(2-3)a Where $\dot{Q}_x = -k A_x \left(\frac{dT}{dx}\right) = -k A_c \left(\frac{dT}{dx}\right)$

Recall, the definition of a derivative

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{d\dot{Q}_x}{dx} dx$$

(2-3)b $\rightarrow \dot{Q}_{x+dx} - \dot{Q}_x = \frac{d\dot{Q}_x}{dx} dx$

Sub. eq. (2-3)b into (2-2) gives :

(2-3)c $\frac{d\dot{Q}_x}{dx} dx + d\dot{Q}_{conv} = 0$

Sub (2-3)a in (2-3)c, gives :

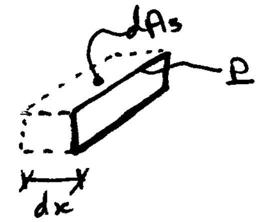
(2-3)d $d/dx \left[-k A_c \, dT/dx \right] dx + \dot{Q}_{conv} = 0$

Sub (2-1) in (2-3)d, yields Note : T(x)

(2-3)e $d/dx \left[-k A_c \, dT/dx \right] dx + h \, dA_s (T - T_\infty) = 0$

(2-4) where, $\boxed{dA_s = P \cdot dx}$

(P called the fin parameter)



Now, considering :

A_c = constant, k = const./uniform (not function of x)

$\rightarrow$ Eq. (2-3)d becomes : NOTE: T is function of x T(x)

$$-k A_c (d/dx)(dT/dx) dx + h P dx (T - T_\infty) = 0$$

Dividing the two terms in this eq. by $(-k A_c \, dx)$, gives

$$d/dx \, (dT/dx) - (hP/kA_c)(T - T_\infty) = 0$$

(2-5) or, $\boxed{\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0}$

This is a linear, homogeneous, 2nd-order, ordinary differential equation with constant coefficients.

$\rightarrow$ In order to be able to solve this differential eq. we introduce the temperature excess $= \theta$;

(2-6) $\boxed{\theta \equiv T - T_\infty}$ (or $\theta(x) = T(x) - T_\infty$)

(2-6)b NOTE : $\theta(x=0) = \theta_b = T_b - T_\infty$

$T_b = T(x=0)$

subscript b for "base"

$$\frac{d^2(\theta + T_\infty)}{dx} = \frac{d}{dx}\left[\frac{d\theta}{dx} + \cancel{\frac{dT_\infty}{dx}}_{\to 0}\right] = 0$$

$$\frac{d}{dx}\left[\frac{d\theta}{dx}\right] = 0 \Rightarrow \boxed{\frac{d^2\theta}{dx^2} = 0}$$

The transformation of variables from $T \to \theta$ in eq. (2-5) gives:

$$\frac{d^2(\overbrace{\theta + T_\infty}^{=T})}{dx^2} - \frac{hP}{kA_c} = 0$$

$$\to \left[\frac{d^2\theta}{dx^2} + \cancel{\frac{d^2T_\infty}{dx^2}}_{\to 0}\right] - \frac{hP\theta}{kA_c} = 0$$

(since T_∞ = const., not function of x)

Simplifying, we have:

(2-7) $$\frac{d^2\theta}{dx^2} - \frac{hP}{kA_c}\theta = 0$$

Introducing a parameter m, given by

(2-8) $$m = \sqrt{\frac{hP}{KA_c}} \quad \text{or} \quad m^2 = \frac{hP}{KA_c}$$

Sub (2-8) in (2-7), gives:

(2-9) $$\boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0}$$

The general solution for this differential eq'n is given by:

(2-10) $$\theta(x) = C_1e^{mx} + C_2e^{-mx}$$

Verification:

Sub. of (2-10) in (2-9) verifies that eq. (2-10) is indeed the Solution for that diff. equation (2-9) (try it!)

Now, to evaluate the constants c_1 & C_2, we need to specify two appropriate boundary conditions.

