$\boxed{\text{Example:}}$ Consider non-linear equation $x^3 = 2x + 1$ which has root on 1.5, 2 → using Fixed point method

$x_0 = 1.5$

$x^3 = 2x+1 \rightarrow f(x) = x^3 - 2x - 1 = 0$

$[1.5,\ 2]$

I $\quad x_{i+1} = \dfrac{x_i^3 - 1}{2} = g_1(x_i)$

$x_0 = 1.5$

II $\quad x_{i+1} = \dfrac{1}{x^2-2} = g_2(x_i)$

	g_1	g_2	g_3	g_4
i	x_i	x_i	x_i	x_i
0	1.5	1.5	1.5	1.5
1	1.1875	4	1.632993	1.5874
2	0.33728	0.071429	1.616282	1.61096
3	−0.4808816	−0.501279	converging	converging
4	−0.5165579	−0.571847		
5	−0.585745	−0.597731		
	DIVERGING	DIVERGING		

III $\quad x_{i+1} = \sqrt{\dfrac{2x+1}{x_i}} = g_3(x_i)$

IV $\quad x_{i+1} = \sqrt[3]{2x+1} = g_4(x_i)$

$\boxed{\text{Example:}}$ $f(x) = e^{-x} - x$ (Initial guess : 0)

$e^{-x} - x = 0$

I $\quad x_{i+1} = e^{-x} = g(x_i)$ ← use

II $\quad x_{i+1} = -\ln x = g(x_i)$

SIDE:

$e^{-x} - x = 0$

$\cancel{\ln} e^{-x} = \ln x$

$x = -\ln x$

i	x_i	$\lvert\varepsilon_a\rvert$ %	$\lvert\varepsilon_t\rvert$ %	$\lvert\varepsilon_t\rvert_i / \lvert\varepsilon_t\rvert_{i-1}$
0	0	—	100.00	
1	1	100		
2	0.3679		○ ○ ○	
3	0.6922			
4	0.5005			

Example Converge Faster to $\sqrt{5}$

$$x_{n+1} = x_n + 1 - x_n^2/5$$
$$x_{n+1} = \frac{1}{3}\left[3x_n + 1 - x_n^2/5\right]$$

$$g_1 = x + 1 - x^2/5$$
$$g_1' = 1 - 2x/5$$
$$\rightarrow \quad g_1'(\sqrt{5}) = \left|1 - \frac{2(\sqrt{5})}{5}\right| = 0.1056 < 1$$

smaller value
(∴ Faster convergence)

$$g_2 = \frac{1}{3}\left(3x + 1 - x^2/5\right)$$
$$g_2' = 1 - \frac{2x}{15}$$
$$\rightarrow \quad g_2'(\sqrt{5}) = \left|1 - \frac{2(\sqrt{5})}{15}\right| = 0.701186 < 1$$

Example Newton-Rhapson method to estimate the root of $f(x) = e^{-x} - x$

Use initial guess $x_0 = 0$

$$f(x) = e^{-x} - x$$
$$f'(x) = -e^{-x} - 1$$
$$x_{i+1} = x_i - \frac{e^{-x} - x_i}{-e^{-x} - 1}$$

i	x_i
0	0
1	0.5
2	0.5663

Example Use Newton-Rhapson method to locate the root of $f(x) = x^{10} - 1$ and initial guess $x = 0.5$

$$f(x) = x^{10} - 1 \qquad x_{i+1} = x_i - \frac{(x^{10} - 1)}{(10x^9)}$$
$$f'(x) = 10x^9$$

i	x_i	ε_a (%)
0	0.5	
1	51.65	99.032
2	46.485	11.111

...

i	x_i	ε_a (%)
41	1.000024	0.229
42	1	0.002

Midterm material (to end of lecture)

↳ true error / approximate error (review for midterm)

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$ backward differencing

$$f'(x) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$ Forward differencing

$\boxed{\text{Example}}$ use secant method

$f(x) = e^{-x} - x$ $\qquad x_{i-1} = 0$

(DONT NEED) $f'(x) = -e^{-x} - 1$ $\qquad x_i = x_0 = 1$

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

For true values:

(Given):

$x_{true} = 0.56714329$

$f(x_{i-1}) = 1$

$f(x_i) = e^{-1} - 1$

$$x_{i+1} = 1 - \frac{(e^{-1} - 1)(0 - 1)}{1 - (e^{-1} - 1)}$$

$$= 1 - \frac{(-0.6321)(-1)}{1 - (-0.6321)} = 0.612670$$

$$\varepsilon_{true} = \left| \frac{App - true}{true} \right| \times 100 \rightarrow \varepsilon_{true} = \left| \frac{0.612670 - 0.567...}{0.567...} \right|$$

↳ 8 %

$$\varepsilon_{app} = \frac{(0.612670 - 1)}{0.612670} = 63.2$$

→ he has to give:

- number of iterations
- initial guesses
- true value (for true error)