^{\frac{1}{2}} = 25.98\ kpsi$$

$$\sigma_{max}' = (\sigma_{max}' + 3\tau_{max}^2)^{\frac{1}{2}} = [(\sigma_a + \sigma_m)^2 + 3(\tau_a + \tau_m)^2]^{\frac{1}{2}}$$
$$= [(25)^2 + (3)(15)^2]^{\frac{1}{2}} = 36.06\ kpsi$$

$$n_y = \frac{S_y}{\sigma_{max}'} = \frac{60}{36.06} = 1.66$$

a) Modified Goodman:

$$n_f = \frac{1}{\left(\frac{\sigma_a}{S_e}\right) + \left(\frac{\sigma_m}{S_{ut}}\right)}$$

$$n_f = \frac{1}{\left(\frac{25}{40}\right) + \left(\frac{35.98}{80}\right)} = 1.05$$

b) Gerber:

$$\frac{n\sigma_a}{S_e} + \left(\frac{n\sigma_m}{S_{ut}}\right)^2 = 1$$

Or:

$$n_f = \left(\frac{1}{2}\right)\left(\frac{S_{ut}}{\sigma_m}\right)^2 \frac{\sigma_a}{S_e}\left[-1 + \sqrt{1 + \left(\frac{2\sigma_m S_e}{S_{ut}\sigma_a}\right)^2}\right]$$

$$n_f = 1.31$$

c) ASME Elliptic:

$$n_f = \sqrt{\frac{1}{\left(\frac{\sigma_a}{S_e}\right)^2 + \left(\frac{\sigma_m}{S_y}\right)^2}}$$

$$n_f = 1.32$$

Lecture (Mar. 19th, 2019)

We have seen how to relate S_e to S_{ut}. The endurance limit of a general mechanical element is obtained from S_e through the use of a variety of modifying factors.

$$S_e = k_a k_b k_c k_d k_e k_f S'_e$$

Where:
S_e = endurance limit
S'_e = endurance limit of test specimen
k_a = surface factor
k_b = size factor
k_c = load factor
k_d = temperature factor
k_e = reliability factor
k_f = miscellaneous effects factor

Surface Factor, k_a

$k_a = aS_{ut}^b$

Where:
k_a = modifying surface factor
S_{ut} = minimum tensile strength
a = a factor from Table 6-2
b = exponent from Table 6-2

Table 6–2

Parameters for Marin Surface Modification Factor, Eq. (6–19)

	Factor a		Exponent
Surface Finish	S_{ut}, kpsi	S_{ut}, MPa	b
Ground	1.34	1.58	−0.085
Machined or cold-drawn	2.70	4.51	−0.265
Hot-rolled	14.4	57.7	−0.718
As-forged	39.9	272.	−0.995

Size Factor, k_b

The results from 133 rotating circular beam tested in bending and torsion may be written as:

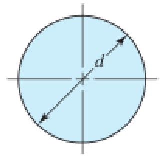
$$k_b = \begin{cases} (d/0.3)^{-0.107} = 0.879d^{-0.107} & 0.11 \le d \le 2 \text{ in} \\ 0.91d^{-0.157} & 2 < d \le 10 \text{ in} \\ (d/7.62)^{-0.107} = 1.24d^{-0.107} & 2.79 \le d \le 51 \text{ mm} \\ 1.51d^{-0.157} & 51 < 254 \text{ mm} \end{cases} \qquad (6\text{–}20)$$

For axial loading, $k_b = 1$

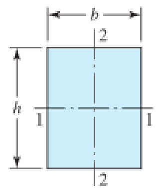
For nonrotating and noncircular cross section an equivalent diameter d_e is used from Table 6-3

Table 6–3

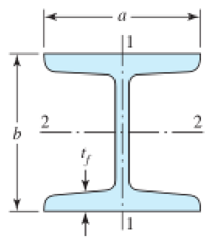
$A_{0.95\sigma}$ Areas of Common Nonrotating Structural Shapes



