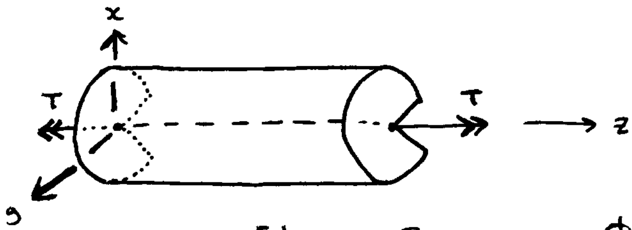
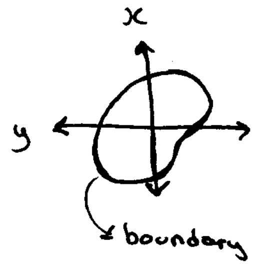


Stress Function $\phi(x,y)$

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = -2G\theta \\ \phi = 0 \quad \text{on the boundary} \end{cases}$$



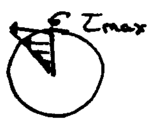
Stress

$$\begin{cases} \sigma_{xz} = \dfrac{\partial \phi}{\partial y} \\ \sigma_{yz} = \dfrac{-\partial \phi}{\partial x} \end{cases}$$

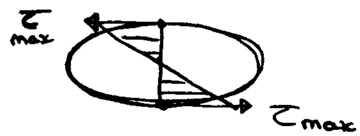
$$T = 2\iint_A \phi \, dx\, dy$$

$$\theta = T/GJ$$

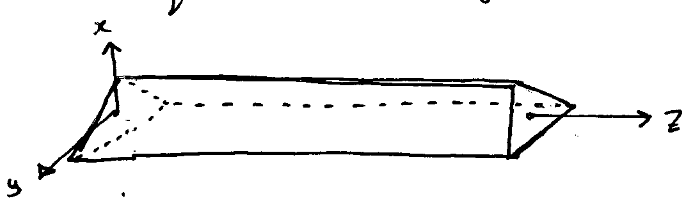
For a circle:

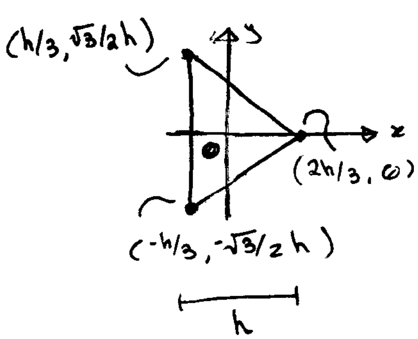


For an elipse:



6.32 - Equilateral Triangle Cross-Section



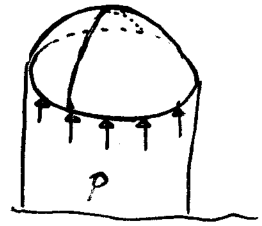


$$\phi = \frac{G\theta}{2h}\left(x - \sqrt{3}\,y - 2h/3\right) \cdot \left(\left(x + \sqrt{3}\,y - 2h/3\right)\right) \cdot \left(x + h/3\right)$$

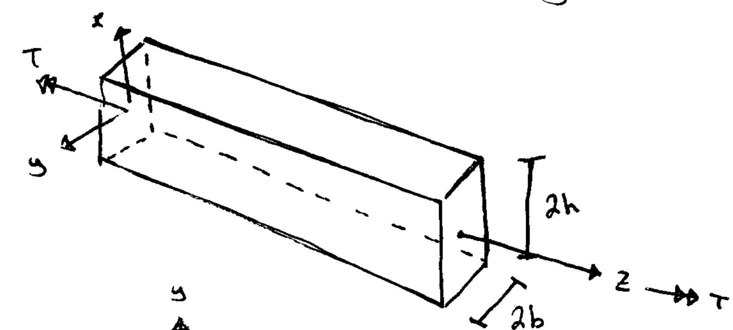
Define $J = \dfrac{h^4}{15\sqrt{3}}$

$$\tau_{max} = \frac{15\sqrt{3}}{2h^3}\, T \qquad \theta = \frac{T}{GJ}$$

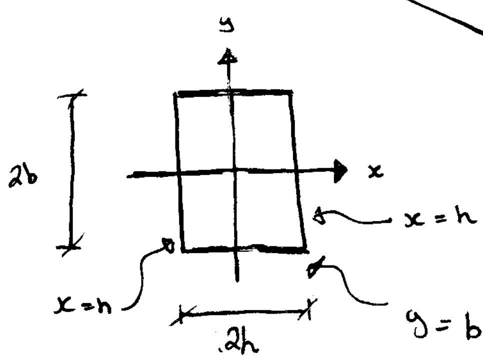
6.4 - The Prandtl elastic-membrane analogy



