Remark :

To a satisfactory approximation, eqs (2-20) & (2-27)a provide equivalent results when :

(2-27)c $\tanh(mL) \geq 0.99$ or $mL \geq 2.65$

(by comparing (2-20) & (2-27)a

Hence, a fin of uniform cross-section may be assumed very long (infinitely long) if :

(2-27)d $L \geq \dfrac{2.65}{m} = L_\infty$

where $m = \sqrt{\dfrac{hP}{kA_c}}$ (refer to eqn (2-8))

Remarks regarding Case A : The convection fin tip

1. The solution of the fin temperature distribution for Case A, given by Eqs (2-13)a & b looks rather complex. An approximate, yet accurate and practical estimations may be obtained by using the adiabatic tip solution (Case B) Eqs. (2-19) & (2-20) but replacing the actual fin length L by a corrected fin length defined by :

(2-28)a $\underset{\text{corrected}}{L_c} = L + \dfrac{A_c}{P}$ cross-section

Note: By multiplying this eq. by the perimeter, gives :

$P \cdot L_c = P \cdot L + A_c$

(2-28)b OR : $A_{corr.} = A_{fin(lateral)} + A_{tip}$

This shows that the fin area determined using the corrected length L_c, is equivalent to the sum of the lateral fin area plus the fin tip area.

Fig (2-7)a

actual fin with convection at the tip (Case A)

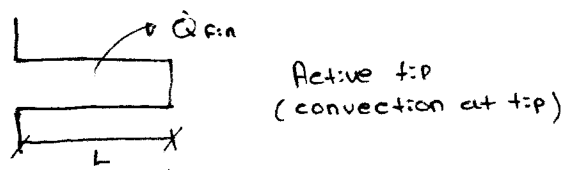
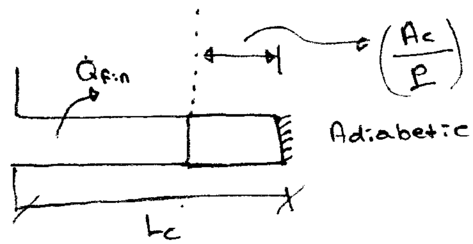


Fig (2-7)b

Equivalent fin modified with new length L_c applied to the solution for case B



(2-28)c ② The corrected length approximation gives very good results when $\boxed{mL \geq 1}$

So, the correction is based on assuming equivalence between heat transfer from the actual fin with tip convection (Case A) and heat transfer from a longer (hypothetical) fin with an adiabatic fin tip (Case B). Thus, the fin heat transfer for a fin with convection at the tip (for Case A) can be approximated as

(2-29) $$\boxed{\left[\dot{Q}_{fin}\right]_{CASE\,A} \simeq M \tanh(mL_c)}$$

(compare to (2-20))

And the temp. distribution is given by:

(2-30)a $$T_x - T_\infty = (T_b - T_\infty)\left[\frac{\cosh(m(L_c - x))}{\cosh(mL_c)}\right]$$

OR

(2-30)b $$\theta(x) = \theta_b\left[\frac{\cosh[m(L_c - x)]}{\cosh(mL_c)}\right]$$

Example (2-1)

Given: Consider a cylindrical fin (pin fin with circular cross-section) made from copper. The fin is very long with a diameter of 5 mm. The base of the fin is maintained at 100 °C. The fin is exposed to convection air at 25 °C and convection heat transfer coefficient 100 $\frac{W}{m^2 \cdot C}$

Required: Determine the following:

(1) Temp. distribution along the fin (plot it)

(2) The heat losses convected from the fin

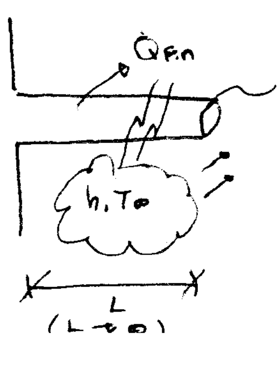
(3) How long the fin must be for the assumption of infinite length to give an accurate estimate of Part (2) above

Solution: This is the case of $L \to \infty$ case D

① the temp distribution $T(x)$ along the fin is given by eq (2-26)b:

$$T(x) = T_\infty + (T_b - T_\infty)e^{-mx} \ldots ①$$

where, $$m \equiv \sqrt{\frac{hP}{kA_c}} \quad \ldots ②$$



D = 5 mm

$h = 100 \frac{W}{m^2 \cdot K}$

$T_\infty = 25\,°C$

Fig (2-1)