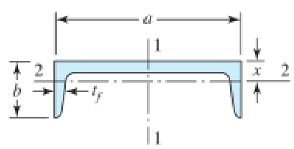
$$A_{0.95\sigma} = 0.01046d^2$$
$$d_e = 0.370d$$



$$A_{0.95\sigma} = 0.05hb$$
$$d_e = 0.808\sqrt{hb}$$



$$A_{0.95\sigma} = \begin{cases} 0.10at_f & & \text{axis 1-1} \\ 0.05ba & t_f > 0.025a & \text{axis 2-2} \end{cases}$$



$$A_{0.95\sigma} = \begin{cases} 0.05ab & \text{axis 1-1} \\ 0.052xa + 0.1t_f(b - x) & \text{axis 2-2} \end{cases}$$

Loading Factor, k_c

$$k_c = \begin{cases} 1 & \text{bending} \\ 0.85 & \text{axial} \\ 0.59 & \text{torsion} \end{cases} \qquad (6\text{–}26)$$

Temperature Factor, k_d

Table 6-4 shows the effect of operating temperature on the tensile strength of steel, where S_{RT} is the strength at room temperature.

Table 6–4

Effect of Operating Temperature on the Tensile Strength of Steel.* (S_T = tensile strength at operating temperature; S_{RT} = tensile strength at room temperature; $0.099 \le \hat{\sigma} \le 0.110$)

Temperature, °C	S_T/S_{RT}	Temperature, °F	S_T/S_{RT}
20	1.000	70	1.000
50	1.010	100	1.008
100	1.020	200	1.020
150	1.025	300	1.024
200	1.020	400	1.018
250	1.000	500	0.995
300	0.975	600	0.963
350	0.943	700	0.927
400	0.900	800	0.872
450	0.843	900	0.797
500	0.768	1000	0.698
550	0.672	1100	0.567
600	0.549		

A fourth order polynomial curve fit gives:

$$k_d = 0.975 + 0.432(10^{-3})T_F - 0.115(10^{-5})T_F^2 + 0.104(10^{-8})T_F^3 - 0.595(10^{-12})T_F^4$$

Where:

$$70 \le T_F \le 1000\ °F$$

Or:

$$k_d = \frac{S_T}{S_{RT}} \quad ; \quad \text{From Table } 6-4$$

Reliability Factor, k_e

$$k_e = 1 - 0.08Z_a$$

Where:

$$Z_a = \frac{x - \mu_x}{\widehat{\sigma_x}} \quad ; \quad \textit{And can be found in Table } A - 10$$

Table 6-5 list values for k_e for some standard reliabilities.

Table 6–5

Reliability Factors k_e Corresponding to 8 Percent Standard Deviation of the Endurance Limit

Reliability, %	Transformation Variate z_a	Reliability Factor k_e
50	0	1.000
90	1.288	0.897
95	1.645	0.868
99	2.326	0.814
99.9	3.091	0.753
99.99	3.719	0.702
99.999	4.265	0.659
99.9999	4.753	0.620

Miscellaneous-Effects Factor, k_f

Actual values of k_f are not always available. However, its presence is a reminder that other effects such as corrosion and others must be considered.

Stress Concentration and Notch Sensitivity

Some materials are not fully sensitive to the presence of discontinuities, such as holes, grooves, or notches. For these, a reduced value of the stress concentration factor k_t or k_{ts} can be used.

The reduced value k_f or k_{fs}, which is called the fatigue stress-concentration factor, is given as:

$$k_f = 1 + q(k_t - 1)$$

Or:

$$k_{fs} = 1 + q_{shear}(k_{ts} - 1)$$

Where q is known as the notch sensitivity. Some known values of q and q_{shear} are shown in Figures 6.20 and 6.21. The functions in these figures are:

$$k_f = 1 + \frac{k_t - 1}{1 + \sqrt{a/r}} \quad ; \quad q = \frac{1}{1 + \sqrt{a/r}}$$