6.6 - Torsion of rectangular cross-section



$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = -2G\theta \\ \phi = 0 \text{ at } \begin{cases} x = h \\ x = -h \\ y = b \\ y = -b \end{cases} \end{cases}$$



$$\Rightarrow \quad \phi = G\theta(h^2 - x^2) - \frac{32 G\theta h^2}{\pi^3} \cdot \frac{\sum_{n=1} (-1)^{\frac{n-1}{2}} \cos\left(\frac{n\pi x}{2h}\right) \cosh\left(\frac{n\pi y}{2b}\right)}{n^3 \cosh\left(\frac{n\pi b}{2h}\right)}$$

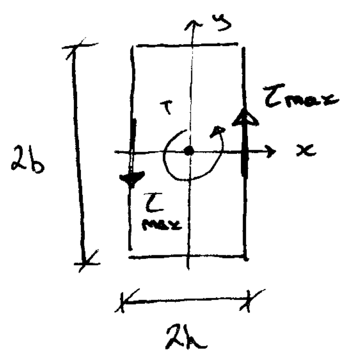
Define $J = k_1 (2h)^3 (2b)$

Then $\theta = T / GJ$

Max shear stress $(b > h)$

$\tau_{max} = G\theta \cdot (2h) \cdot (k_1 / k_2)$

The max. shear occurs at the location with the shortest distance to centre (at $x = \pm h$, $y = 0$)



b/h	1.0	2.0	... ∞
k_1	0.141	0.229	... 0.333
k_2	0.208	0.246	0.333

$b/h \geq 3$

$k_1 \simeq k_2 = 1/3 - 0.210\, h/b$

6.5 Narrow rectangular cross-section

$b \geq 10h$ $\qquad k_1 = k_2 = 1/3$

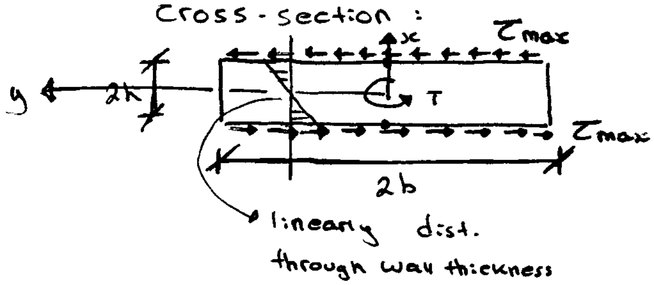
$J = 1/2(2h)^3(2b)$

$\theta = T/GJ$

$\tau_{max} = G\theta \cdot 2h \cdot k_1/k_2 = G\theta \cdot 2h$

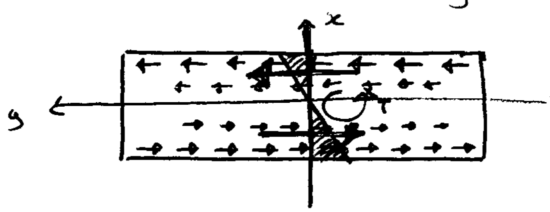
$= 2G\theta h$

Cross-section :



