Systems with Negligible Internal Resistance

(Approx. Solutions)

If temp. Gradients within the solid (system) may be neglected, a comparatively simple approach, termed the lumped capacitance method (LCM) can be used to determine the variation of temp. with time

Example of Application : cooling of a hot metal forging

Problem Formulation : Consider the cooling of a small metal billet in a quenching bath after its removal from a hot furnace. Suppose that the billet is removed from the furnace at a uniform initial temperature T_i ($T(t=0) = T_i$) and is quenched or exposed suddenly to an environmental temp. T_∞, as shown in Fig (4-1).

$\boxed{T_i > T_\infty}$

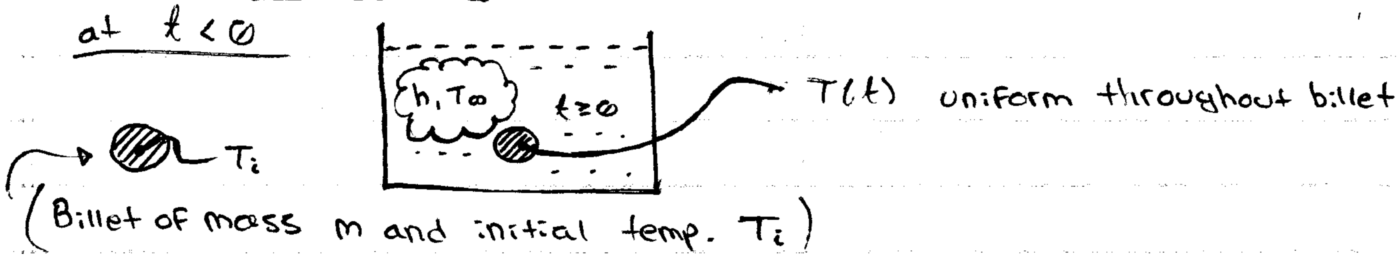
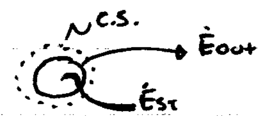


Figure (4-1): Cooling of Billet

(*) Example (4-1), Example (4-2)

Applying Energy Balance over the billet gives:

$$\cancel{\dot{E}_{in}}^{0} - \dot{E}_{out} + \cancel{\dot{E}_{gen}}^{0} = \dot{E}_{ST}$$



(Closed System Analysis)

(4-2) $\rightarrow \dot{E}_{ST} = -\dot{E}_{out}$

<u>NOTE: $\dot{E}_{gen} = \emptyset$</u>

(4-3) $\dot{E}_{out} = \dot{Q}_{conv} = hA_s\left[T(t) - T_\infty\right]$

(4-4)a $\dot{E}_{ST} = dU/dt$

(4-4)b Recall, $dU = mC_v\,dT$

(4-4)c <u>NOTE:</u> For solids, $C_v \simeq C_p = C$

(4-4)d $\therefore\ dU = mC\,dT$

But, $m = \rho V$

Sub in (4-4)d, yields

(4-4)e $dU = \rho V C\,dT$

(4-4)f or $dU/dt = \rho V C\, dT/dt$

Sub eqs (4-4)f & (4-3) back in Eq (4-2) gives

$$\rho V C\, dT/dt = -hA_s\left[T(t) - T_\infty\right]$$ rearranging gives

(4-5)a
$$\boxed{\frac{dT}{dt} + \frac{hA_s}{\rho V C}(T - T_\infty) = \emptyset}$$

- 1^{ST} order ordinary diff. eq'n
  - ↳ need initial condition

(4-5)b
- Initial condition, $T(t=0) = T_i$

The solution to eq (4-5)a can be obtained by <u>Separating</u> the variables T and t, as follows:

(4-6)a $$dT/(T - T_\infty) = -\left(hA_s/\rho V C\right) dt$$

Noting that dT can be written as:

(4-6)b $dT = d(T - T_\infty)$ (since $T_\infty = \text{const.}$, $dT_\infty = 0$)

Sub (4-6)b in (4-6)a, gives

(4-6)c $$\frac{d(T - T_\infty)}{(T - T_\infty)} = -\left(\frac{hA_s}{\rho \forall c}\right) dt$$

Now integrating this eq. from $t = 0$, at which $T = T_i$, to anytime t, at which $T = T(t)$, gives

(4-7)a $$\ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = -\left(\frac{hA_s}{\rho \forall c}\right) t$$

(4-7)b or, $$\boxed{\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\left(\frac{hA_s}{\rho \forall c}\right) t}}$$

(Check when $t = 0$, $\rightarrow T = T_i$)