Where:
r = notch radius
$\sqrt{a}$ = Neuber constant given as

For bending and axial:
$\sqrt{a} = 0.246 - 3.08(10^{-3})S_{ut} + 1.51(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3$

For torsion:
$\sqrt{a} = 0.190 - 2.51(10^{-3})S_{ut} + 1.35(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3$

Figure 6–20

Notch-sensitivity charts for steels and UNS A92024-T wrought aluminum alloys subjected to reversed bending or reversed axial loads. For larger notch radii, use the values of q corresponding to the r = 0.16-in (4-mm) ordinate. *(From George Sines and J. L. Waisman (eds.),* Metal Fatigue, *McGraw-Hill, New York. Copyright © 1969 by The McGraw-Hill Companies, Inc. Reprinted by permission.)*

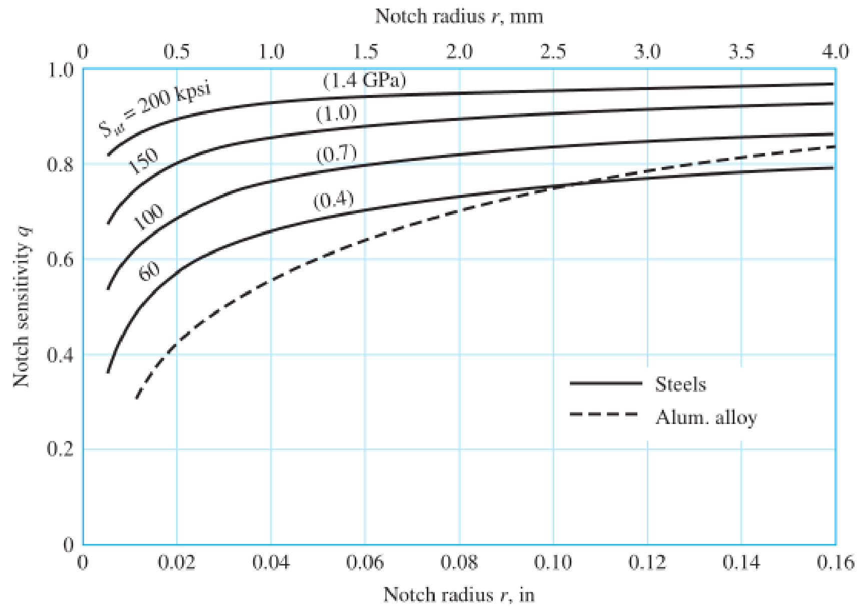
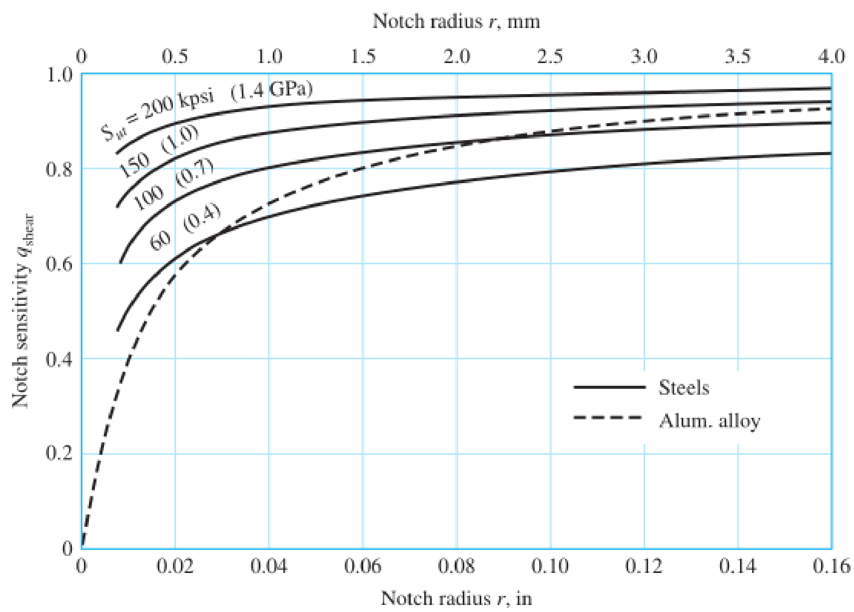


Figure 6–21

Notch-sensitivity curves for materials in reversed torsion. For larger notch radii, use the values of q_{shear} corresponding to r = 0.16 in (4 mm).