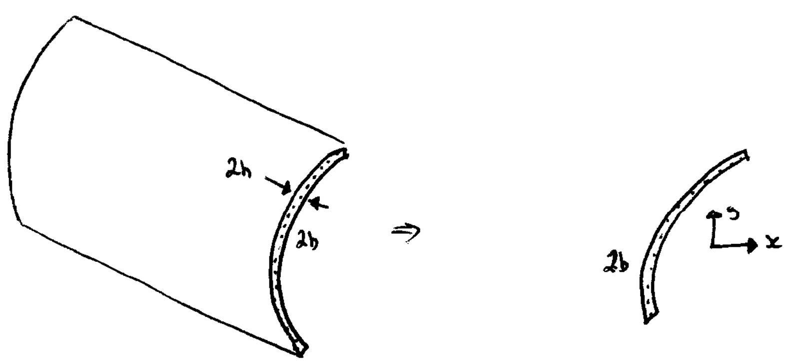
Shear stresses

$\sigma_{zx} = 0 \qquad \sigma_{zy} = 2G\theta x$

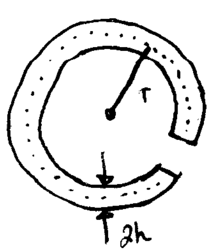


Resultant of the shear stress through the wall thickness is zero

Summation of shear stress (moment)
$= T/2$

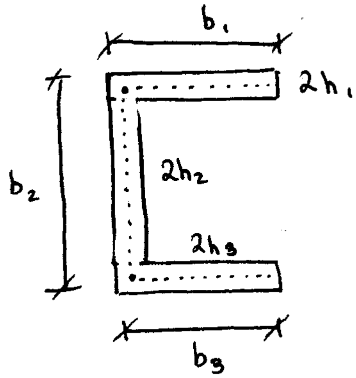


$J = 1/3(2h)^3(2b)$



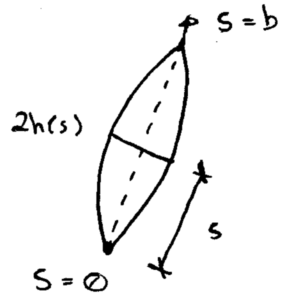
$2b = 2\pi r$

$J = (1/3)(2\pi r)(2h)^3$



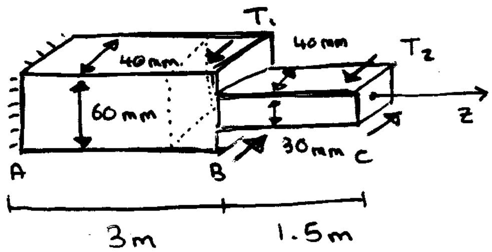
$$J = \tfrac{1}{3}(2h_1)^3(b_1) + (\tfrac{1}{3})(2h_2)^3(b_2) + (\tfrac{1}{3})(2h_3)^3(b_3)$$

$$\tau_{max,1} = 2G\theta h_1$$
$$\tau_{max,2} = 2G\theta h_2$$
$$\tau_{max,3} = 2G\theta h_3$$
$$\therefore \quad \tau_{max} = 2G\theta \cdot h_{max}$$



$$J = \tfrac{1}{3}\int_0^b (2h)^3\, ds$$

Example :



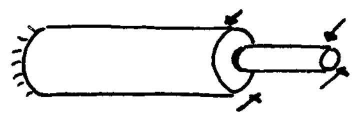
$T_1 = 750 \text{ N·m}$

$T_2 = 400 \text{ N·m}$

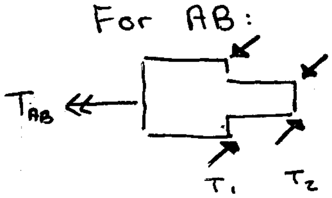
$G = 77.5 \text{ GPa}$
$= 77500 \text{ N/mm}^2$

Find τ_{max} and angle of twist of the free end.

Solution :



For AB:

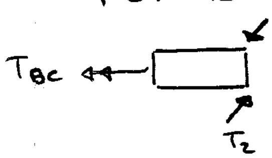


$\sum M_z = 0$

$T_1 + T_2 - T_{AB} = 0$

$T_{AB} = T_1 + T_2 \Rightarrow T_{AB} = 1150 \text{ N·m}$

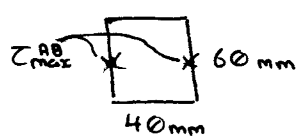
For BC:



$\sum M_z = 0$

$T_{BC} = T_2 \Rightarrow T_{BC} = 400 \text{ N·m}$

For AB:



$$\left.\begin{aligned} 2b &= 60 \\ 2h &= 40 \end{aligned}\right\} \frac{b}{h} = 1.5$$

$$\left.\begin{aligned} 1\text{m} &= 10^3\text{mm} \\ 1\text{N/m}^2 &= 1\text{Pa} \end{aligned}\right\} 1\text{N/mm}^2 = 1\text{MPa}$$

