Perpendicular

Here, the conduction heat rates $\perp$ to each of the central surfaces at the x-, y-, and z- directions are indicated by the quantities q_x, q_y, and q_z, respectively. The quantities q_{x+dx}, q_{y+dy}, and q_{z+dz} at the opposite surfaces can be expressed as a Taylor series expansion with neglecting higher-order terms, as follows:

(10-2)
$$\begin{cases} q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx \\ q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy \\ q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz \end{cases}$$

In this case there is a heat source term associated with the rate of thermal energy generation, the term is expressed as:

(11-2) $$\dot{E}_g = \dot{q}_{gen}\, d\forall$$

$\dot{q} \equiv$ Rate of heat generation per unit volume of the medium (W/m^3)

(12-2) or $$\dot{E}_g = \dot{q}(dx \cdot dy \cdot dz)$$

The energy storage term $\dot{E}_{st}$ can be expressed as:

(13-2) $$\dot{E}_{st} = \frac{\partial \dot{E}_{c.v.}}{\partial t}$$

(14-2) When $E_{c.v.} = dm \cdot C_p T$

Assuming there is no change of phase in the medium (latent energy is ignored)

→ Sub eqn (14-2) in (13-2), gives

(15-2)
$$\begin{cases} \dot{E}_{st} = \frac{\partial}{\partial t}\left[dm \cdot C_p T\right] \\ \dot{E}_{st} = dm\, C_p \frac{\partial T}{\partial t} \end{cases}$$

But $dm = \text{const.} = \rho\, d\forall = \rho(dx \cdot dy \cdot dz)$

Sub in (15-2), gives

(16-2) $$\dot{E}_{st.} = \underbrace{\rho C_p \left(\frac{\partial T}{\partial t}\right)}_{\text{Rate of change of the sensible (thermal) energy of the medium per unit volume.}} dx \cdot dy \cdot dz \quad (*)$$

$\dot{E}_g$ = Representation of some Energy conversion processes, i.e. chemical energy → heat
nuclear energy → heat (thermal energy)

(17-2) Recall, $\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$

and applying this energy balance for Fig (2-2) gives,

(18-2) $\begin{cases} \dot{E}_{in} = q_x + q_y + q_z \\ \dot{E}_{out} = q_{x+dx} + q_{y+dy} + q_{z+dz} \end{cases}$

Sub. eqs. (12-2), (18-2), (16-2) in (17-2), gives

(19-2) $(q_x + q_y + q_z) - (q_{x+dx} + q_{y+dy} + q_{z+dz}) + \dot{q}(dx \cdot dy \cdot dz) = \rho C_p \frac{\partial T}{\partial t}(dx \cdot dy \cdot dz)$

Since, $\vec{q}_n'' = q''\vec{n} = -k\frac{\partial T}{\partial n}\vec{n}$

or $\vec{q}_n = q''_{\vec{n}}\, dA_{\perp n}\vec{n}$ differential area $\perp$ to q_n

(20-2) $q_n = -k\, dA_\perp \frac{\partial T}{\partial n}\vec{n}$

OR, considering the magnitudes only, we have

(21-2) $\boxed{q_n = -k\, dA_T \frac{\partial T}{\partial n}}$ (generalized Fourier's law)

So, in x-direction → conduction heat rates

(22-2) $\begin{cases} q_x = -k(dy \cdot dz)\frac{\partial T}{\partial x} \\ q_y = -k(dx \cdot dz)\frac{\partial T}{\partial y} \\ q_z = -k(dy \cdot dx)\frac{\partial T}{\partial z} \end{cases}$

Simplifying eq'n (19-2), and recognizing eq'n (10-2) gives:
Eq'n (10-2)

(23-2) $-\frac{\partial q_x}{\partial x}dx - \frac{\partial q_y}{\partial y}dy - \frac{\partial q_z}{\partial z}dz + \dot{q}(dx \cdot dy \cdot dz) = \rho C_p \frac{\partial T}{\partial t}(dx \cdot dy \cdot dz)$

Sub (22-2) in (23-2) and rearranging gives

(24-2) $$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho C_p \frac{\partial T}{\partial t}}$$

- The general form of the Heat Diffusion (conduction) Equation in Cartesian coordinates.