The notch sensitivity of cast iron is very low. However, to be on the conservative side, it is recommended that $q = 0.20$ be used for all grades of cast iron.

Reading Assignment:
Exampes 6.6 to 6.9
Section 6.17

Load Line Concept

When Goodman relation is used, the safe-stress line through the working stress point A, as shown in the figure, is constructed parallel to the Goodman line. Note that the sage-stress-line is the locus of all sets of σ_a, σ_m stresses having a factor of safety n and that $S_m = n\sigma_m$ and $S_a = n\sigma_a$.

(TODO – Picture)

Let:
σ_p = the static component of a working stress
σ_m = the mean stress corresponding to $k_f \sigma_a$
S_m = the critical mean stress corresponding to S_a

The line going through the point $(0, \sigma_p)$, $(k_f \sigma_a, \sigma_m)$ and (S_a, S_m) is the load line L of slope m.

Have:

$$S_o = n\sigma_a k_f \quad ; \quad n = \frac{S_o}{\sigma_a k_f} = \frac{S_m - \sigma_p}{\sigma_m - \sigma_p} \quad ; \quad (By\ similar\ triangles)$$

From Goodman relation:

$$S_m = S_{ut}\left(1 - \frac{S_a}{S_e}\right)$$

The load line slope is:

$$m = \frac{\sigma_a k_f}{\sigma_m - \sigma_p} = \frac{S_a}{S_m - \sigma_p}$$

$$\therefore S_a = m(S_m - \sigma_p) = mS_{ut}\left(1 - \frac{S_a}{S_e}\right) - m\sigma_p$$

$$= m(S_{ut} - \sigma_p) - \frac{mS_a S_{ut}}{S_e}$$

Or:

$$S_a + \frac{mS_a S_{ut}}{S_e} = m(S_{ut} - \sigma_p)$$

$$S_a\left(1 + \frac{mS_{ut}}{S_e}\right) = m(S_{ut} - \sigma_p)$$

And:

$$S_a = \frac{m(S_{ut} - \sigma_p)}{1 + \frac{mS_{ut}}{S_e}}$$

Where:

$$m = \frac{k_f \sigma_a}{\sigma_m - \sigma_p}$$

But:

$$(\sigma_m - \sigma_p) = \sigma_a$$

$$\therefore m = k_f$$

(TODO – Picture)

The concept of the load line in conjunction with the failure criteria is used to tabulate the principal intersection in Table 6-6 to 6-8, where $r = S_a/S_m = \sigma_a/\sigma_m$. The first column in each table gives the intersection equations and the second column gives the intersection coordinates.

Reading Assignment:
Examples 6-10 to 6-12

Fatigue Failure of Brittle Materials

(TODO – Picture)

Not enough work has been done on brittle fatigue. Consequently, designed stay in the first and a bit in the second quadrant in the range from:

$$-S_{ut} \leq \sigma_m \leq S_{ut}$$

Lecture (Mar. 21st, 2019)

The first quadrant failure fatigue criteria for many brittle materials follows the Smith-Dolan locus given by:

$$\frac{S_a}{S_e} = \frac{1 - \frac{S_m}{S_{ut}}}{1 + \frac{S_m}{S_{ut}}}$$

Or:

$$\frac{n\sigma_a}{S_e} = \frac{1 - \frac{n\sigma_m}{S_{ut}}}{1 + \frac{n\sigma_m}{S_{ut}}}$$

If r is the load line slope $r = S_a/S_m$, substituting S_a/r for S_m and solving for S_a we get,

$$S_a = \frac{rS_{ut} + S_e}{2}\left[-1 + \sqrt{1 + \frac{4rS_{ut}S_e}{(rS_{ut} + S_e)^2}}\right]$$