From table $K_1 = 0.196$
$K_2 = 0.231$

$$\theta_{AB} = \frac{T_{AB}}{GJ_{AB}} = \frac{1150 \times 10^3}{(77.5)(\times 10^3)(752640)}$$

$$= 1.9716 (10^{-5})$$

$$J_{AB} = K_1(2b)(2h)^3$$
$$= (0.196)(60)(40)^3$$
$$= 752640 \text{ mm}^4$$

$$\tau_{max}^{AB} = \frac{T_{AB}}{K_2(2b)(2h)^3} = \frac{1150 \times 10^3}{(0.231)(60)(40)^2} = 51.9 \text{ MPa}$$

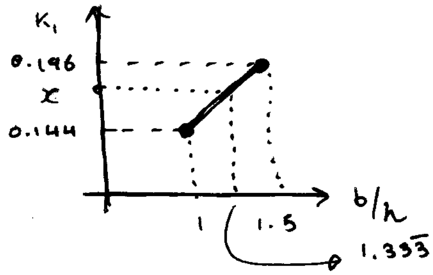
For BC :

$2b = 40$

$2h = 30$

$b/h = 40/30 = 1.33\overline{3}$

$K_1 = k_1(b/h)$



Assuming linearly distributed.

$$\frac{0.196 - 0.144}{1.5 - 1} = \frac{x - 0.141}{1.333 - 1}$$

$x = 0.1776 = K_1$

$K_2 = 0.2233$

$$J_{BC} = K_1(2b)(2h)^3$$

$$= (0.1776)(40)(30)^3$$

$$= 19180.8 \text{ mm}^4$$

$$\therefore \theta_{BC} = \frac{T_{BC}}{GJ_{BC}} = \frac{400(10^3)}{(77.5\times10^3)(19180.8)}$$

$$= 2.6909(10^{-5}) \text{ rad/mm}$$

$$\tau_{max}^{BC} = \frac{T_{BC}}{K_2(2b)(2h)^2} = \ldots = 49.8 \text{ MPa}$$

$\therefore \tau_{max} = 51.9$ MPa

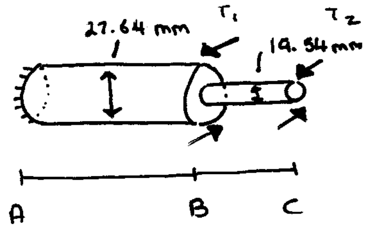
it occurs in the AB segment.

$$\beta_{C/A} = \beta_{C/B} + \beta_{B/A}$$

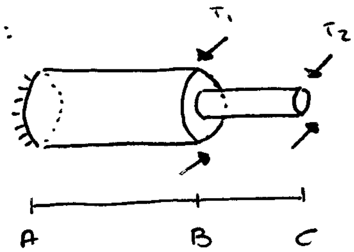
$$\beta_{C/A} = (BC \cdot \theta_{BC}) + (AB \cdot \theta_{AB})$$

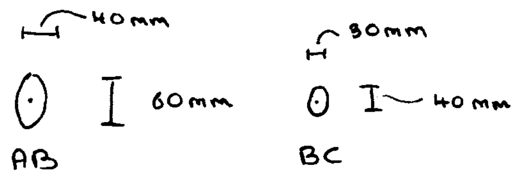
$$\Rightarrow (1500)(2.6909\times10^{-5}) + (3000)(9.9512\times10^{-6}) \text{ rad}$$

Case 1 :



Case 2 :





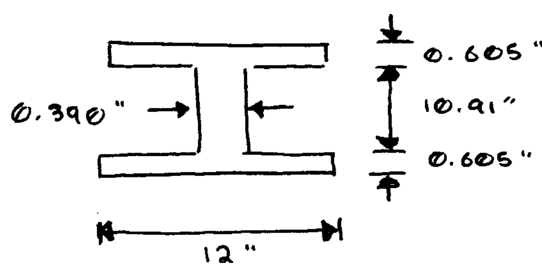
Example : For the given I-beam :

a) Find the torsional constant J

b) Find the maximum torque that the beam can take if the yield shear stress is $\tau_y = 36$ ksi