$P = \pi D = \pi(0.005)$
$\cong 0.01571$ m

Assumptions:
- 1-D HT (x-dir)
- uniform prop.
- Radiation HT negligible
- copper is pure

The thermal cond. for copper is from Table (A-3) → Pg. 910

$$T_{avg} = \frac{T_b + T_\infty}{2} = 62.5\,°C \approx 335\,K$$

$k \cong 398 \ \frac{W}{m \cdot K}$ (by interpolation)

$A_c = (\pi/4)D^2 = (\pi/4)(0.005)^2 = 1.9634 \times 10^{-5}\ m^2$

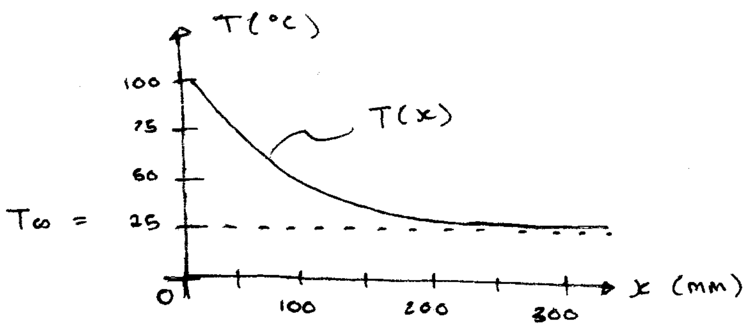
Sub the foregoing results in Eq (2), yields

$$m = \sqrt{\frac{100 \times 0.01571}{398 \times 1.9634 \times 10^{-5}}} = 14.178 \ \frac{1}{m}$$ (unit for m is $1/m$)

Sub back in Eq (1), gives

$$\boxed{T(x) = 25 + 75 e^{-14.178 x}} \ (T \text{ in } °C) \quad (x \text{ in m - why?}) \quad --- \ (3)$$

(2) the heat losses from the fin (heat convected from the fin) can be estimated using the fin equation (case D) given by Eq (2-27)b



Recall,
$$\dot{Q}_{fin} = \sqrt{h P k A_c}\,(T_b - T_\infty)$$
$$= \sqrt{100 \times 0.01571 \times 398 \times 1.9634 \times 10^{-5}} \times (100 - 25)$$
$$\cong 8.310 \ W$$

(3) The fin in this problem may be assumed to be infinitely long (very long) if $L \geq L_\infty = 2.65/m$ (see eq(2-27)d)

Sub in L_∞, gives: $L_\infty = 0.187$ (187 mm)

Performance Indicators for Evaluation of Fins :

(1) The Fin Effectiveness ε_{Fin}

It is defined as

(2-31)a where, $$\boxed{\varepsilon_{Fin} = \frac{\dot{Q}_{Fin}}{\dot{Q}_{no\,Fin}}}$$

$\dot{Q}_{Fin}$ = The heat transfer rate from the entire Fin

$\dot{Q}_{no\,Fin}$ = The heat transfer rate that would exist <u>without</u> the Fin, given by :

(2-31)b
$$\begin{cases} \dot{Q}_{no\,Fin} = hA_{c,b}(T_b - T_\infty) \\ \quad\quad\;\; = hA_{c,b}\,\theta_b \end{cases}$$

$A_{c,b}$ = the Fin cross-sectional area at the base (i.e. @ $x = 0$)

→ Justified unless $\varepsilon_{Fin} \geq 2$

(2) The Fin Efficiency η_{Fin}

Another measure of the Fin performance is provided by the Fin efficiency η_{Fin}. It is defined as :

(2-33)a
$$\boxed{\eta_{Fin} = \frac{\dot{Q}_{Fin}}{\dot{Q}_{Fin,max}}}$$

$\dot{Q}_{Fin,max}$ = The max rate at which a fin could dissipate heat that would exist if the entire Fin surface area (A_{Fin} or A_F) were at the T_b. This is because of the max driving potential fin convection is when

$\Delta T_{max} = T_b - T_\infty$ at $x = 0$ $(= \theta_b)$

However, since any Fin is characterized by a finite conduction resistance (as mentioned before), a temp. gradient must exist along the fin.