The portion of the second quadrant is represented by a straight line between the two points $(-S_{ut}, S_{ut})$ and $(0, S_e)$, which is represented by,

$$S_a = S_e + \left(\frac{S_e}{S_{ut}} - 1\right)S_m \quad ; \quad -S_{ut} \le S_m \le 0$$

Properties of gray cast iron are found in Table A-24, where the endurance limit stated include k_a and k_b

Table A–24

Mechanical Properties of Three Non-Steel Metals
(a) Typical Properties of Gray Cast Iron
[The American Society for Testing and Materials (ASTM) numbering system for gray cast iron is such that the numbers correspond to the *minimum tensile strength* in kpsi. Thus an ASTM No. 20 cast iron has a minimum tensile strength of 20 kpsi. Note particularly that the tabulations are *typical* of several heats.]

ASTM Number	Tensile Strength S_{ut}, kpsi	Compressive Strength S_{uc}, kpsi	Shear Modulus of Rupture S_{su}, kpsi	Modulus of Elasticity, Mpsi: Tension†	Modulus of Elasticity, Mpsi: Torsion	Endurance Limit* S_e, kpsi	Brinell Hardness H_B	Fatigue Stress-Concentration Factor K_f
20	22	83	26	9.6–14	3.9–5.6	10	156	1.00
25	26	97	32	11.5–14.8	4.6–6.0	11.5	174	1.05
30	31	109	40	13–16.4	5.2–6.6	14	201	1.10
35	36.5	124	48.5	14.5–17.2	5.8–6.9	16	212	1.15
40	42.5	140	57	16–20	6.4–7.8	18.5	235	1.25
50	52.5	164	73	18.8–22.8	7.2–8.0	21.5	262	1.35
60	62.5	187.5	88.5	20.4–23.5	7.8–8.5	24.5	302	1.50

*Polished or machined specimens.

†The modulus of elasticity of cast iron in compression corresponds closely to the upper value in the range given for tension and is a more constant value than that for tension.

Reading Assignment:
Example 6.13
Section 6-13

Combination of Loading Modes

As we have seen, the load factor k_c depends on the type of loading. There may also be stress-concentration factors, which may depend on the type of loading. The question is therefore, 'How do we proceed when the loading is a mixture of axial, bending, and torsional loads?"

To answer this question, we first generate the two stress elements σ_a and σ_m and apply the appropriate fatigue stress-concentration factors to them.

Second, we calculate the equivalent von Mises stress, for each of these two stress elements, σ_a' and σ_m'

Finally, select a fatigue failure criterion to complete the fatigue analysis.

For the endurance limit, S_e, we only use k_a, k_b, and k_c for bending and account for the axial load factor by dividing the alternating axial stress by 0.85.

In the common case of a shaft with bending stresses, torsional shear stresses and axial stresses, the von Mises stress is:

$$\sigma' = \left(\sigma_x^2 + 3\tau_{xy}^2\right)^{\frac{1}{2}}$$

And therefore:

$$\sigma_a' = \left\{\left[(K_f)_{\text{bending}}(\sigma_a)_{\text{bending}} + (K_f)_{\text{axial}}\frac{(\sigma_a)_{\text{axial}}}{0.85}\right]^2 + 3\left[(K_{fs})_{\text{torsion}}(\tau_a)_{\text{torsion}}\right]^2\right\}^{1/2} \quad (6\text{–}55)$$

$$\sigma_m' = \left\{\left[(K_f)_{\text{bending}}(\sigma_m)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_m)_{\text{axial}}\right]^2 + 3\left[(K_{fs})_{\text{torsion}}(\tau_m)_{\text{torsion}}\right]^2\right\}^{1/2} \quad (6\text{–}56)$$

For first-cycle localized yielding,

First add the axial and bending alternating and midrange stresses to obtain σ_{max}

Second add the alternating stress to the midrange shear stresses to obtain τ_{max}