Eq. (4-7)b can be written as:

(4-7)c $$T(t) = T_\infty + (T_i - T_\infty)\, e^{-\left(\frac{hA_s}{\rho \forall c}\right) t}$$

<u>Determination of the Heat Transfer Rate Convected From the Surface at any instant t:</u>

Recall, Eq. (4-3)

(4-8)a $\dot{Q}_{conv} = hA_s(T - T_\infty)$

Sub. Eq (4-7)b in (4-8)a, gives

(4-9)a $$\boxed{\dot{Q}_{conv.} = hA(T - T_\infty)\, e^{-\left(\frac{hA_s}{\rho \forall c}\right) t}}$$

$$Q_{conv} = \int_0^t \dot{Q}_{conv}\, dt = \int_0^t \left[hA_s(T_i - T_\infty)\, e^{-\left(\frac{hA_s}{\rho \forall c}\right) t}\right] dt$$

$$= hA_s(T_i - T_\infty) \int_0^t e^{-\left(\frac{hA_s}{\rho \forall c}\right) t} \cdot dt$$

$$= hA_s(T_i - T_\infty) \left(\frac{-\rho \forall c}{hA_s}\right) \cdot \int_0^t e^{-\left(\frac{hA_s}{\rho \forall c}\right) t} \cdot \left(\frac{-hA_s}{\rho \forall c}\right) dt$$

$$= -(\rho \forall c)(T_i - T_\infty) \left[e^{-(hA_s/\rho \forall c) t}\right]_0^t$$

$$= -(\rho \forall c)(T_i - T_\infty) \left[e^{-\left(\frac{hA_s}{\rho \forall c}\right)} - 1\right]$$

(4-10) $$\boxed{\dot{Q}_{conv} = \rho \forall c\, (T_i - T_\infty)\left[1 - e^{-\left(\frac{hA_s}{\rho \forall c}\right) t}\right]}$$

Remarks:

1. The max total amount of internal energy that can be convected to the surrounding fluid is obtained by allowing $t \to \infty$ in Eq (4-10), i.e.

$$Q_{max} = \rho V c (T_i - T_\infty)\left[1 - \cancel{e^{-\infty}}\right] \quad (J \text{ or } kJ)$$

(4-11) $$\therefore \boxed{Q_{max} = \rho V c (T_i - T_\infty)}$$

2. Q_{max} is equal to the relative internal energy possessed by the body at $t = 0$

3. Eq. (4-10) can also be written as

$$Q_{conv} = \rho V c (T_i - T_\infty) - \underbrace{\rho V c (T_i - T_\infty)\, e^{-\frac{hA_s}{\rho V c} t}}_{(= T(t) - T_\infty,\ \text{eq. } (4\text{-}7))}$$

$$= \rho V c (T_i - T_\infty) - \rho V c (T(t) - T_\infty))$$

$$= \rho V c \left[T_i - T_\infty - T(t) + T_\infty\right]$$

$$\left.\begin{aligned} \therefore Q_{conv} &= \rho V c (T_i - T) \\ \text{or} \quad Q_{conv} &= m c (T_i - T) \end{aligned}\right\} \text{(4-12) a \& b}$$

4. The dimensionless ratio $\frac{Q_{conv}}{Q_{max}}$ is given by dividing Eq. (4-10) by Eq (4-11), i.e.

(4-13) $$Q_{conv}/Q_{max} = 1 - e^{-\left(\frac{hA_s}{\rho V c}\right)} = 1 - \mathrm{Exp}\left(-\frac{hA_s}{\rho V c} t\right)$$

5. The quantity $\left(\frac{\rho V c}{hA_s}\right)$ appearing in Eq (4-7)b and associated equations is known as the thermal time constant of the system, i.e.

(4-14)a $$t_c \equiv \left(\frac{\rho V c}{hA_s}\right)$$

(4-14)b (NOTE: in the textbook $t_c = \frac{1}{b}$, so that $b \equiv \frac{hA_s}{\rho V c}$

(*) t_c provides an indication of the length of time required for a system to approach thermal equilibrium ; i.e. the smaller t_c, the faster the system response.

Sub for $t = t_c$ in Eq. (4-7)b, yields

(4-15) $$\frac{T - T_\infty}{T_i - T_\infty} = e^{-1} = 0.368$$

Criteria (conditions for using LCM):
Defining Biot number by

(4-16)a $$\boxed{Bi \equiv \frac{R_{internal}}{R_{external}} = \frac{R_{cond.}}{R_{conv.}} = \frac{hL_c}{k}}$$

where h = avg. heat trans. coeff.
L_c = some characteristic length, given by
(4-16)b $L_c \equiv \forall / A_s$
k = thermal conductivity of the solid body (system)

so, for systems where $R_{cond} \lesssim R_{conv.}$ (typically $R_{cond} < 10\% R_{conv.}$)
$\boxed{Bi \leq 0.1}$ and for this limiting case the lumped system analysis (LSA) (or lumped capacitance method (LCM)) can be used with approx. and the variation of temp. is with time, whereas the variation of temp. with the spatial coordinate becomes negligible, as discussed earlier.