→ The 1st BC is that at $x = 0$, $\theta = \theta_b$ as given by Eq. (2-6)b

→ The 2ND BC, specified at the fin tip $(x = L)$, may correspond to one of four different physical cases (situations), as follows:

$\boxed{\text{Case A:}}$ Convection from the fin tip:

This case considers convection heat transfer from the fin tip: Applying surface energy balance as shown in Fig (2-6)a, gives:

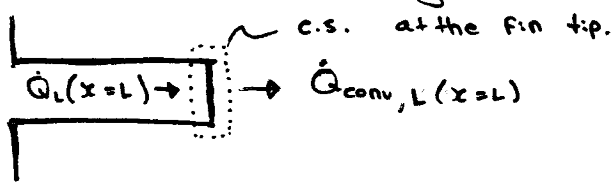


Figure (2-6)a: Energy balance at the tip.

$$\dot{Q}_{cond,L} \ (\text{at } x = L) = \dot{Q}_{conv,L} \ (\text{at } x = L)$$

(2-11)a $$-kA_c \frac{dT}{dx}\Big|_{x=L} = hA_c\,[T(L) - T_\infty]$$

or, in terms of θ, we have

$$-kA_c \frac{d\theta}{dx}\Big|_{x=L} = hA_c\,[\theta(L)]$$

And dividing by A_c, yields

(2-11)b $$-k\frac{d\theta}{dx}\Big|_{x=L} = h\,\theta(L)$$

Now, sub the 1st BC $\theta = \theta_b$ at $x = 0$ into (2-10) we get

$$\theta(x=0) = \theta_b = c_1 \cancel{e^{m(0)}}^{\,1} + C_2\,\cancel{e^{-m(0)}}^{\,1}$$

(2-12)a $$\therefore\ \theta_b = c_1 + C_2$$

Sub. in the 2nd B.C. (Case A) eq (2-11)b, gives eq. (2-10) θ.

$$-k\left[\frac{d}{dx}\overbrace{(c_1 e^{mx} + C_2 e^{-mx})}^{\theta}\right] = h\overbrace{(c_1 e^{mL} + C_2 e^{-mL})}^{=\theta_L}$$

$$\rightarrow -k\left[C_1 e^{mx}\cdot m + C_2 e^{-mx}(-m)\right] = h\left[C_1 e^{mL} + C_2 e^{-mL}\right]$$

$$km\left[C_2 e^{-mL} - C_1 e^{mL}\right] = h\left[C_1 e^{mL} + C_2 e^{-mL}\right]$$

Solving for C_1 & C_2 using eqs. (2-12)a & b, it may be shown after some manipulation, that

(called cosine hyperbolic cosh) (called sine hyperbolic sinh)

(2-13)a $$\theta = \theta_b\left[\frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh(mL) + (h/mk)\sinh(mL)}\right]$$

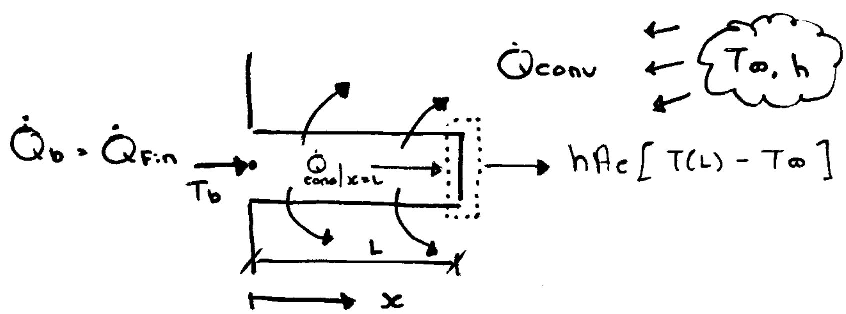
Now, transform back $\theta \rightarrow T$, gives