Then substitute σ_{max} and τ_{max} into the von-mises stress equation

A simpler and more conservative method is to add σ_a' and σ_m' to find σ_{max}'

$$\sigma_{max}' = \sigma_a' + \sigma_m' = \frac{S_y}{n}$$

Reading Assignment:
Example 6.14

The fatigue factor of safety is then found using one of the following:

Modified Goodman and Langer Failure Criteria

$$n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}$$

For Gerber and Langer Failure Criteria

$$n_f = \frac{1}{2}\left(\frac{S_{ut}}{\sigma_m}\right)^2 \left(\frac{\sigma_e}{S_e}\right)\left[-1 + \sqrt{1 + \left(\frac{2\sigma_m S_e}{S_{ut}\sigma_a}\right)^2}\right] \quad ; \quad \sigma_m > 0$$

For ASME Elliptic and Langer Failure Criteria

$$n_f = \sqrt{\frac{1}{(\sigma_m/S_e)^2 + \left(\sigma_m/S_y\right)^2}}$$

For First-cycle failure

$$n_y = \frac{S_y}{\sigma_a + \sigma_m} = \frac{S_y}{\sigma_{max}}$$

In the case of pure shear, it is convenient to use the maximum shear-stress theory and replacing σ by τ in the above equations and S_{ut} by S_{su} from Equation 6.54:

$$S_{su} = 0.67 S_u$$

And:

$$n_f = \frac{\frac{S_y}{2}}{\tau_{max}}$$

Section 6.17 pp 338-341 provides a good summary for fatigue failure.

Surface Endurance Shear (Buckingham Wear Factor)

When two surfaces roll, slide, or roll and slide against each other with sufficient force, a pitting failure will occur after a certain number of cycles of operation. To determine the surface strength of mating materials, Buckingham conducted many tests which were later extended by Talbourdet. Based on the data obtained, and using Heart constant stresses equations, Buckingham defined a wear factor, which is also known as load-stress factor, as follow:

Hearty equations for contacting cylinders are:

$$b = \left[\frac{2F}{\pi l} \frac{\frac{(1 - v_1^2)}{E_1} + \frac{(1 - v_2^2)}{E_2}}{\left(\frac{1}{d_1}\right) + \left(\frac{1}{d_2}\right)}\right]^{\frac{1}{2}}$$

$$P_{max} = 2F/\pi bl$$

Where:
$b =$ half width of rectangular contact area
$F =$ contact force
$w \; or \; l =$ width of cylinders (length of contact)
$v =$ poisson's ratio
$E =$ modulus of elasticity
$d =$ cylinder diameter

Replacing d by $2r$, l by w and using an average value of 0.3 for v_1 and v_2 to get:

$$b^2 = 1.16\left(\frac{F}{w}\right)\frac{\left(\frac{1}{E_1}\right)+\left(\frac{1}{E_2}\right)}{\left(\frac{1}{r_1}\right)+\left(\frac{1}{r_2}\right)}$$

Defining surface strength as the maximum pressure at which surface fatigue failure starts to get:

$$S_e = \frac{2F}{\pi bw}$$

This is also known as the contact strength, the contact fatigue strength, or the Hertigan endurance strength.

Substituting b and rearranging to get:

$$2.857S_e^2\left(\frac{1}{E_1}+\frac{1}{E_2}\right) = \frac{F}{w}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

Buckingham's load-stress factor k_1 is defined as:

$$k_1 = 2.857S_c^2\left(\frac{1}{E_1}+\frac{1}{E_2}\right)$$

The design equation for surface fatigue strength is then:

$$k_1 = \frac{F}{w}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

In the presence of a factor of safety n this equation is written as:

$$\frac{k_1}{n} = \frac{F}{w}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

The Hertigian endurance strength of steels for 10^8 cycles of repeated contact stress is obtained from the following equation:

$$S_e = \begin{cases} 0.4H_B - 10kpsi \\ 2.76H_B - 70\ MPa \end{cases}$$

Where H_B is the Brinnel hardness number.