Remarks:

1. The parameter $\left(hA_s / \rho \forall c\right)$ (or $= b = 1/t_c$) is related to the biot number by

$$\frac{hA_s}{\rho \forall c} = \frac{h(\forall / A_s)}{k} \cdot \frac{(k/\rho c)}{(\forall / A_s)^2} = Bi\left(\frac{\alpha}{L_c^2}\right)$$

(4-17)a or $$\frac{hA_s}{\rho \forall c} t = Bi \frac{\alpha t}{L_c^2}$$

(4-17)b where α = thermal diffusivity $= k/\rho c$
and the parameter $\alpha t / L_c^2$ is dimensionless, known as Fourier number, F_o, i.e.

(4-17)c $$\boxed{F_o \equiv \frac{\alpha t}{L_c^2}}$$

2. The parameter $\rho \forall c$ $(= mc)$ is sometimes written
(4-17)d as $C = \rho \forall c$
↳ called lumped thermal capacitance of the solid

Thermal Circuit Method

An electrical network analogous to the thermal network for a lumped-single-capacity system is shown Fig. (4-2)a, b, c

$T_i > T_\infty$

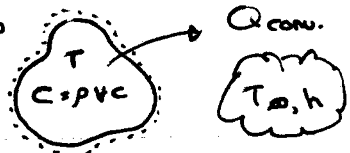


Fig (4-2)a

Recall,

$$\dot{Q}_{conv} = -\dot{E}_{st.} = -\rho V c \frac{dT}{dt}$$

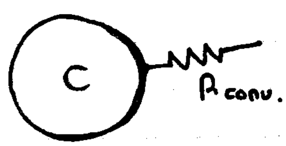
$$\dot{Q}_{conv} = \frac{T - T_\infty}{R_{conv.}} = -C \frac{dT}{dt}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\left(\frac{1}{R_{conv.}C}\right)t}$$

$t = 0$ when the hot billet is immersed in fluid and heat begins to flow

About Thermal circuits:

$t < 0$ $\qquad$ $t \geq 0$



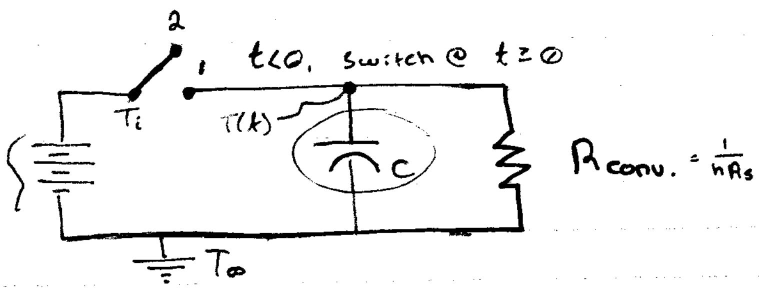


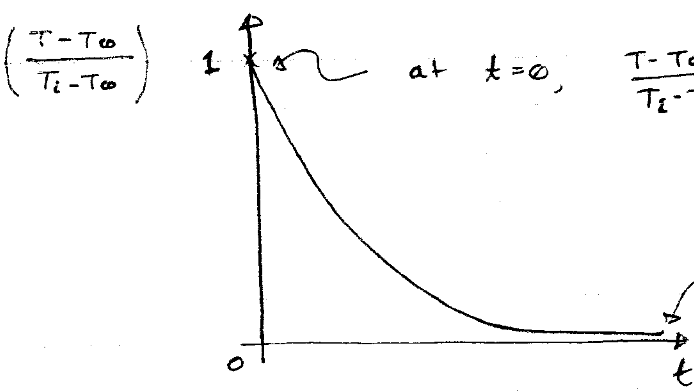
Fig. (4-2)b: Transient Lumped Capacity System Thermal Network

Switch Position

1 for $t < 0$

2 for $t \geq 0$

$t = 0$ when switch is opened and the capacitor begins to discharge



at $t = 0$, $\frac{T - T_\infty}{T_i - T_\infty} = 1.0 \rightarrow T = T_i$

$t = \infty$, $\frac{T - T_\infty}{T_i - T_\infty} \simeq 0$

$\rightarrow T \cong T_\infty$

Fig (4-2)c: Temp. Profile of the lumped capacity system

Remarks :