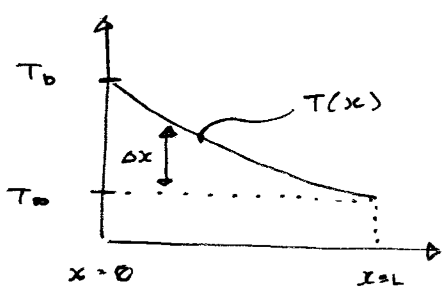
(2-13)b $$T - T_\infty = (T_b - T_\infty)\left[\frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh(mL) + (h/mk)\sinh(mL)}\right]$$

Re-arranging (dividing both sides by $(T_b - T_\infty)$), gives

(2-14) $$\boxed{\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh(m(L-x)) + (h/mk)\sinh(m(L-x))}{\cosh(mL) + (h/mk)\sinh(mL)}}$$

→ temp. dist. for Case A





$\dot{Q}_{fin} = ?$

We are particularly interested to determine the amount of heat transferred from the entire fin.

$\dot{Q}_{fin}$ can be evaluated in two different ways, both of which involve the use of temp. distribution.

1 The simpler one involves applying Fourier's Law at the fin base, as follows:

(energy balance at the fin-base)

(2-15) $\dot{Q}_{fin} = \dot{Q}_b = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA \frac{d\theta}{dx}\Big|_{x=0}$

Using eq (2-13)a (or 2-14) gives [CASE A]:

(2-16)a $$\boxed{\dot{Q}_{fin} = \sqrt{hPkA_c}\,\theta_b \left[\frac{\sinh(mL) + (h/mk)\cosh(mL)}{\cosh(mL) + (h/mk)\sinh(mL)}\right]}$$

For CASE A

From eq (2-6)b

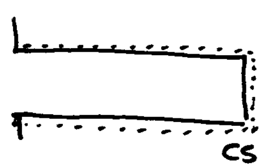
(2-16)b Defining, $M \equiv \left(\sqrt{hPkA_c}\right)\theta_b$

2 The alternative way to find $\dot{Q}_{fin}$ is by using energy balance over the entire fin, as shown:

(2-17)a $\dot{Q}_b \ (= \dot{Q}_{fin}) = \dot{Q}_{conv}$

$\dot{E}_{in} = \dot{E}_{out}$

This eq



This eqn (2-17) says that the rate at which heat is transferred by convection from the fin must equal the rate at which it is conducted through the base of the fin. ($\dot{Q}_b$)

(2-17)b But, $d\dot{Q}_{conv} = h\,dA_s\,[T(x) - T_\infty]$

Integrating both sides of Eq (2-17)b

where, the integration on the RHS of (2-17)b is over the whole fin area = A_F (including the tip)
eq (2-17)c can also be written as:

(2-17)d $\int d\dot{Q}_{conv} = \int_{A_f} h\,\theta(x)\, dA_s$
see (2-6)A

(2-17)e $\rightarrow \quad \dot{Q}_{conv} = \int_{A_f} h\,\theta(x)\, dA_s$

but from eq. (2-17)a, $\dot{Q}_{conv} = \dot{Q}_{fin}$, thus eq (2-17)e becomes:

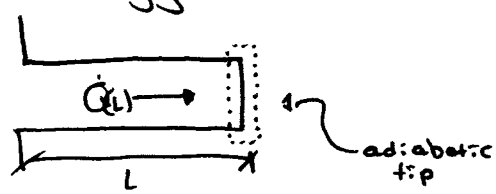
(2-17)f $\dot{Q}_{fin} = \int_{A_f} h\,\theta(x)\, dA_s$

Now sub eq (2-13)a for $\theta(x)$ in eq (2-17)f and integrating, would yield exactly eq (2-16)a

$\boxed{\text{Case B}}$ (The 2nd Tip Condition): Adiabatic tip

This condition corresponds to the assumption that the convection heat loss from the tip is negligible, in which case the tip may be treated as adiabatic. This leads to using energy balance over the tip surface as shown.