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Remarks (special cases):

- For an isotropic medium k is independent of direction, k = const.
  - isotropic: same in each direction

The Heat Equation, becomes

(25-2) $$k\left[\partial/\partial x(\partial T/\partial x) + \partial/\partial y(\partial T/\partial y) + \partial/\partial z(\partial T/\partial z)\right] + \dot{q} = \rho C_p\left(\frac{\partial T}{\partial t}\right)$$

or $$(\partial^2 T/\partial x^2) + (\partial^2 T/\partial y^2) + (\partial^2 T/\partial z^2) + (\dot{q}/k) = (\rho C_p/k)(\partial T/\partial t)$$

(26-2) Recognizing that $\alpha = \frac{k}{\rho C_p}$ = Thermal Diffusivity (m^2/s)

Eq. (25-2) becomes

(27-2) $$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

($\dot{q}_{gen}$)

- For a steady state heat conduction $\Rightarrow \frac{\partial T}{\partial t} = 0$

Eq'n (24-2) or (27-2) leads to

(28-2) $$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = 0}$$

- For a steady-state, 1-D (in x-dir), with no heat generation, we have:

(29-2) $$\boxed{\frac{d^2 T}{dx^2} = 0} \qquad k = \text{const.}$$

(30-2)a or $$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0} \qquad k \neq \text{const.}$$

with a single variable, you can write the derivative like normal (not partial)

1-D (in x-dir) Heat Equation, steady-state, $\dot{q} = 0$

- 3-D, steady-state, $\dot{q} = 0$, k = const.

Eq. (27-2), gives

(30-2)b $$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0}$$

This eq'n is known as Laplace's Eq'n

- It is convenient to introduce the del-squared operator (or known as the Laplacian operator):

(30-2)c $$\nabla^2 = \partial/\partial x^2 + \partial/\partial y^2 + \partial/\partial z^2$$

∴ Laplace eq'n can be written as:

(30-2)d $$\boxed{\nabla^2 T = 0}$$

laplace eq'n in short form

- Eq (27-2) can then be written as:

$$\boxed{\nabla^2 T + \dot{q}/k = (1/\alpha)(\partial T/\partial t)}$$

3-D, unsteady heat eq'n, k = const.

* where ∇^2 is given by Eq (30-2)c above

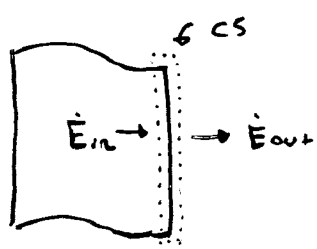
Energy Balance over a surface

In certain applications, it is convenient to carry-out a surface energy balance.

A surface contains no volume or mass, and thus no energy. Hence, a surface can be viewed as a fictitious system whose energy content remains constant during a process. Then the energy balance for a surface can be expressed as:

(1-21) $$\dot{E}_{in} = \dot{E}_{out}$$

In this case (ie a surface energy balance), it is valid for both steady and transient conditions (why?), and the surface energy balance does not involve heat generation. (why?)



$\dot{E}_{in} - \dot{E}_{out}$

Fig (1-5)a : Energy Balance over a surface of a plane

Applications For Surface Energy Balance

Example: Heat Transfer across Human skin (Thermoregulation Process)

Heat generation and heat loss rates are controlled by Human skins in order to maintain a nearly constant core temperature of $T_c = 37\,°C$ under a wide range of environmental conditions. This phenomenom is called thermoregulation.

- Now, consider heat transfer analysis involving a human body and its surroundings. in this case the human skin/Fat is considered where it's outer surface is exposed to the environment and its inner surface at a temp. slightly less than T_c, with $T_i = 35°C$ $(= 308\ K)$
- A simplified model of the skin/Fat is given as follows:

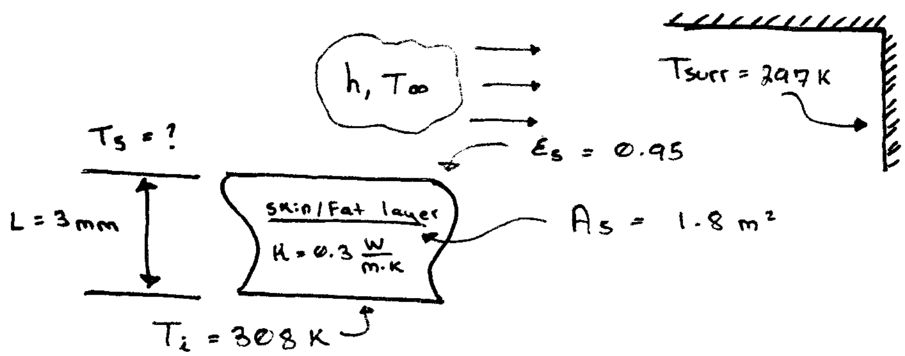


Fig (1-2)a : Skin/Fat layer conditions

Required: Determine the skin surface temperature and the rate of heat loss to the environment (air with $T_\infty = 297\ K$ and $h = 2.0\,(W/m^2 \cdot K)$)

Assumptions:

- Steady-state conditions
- Thermal conductivity of the skin is uniform
- 1-D heat transfer by conduction
- Radiation heat transfer is considered between the skin surface (small surface) and the surroundings (large enclosure)
- Solar radiation is absent (or negligible)