(2-33)b
$$\boxed{\dot{Q}_{Fin,max} = hA_{Fin}\Delta T_{max} = hA_{Fin}(T_b - T_\infty) = hA_{Fin}\theta_b}$$

For example, For a fin with an adiabatic tip (case B)

(i) $$\eta_{Fin} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)\tanh(mL)}{hA_{Fin}(T_b - T_\infty)}$$

(2-34)a
$$\boxed{\eta_{Fin} = \frac{\tanh(mL)}{mL}}$$ prove it!

m is given by (2-8)

2 For case D - an infinitely long fin, we have

$$\eta_{fin} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{fin}(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}}$$

(2-34)b $$\boxed{\eta_{fin} = \frac{1}{mL}}$$

3 For Case A - A fin with tip convection, we have

$$\eta_{fin} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh(mL_c)}{hA_{fin}(T_b - T_\infty)}$$

(2-34)c $$\boxed{\eta_{fin} = \frac{\tanh(mL_c)}{mL_c}}$$

3 A Fin thermal resistance $(R_{t,fin})$

Fin thermal performance can also be evaluated in terms of a fin thermal resistance $R_{t,fin}$. Recall, (From Heat I course), the analogy made between heat transfer rate occuring due to a temp. diff. potential and a DC electric current flowing due to Voltage potential, i.e.

(2-36)a $$\dot{Q} \equiv \frac{\Delta T}{R_t} \qquad \left(\text{or} \quad R_T = \frac{\Delta T}{\dot{Q}} \right)$$

A fin thermal resistance may be defined as

(2-36)b $$\boxed{R_{t,fin} \equiv \frac{\Delta T}{\dot{Q}_{fin}} \equiv \frac{T_b - T_\infty}{\dot{Q}_{fin}} = \frac{\theta_b}{\dot{Q}_{fin}}}$$ → See Fig (2-9)

<u>Note</u>: ①: $\dot{Q}_{fin}$ can be determined depending on tip condition (Case A, B, C, D)

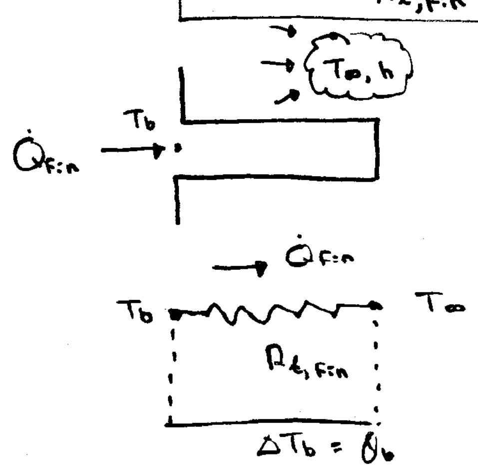
②: Eq(2-36)b is very useful, particularly when representing a finned surface by a thermal circuit network.

Defining, the thermal resistance due to convection at the exposed base $R_{t,b}$, as

(2-36)c $$R_{t,b} = \frac{(T_b - T_\infty)}{hA_{c,b}(T_b - T_\infty)} = \frac{1}{hA_{c,b}}$$

Sub (2-36)c in (2-31) a & b, yields

(2-37) $$\boxed{\varepsilon_{fin} = \frac{R_{t,b}}{R_{t,fin}}}$$



$$R_{t,fin} = \frac{\theta_b}{\dot{Q}_{fin}}$$

$$\theta_b = T_b - T_\infty$$

<u>Fig (2-9)</u>: Thermal resistance circuit for a single fin.

Performance of an Array of Fins

An array of fins and the base surface to which they are attached may be characterized, from performance point of view, using the overall surface efficiency "η_o" of a multiple fin array can be expressed by:

(2-38)a $$\eta_o \equiv \frac{\dot{Q}_{total}}{\dot{Q}_{max}} = \frac{\dot{Q}_t}{\dot{Q}_{max}}$$

(2-38)b $$\eta_o = \frac{\dot{Q}_t}{hA_t\theta_b}$$

Where,

$\dot{Q}_t \equiv$ the total heat transfer from the surface area A_t associated with <u>both</u> (1) the fins and (2) the exposed portion of the base (also known as the prime surface) or inter fins area.