Given $G = 12 \times 10^3$ ksi

→ a)



$$J = \left(\tfrac{1}{3}\right)(2b)(2h)^3$$

$$\rightarrow \; J = \left(\tfrac{1}{3}\right)(12)(0.605)^3 \times 2 \;\cdots$$

$$\cdots + \left(\tfrac{1}{3}\right)(10.91)(0.390)^3$$

$$J = 1.99 \text{ in}^4$$

→ b) $$\tau_{max} = \frac{T}{J}(2h)_{max}$$

$$\Rightarrow \tau_{max} = \frac{T}{J}(0.605)$$

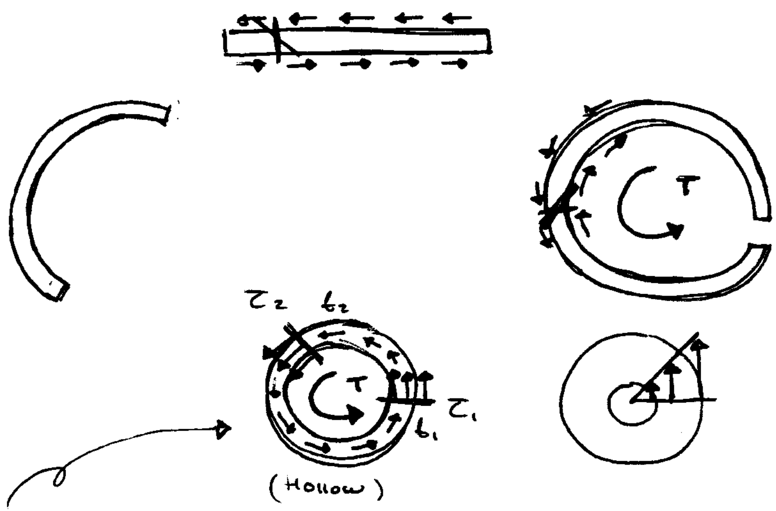
Since $\tau_{max} \leq \tau_y$

$$\Rightarrow \frac{\tau_{max}}{J}(0.605) = \tau_y$$

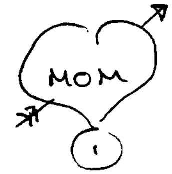
$$\Rightarrow \tau_{max} = \left(\frac{36 \times 1.99}{0.605}\right) \text{ kip-in}$$

6.7 Hollow thin-wall torsion members and multiply connected cross-section.

Narrow rectangular cross-section.

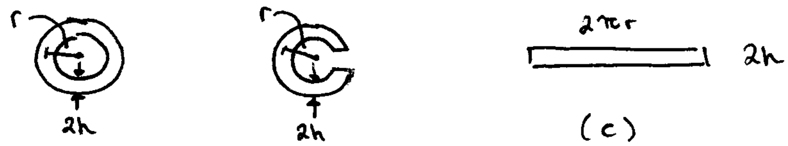


- Shear stress is practically constant through the wall thickness.
- Shear stress is parallel to the boundary of the section.
- $q = \tau t$ = Shear Flow
- q = const.



Oct. 19/18

Example :

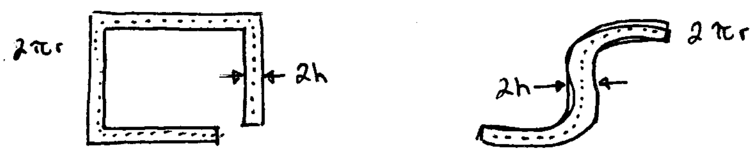


(a) (b) (c)

→ three thin wall members, Find the ratio of the largest shear stress and the ratio of the angle of twist per unit length.

Solution : b and c are the same

$I_b = \frac{1}{3}(2\pi r)(2h)^3$



(d) (e)

For (a) outer radius : $r+h$

inner radius : $r-h$

$$J_a = \frac{\pi}{2}\left[(r+h)^4 - (r-h)^4\right]$$

$$\Rightarrow \pi/2 \left[(r^4 + 4r^3 + 6r^2h^2 + 4rh^3 + h^4) - (r^4 - 4r^3h + 6r^2h^2 - 4rh^3 + h^4)\right]$$

$$\Rightarrow (\pi/2)\left[8r^3h + 8rh^3\right]$$

$$\approx 4\pi r^3 h$$

o neglect

$$\tau^a_{max} = T/J_a \cdot (r + \cancel{h}) = (T/4\pi r^3 h)\cdot r) = \frac{T}{4\pi r^2 h}$$

$$\tau^b_{max} = T/J_b \cdot (2h) = \left(T/(1/3)2\pi r(2h)^3\right) = \frac{3T}{8\pi r h^2}$$

$$\hookrightarrow \frac{\tau^a_{max}}{\tau^b_{max}} = \left(\frac{T}{4\pi r^2 h}\right)\left(\frac{8\pi r h^2}{3T}\right) \Rightarrow \left(\frac{2}{3}\cdot\frac{h}{r}\right)$$