$\dot{Q}$ at $x=L$ = $\underbrace{\dot{Q}_L}_{\dot{E}_{in}} = \underbrace{0}_{\dot{E}_{out}}$



Using Fourier's Law for $\dot{Q}(x=L)$, gives

(2-18)a $\dot{Q}(\text{at } x=L) = -kA_c \frac{dT}{dx}\Big|_{x=L} = 0$

OR in terms of $\underbrace{\theta(=T-T_\infty)}$, eq. (2-18)a can be written as $\rightarrow (T = \theta + T_\infty)$

$\dot{Q}(x=L) = -kA_c \frac{d}{dx}(\theta + T_\infty)\Big|_{x=L} = 0$

$= -kA_c \left[\frac{d\theta}{dx} + \cancel{\frac{dT_\infty}{dx}}^{\,0}\right] = -kA_c \frac{d\theta}{dx}\Big|_{x=L} = 0$

Dividing by $-kA_c$, we get

(2-18)b $\frac{d\theta}{dx}\Big|_{x=L} = 0$

But From eq. (2-10) $\longrightarrow \theta(x) = C_1 e^{mx} + C_2 e^{-mx}$

So differentiating (2-10) and equating to zero as eq (3-18)b says, gives:

$$\left.\frac{d\theta}{dx}\right|_{x=L} = \left[C_1 e^{mx} \cdot m + C_2 e^{-mx} \cdot (-m)\right]_{x=L} = 0$$

$$\rightarrow \quad m\left[C_1 e^{mL} - C_2 e^{-mL}\right] = 0$$

(2-18)c OR $C_1 e^{mL} - C_2 e^{-mL} = 0$

Using eq(2-18)c, with (2-12)a : $[\theta_b = C_1 + C_2]$, to solve for C_1 & C_2 and sub the results in (2-10):

(2-19)ab
$$\boxed{\theta(x) = \theta_b \left[\frac{\cosh m(L-x)}{\cosh mL}\right] \quad \text{or} \quad \frac{\theta(x)}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}}$$

where θ & θ_b are given in eqs. (2-6) a&b

The amount of heat transferred from the entire Fin $\dot{Q}_{Fin}$ for this case (Case B), can be obtained using eq (2-19)a in (2-15). This gives:

(2-20)
$$\boxed{\begin{aligned}\left[\dot{Q}_{Fin}\right]_{CASE\,B} &= \sqrt{h\,P\,k\,A_c}\;\theta_b \tanh(mL) \\ &= M \tanh(mL) \\ &\text{(where, } M \text{ from eq. (2-16)b} = \sqrt{hPkA_c}\,\theta_b\text{)}\end{aligned}}$$

Case C (The 3rd tip condition): The temp is prescribed at the tip $T(x=L) = T_L$

In the same way, we can obtain the fin temp. distribution. ($T(x)$ or $\theta(x)$) and $\dot{Q}_{fin}$ for this case C, where T_L is prescribed at the fin tip.

Defining, $\theta(\text{at } x=L) = \theta(L) = \theta_L$ as

(2-21) $$\boxed{\theta_L = T(x=L) - T_\infty \quad \text{or} \quad \theta_L = T_L - T_\infty}$$

this is the 2ND BC (case C)

Solving for C_1 & C_2 and performing some algebra, the final result is:

(2-22) $$\theta(x) = \theta_b \left[\frac{(\theta_L/\theta_b)\sinh(mx) + \sinh m(L-x)}{\sinh(mL)} \right]$$

CASE C

or sub for $\theta(x) = T(x) - T_\infty$ (Eq (2-6)a, previously) yields:

(2-23) $$T(x) - T_\infty = (T_b - T_\infty)\left[\frac{(T_L - T_\infty)(T_b - T_\infty)\sinh(mx) + \sinh m(L-x)}{\sinh(mL)} \right]$$