If there are N fins in the array, each of surface area A_{fin} (or A_f), and the area of the prime surface (interfins area) is given by A_b (or A_{unfin})

[Recall, Fig (2-3)], A_t is given by:

(2-39) $$A_t = \underbrace{NA_{fin}}_{\text{finned}} + \underbrace{A_b}_{\text{unfinned (primary)}}$$

$\dot{Q}_{max}$ = The max. possible heat transfer rate that would result if the entire fin surface, as well as the exposed base, were maintained at T_b

The total rate of heat transfer by convection from the fins and the prime (unfinned surface) can be expressed as:

(2-40)a $$\dot{Q}_t = \underbrace{N\eta_{fin}hA_{fin}\theta_b}_{=\dot{Q}_{fin}} + \underbrace{hA_b\theta_b}_{\dot{Q}_{unfin} = \dot{Q}_{prime} = \dot{Q}_b}$$

(2-40)b $$= N\dot{Q}_{fin} + \dot{Q}_{unfin}$$

single fin

<u>NOTE:</u> h is assumed here to be equivalent for finned and unfinned (prime) surfaces

Eq. (2-40)a can be written as: $\underbrace{(A_t - NA_{Fin})}_{= A_b = A_{unfin}}$

(2-40)c $$\dot{Q}_t = h[N\eta_{Fin}A_{Fin} + (A_t - NA_{Fin})]\,\theta_b$$

(2-40)d or $$\dot{Q}_t = hA_t\left[1 - \frac{NA_{Fin}}{A_t}(1 - \eta_{Fin})\right]\theta_b$$

Now, sub Eq(2-40)d in (2-38)b, yields

(2-41) $$\boxed{\eta_o = 1 - \frac{NA_{Fin}}{A_t}(1 - \eta_{Fin})}$$

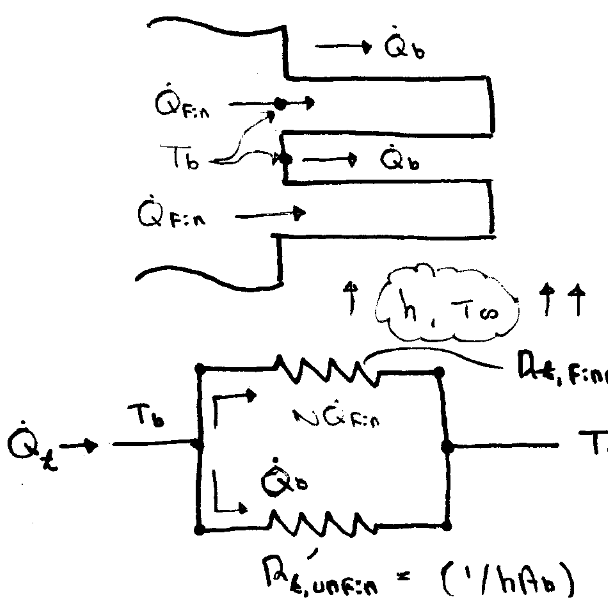
Note: 1. knowledge of η_o, the total heat transfer rate $\dot{Q}_t$ for an array of fins can be estimated using Eq. (2-38)b

2. Check for $(N = 1)$? (Eq. 2-41)

Recalling the fin thermal resistance $R_{t,Fin}$ (Eq. (2-36)b), Eq(2-38)b can be used to suggest an expression for the thermal resistance of a fin array, as:

(2-42) $$R_{t,\,Fin\,array} = R_{t,o} = \frac{\theta_b}{\dot{Q}_t} = \frac{1}{\eta_o h A_t}$$ verify!

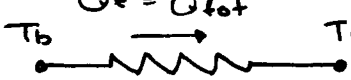
$R_{t,\,Fin\,array}$ is an effective resistance that accounts for parallel heat flow paths by conduction / convection in the fins and by convection from the prime (unfinned) surface.



Note: energy balance around T_b node gives:

$$\dot{Q}_t = N\dot{Q}_{Fin} + \dot{Q}_b$$

$$\dot{Q}_t = \dot{Q}_{tot}$$



$$R_{t,\,Fin\,array} = R_{t,o} = R_{t,\,equivalent} = \left(\frac{1}{\eta_o h A_t}\right)$$

same

Fig (2-10): Thermal Circuit of a Fin Array (with N multiple Fins)

The overall effectiveness of a Finned Surface

The $\mathcal{E}_o$ is defined as

(2-43)a $$\mathcal{E}_o \equiv \frac{\dot{Q}_{t,\text{with Fins}}}{\dot{Q}_{t,\text{no Fins}}}$$

Recall, Eq. (2-40)c

Remember A_b is the unfinned part surface area ($= A_{unfinned}$) of the fin array

$$\dot{Q}_{t,\text{with Fins}} = h(N \eta_{Fin} A_{Fin} + A_b)\theta_b$$

($\theta_b = T_b - T_\infty$, always)

The $\dot{Q}_{t,\text{no Fins}}$ is given by