$\theta = \dfrac{T}{GJ}$ → where T, G don't change.

$$\therefore \frac{\theta_a}{\theta_b} = \frac{J_b}{J_a} = \frac{(\frac{1}{3})\,2\pi r\,(2h)^3}{4\pi r^3 h} = \frac{4}{3}\left(\frac{h}{r}\right)^2$$

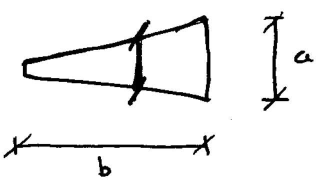
A special case:

$r = 400$ mm $\qquad 2h = 30$ mm

$$\frac{\tau^a_{max}}{\tau^b_{max}} = \frac{2}{3}\cdot\left(\frac{15}{400}\right) = \frac{1}{40}$$

$$\frac{\theta_a}{\theta_b} = \frac{4}{3}\left(\frac{15}{400}\right)^2 = \frac{2}{1067}$$

$\boxed{\text{Example}}$

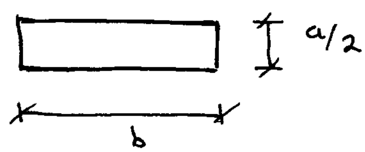


$b \gg a$

a) Find the max shear stress in terms of:

T, a, b, G

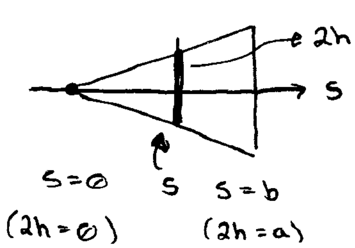
b)



what are the percentage errors of τ_{max} and θ if using $J = (1/3)(b)(a/2)^3 = \left(\frac{1}{24}\right)a^3 b$

Solution: a)

$$J = \tfrac{1}{3}\int_0^b (2h)^3\,ds$$



$$\frac{s}{b} = \frac{2h}{a} \Rightarrow 2h = \frac{a}{b}\cdot s$$

$$\Rightarrow J = \tfrac{1}{3}\int_0^b \left(\frac{a}{b}s\right)^3 ds$$
$$= \tfrac{1}{3}\, a^3/b^3 \int_0^b s^3\,ds$$
$$= (1/3)(a^3/b^3)(1/4)(b^4) = 1/12\, a^3 b$$

$$\Rightarrow \tau_{max} = \frac{T}{J}(2h)_{max}$$

$$\Rightarrow \tau_{max} = T/J\,(2h)_{max}$$

$$= \frac{T}{\left(\frac{1}{12}\right)a^3 b} \cdot a = \frac{12T}{a^2 b}$$

$$\theta = \frac{T}{GJ} = \frac{T}{G\left(\frac{1}{12}\right)a^3 b} = \frac{12T}{Ga^3 b}$$

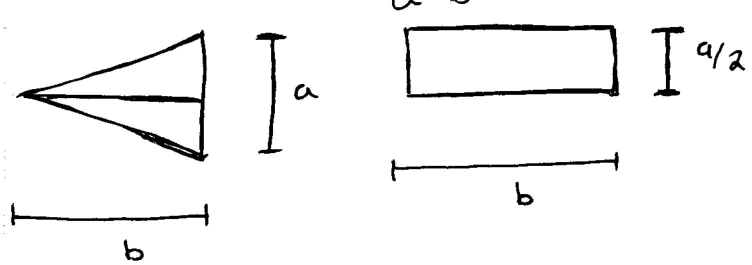
b) $J = \left(\frac{1}{12}\right)a^3 b$

$$J_{rec} = \left(\frac{1}{24}\right)a^3 b$$

$$\tau_{max}^{rec} = \frac{T}{J_{rec}} \cdot (2h)_{max}$$

$$= \frac{T}{\left(1/24\right)a^3 b} \cdot \left(\frac{a}{2}\right)$$

$$= \frac{12T}{a^2 b}$$



$\theta = ?$