CASE C

The heat transfer rate from the entire fin for case C is obtained using the Fourier's Law of Conduction at the base ($x=0$), this gives:

see eq(2-15)

$$\dot{Q}_{fin} = \dot{Q}_{\substack{cond \\ at\ x=0}} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} \quad \left(\text{or} \quad \dot{Q}_{\substack{cond \\ x=0}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0}\right)$$

$$\left[\dot{Q}_{fin}\right]_{case\ c} = \sqrt{h\ell k A_c}\ \theta_b \left[\frac{\cosh(mL) - \left(\frac{\theta_L}{\theta_b}\right)}{\sinh(mL)}\right] \qquad (2\text{-}24)a$$

or

$$\left[\dot{Q}_{fin}\right]_{case\ c} = M\left[\frac{\cosh(mL) - (\theta_L/\theta_b)}{\sinh(mL)}\right]$$

$(M = \sqrt{h\ell k A_c}\ \theta_b$ as before$)$

In terms of T, this yields

$$(2\text{-}24)b \qquad \left[\dot{Q}_{fin}\right]_{case\ c} = \sqrt{h\ell k A_c}\ (T_b - T_\infty)\left[\frac{\cosh(mL) - \left(\frac{T_L - T_\infty}{T_b - T_\infty}\right)}{\sinh(mL)}\right]$$

CASE D Infinitely (very long) fin, (as $L \to \infty$)

For a sufficiently long fin of uniform cross section (A_c = const.), the tip temp. approaches T_∞ i.e. as $L \to \infty \Rightarrow T_L \to T_\infty$, thus

(2-25) $\theta(\text{at } x = L) = \theta_L = T_\infty - T_\infty = 0$

This would be the second boundary condition at the fin tip. The solution of the temp. can easily be verified that:

(2-26)a $\theta(x) = \theta_b e^{-mx}$

(2-26)b or $T(x) - T_\infty = (T_b - T_\infty) e^{-mx}$

CASE D

where $\underline{\underline{m}}$ is given by eq (2-8)

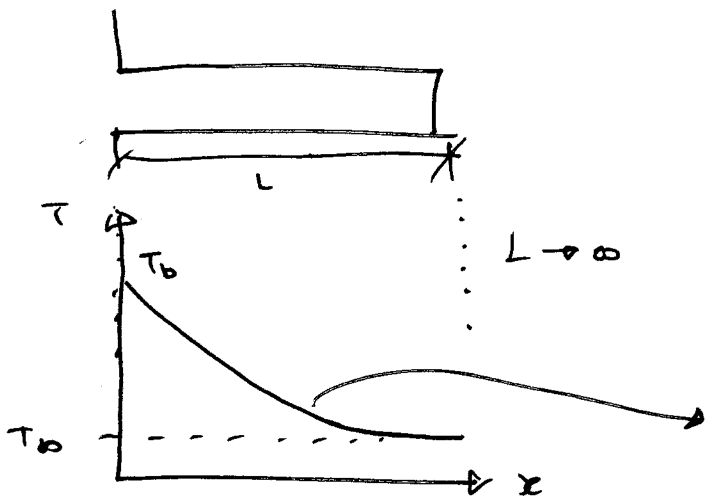
Similarly, $\dot{Q}_{fin}$ for case D can be obtained using $\dot{Q}_{cond}$ at $x = \infty$ (eq. 2-15)

This gives:

$$\dot{Q}_{fin} = \sqrt{h\ell k A_c}\ \theta_b = M \qquad (2\text{-}27)a$$

$$\dot{Q}_{fin} = \sqrt{h\ell k A_c}\ (T_b - T_\infty)$$

CASE D



$$T(x) = T_\infty + (T_b - T_\infty)e^{-mx}$$

Fig (2-6)e : Temp. distrib. $T(x)$ for case D