(4-17)e ① $t_c \equiv \frac{\rho V c}{h A_s} = \overbrace{\frac{1}{h A_s}}^{R_{conv.}} \overbrace{(\rho V c)}^{= C} \Rightarrow R_{conv.} \times C$

② Sub. Eq (4-17)c in (4-17)a

(4-17)f $\rightarrow \frac{h A_s}{\rho V c} t = Bi \cdot Fo$

③ Sub. Eq (4-17)f in (4-7)a

(4-18) $$\boxed{\frac{T - T_\infty}{T_i - T_\infty} = e^{-(Bi \cdot Fo)}}$$

→ Example 4-1

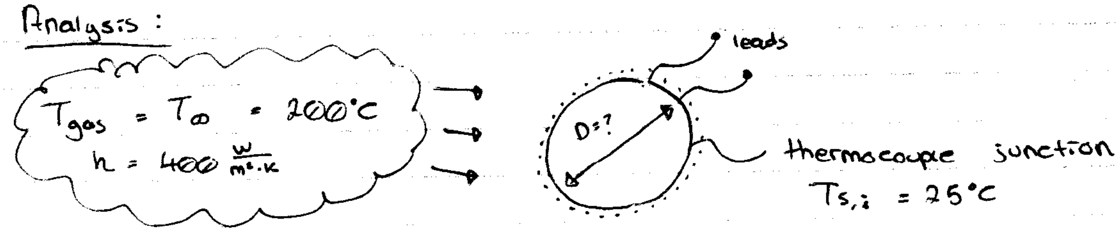
Consider a thermocouple junction spherical in shape (approximation) used for temp. measurement in a gas stream

- $h = 400 \frac{W}{m^2 \cdot K}$ (between junction surface and gas)
- The junction thermophysical properties are:
  $k = 20 \frac{W}{m \cdot K}$; $c = 400 \ J/kg \cdot K$; $\rho = 8500 \ kg/m^3$
- $T_{s,i} = 25°C$; $T_g (= T_\infty) = 200°C$

Required : ① Determine the junction diameter needed for the thermocouple to have $\boxed{t_c = 1 \text{ sec}}$

② Determine how long it will take for the junction to reach $\boxed{199°C}$

Analysis :



Properties :

$k = 20 \ W/m \cdot K$

$c = 400 \ J/kg \cdot K$

$\rho = 8500 \ kg/m^3$

Assumptions :

- Temp. of the junction is uniform at any time instant t
- Radiation exchange is negligible
- conduction losses through the leads are negligible
- constant thermophysical properties

Recall, Eq (4-14)a , $t_c = \frac{1}{hA_s}(\rho \forall c)$ - - - ①

where $(A_s)_{sphere} = \pi D^2$ & $\forall = \pi/6\, D^3$ (or $4/3\, \pi R^3$)

Sub we get $t_c = \frac{1}{h\pi D^2}\left(\frac{\rho \pi D^3}{6}\right)c = \frac{\rho D}{6h} c$

Rearrange : $D = \frac{6 h t_c}{\rho c}$ - - - ②

Sub the values ② yields:

$D = \frac{6 \times 400 \times 1}{8500 \times 400} \cong 7.06 \times 10^{-4}\ m$

Calculating B_i using Eq (4-16)a, gives

$B_i = hL_c/k$ - - - ③ , where L_c is given by

Eq (4-16)b , $L_c = \forall/A_s = (\pi/6\, D^3)/\pi D^2 = D/6$ - - - ④

$B_i = \frac{hD}{6k} = \frac{400 \times 7.06 \times 10^{-4}}{6 \times 20} \cong 2.35 \times 10^{-3} < 0.1$

∴ the criterion of $B \leq 0.1$ is satisfied and the LCM can be used with excellent approx.

② $t = ?$ for $T(t) = 199\ °C$

NOTE $\frac{A_s}{\forall} = \frac{1}{L_c}$

Recall, Eq (4-7)a

$\ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = -\left(\frac{hA_s}{\rho \forall c}\right)t = -\left(\frac{h}{\rho L_c c}\right)t$

Solving for t, gives :

$t = \frac{-L_c \rho c}{h} \ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = \frac{-D\rho c}{6h} \ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right)$

$= \frac{-7 \times 0.6 \times 10^{-4} \times 8500 \times 400}{6 \times 400} \ln\left(\frac{199 - 200}{25 - 200}\right)$

∴ $t \simeq 5.2\ s$ $(\simeq 5\, t_c)$