MOM

1

Oct. 2/18

Chapter 6 : Torsion

Displacement $\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$

Strains :

$$\epsilon_{xx} = \partial u / \partial x$$

$$\epsilon_{yy} = \partial v / \partial y$$

$$\epsilon_{zz} = \partial w / \partial z$$

$$\epsilon_{yz} = \left(\tfrac{1}{2}\right)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\epsilon_{zx} = \left(\tfrac{1}{2}\right)\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\epsilon_{xy} = \left(\tfrac{1}{2}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Stresses : $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$

Equilibrium

$$\left\{ \begin{array}{l} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \cdots \\ \vdots \end{array} \right\}$$

Hooke's Law for isotropic :

$$\sigma_{xx} = \lambda e + 2G\epsilon_{xx}$$

$$\sigma_{yy} = \lambda e + 2G\epsilon_{yy}$$

$$\sigma_{zz} = \lambda e + 2G\epsilon_{zz}$$

$$\sigma_{yz} = 2G\epsilon_{yz}$$

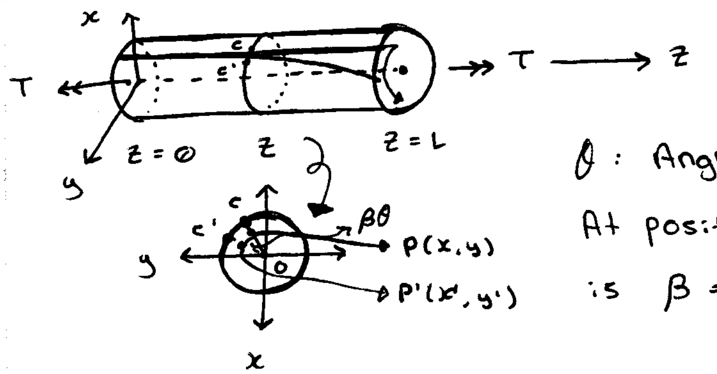
$$\sigma_{zx} = 2G\epsilon_{zx}$$

$$\sigma_{xy} = 2G\epsilon_{xy}$$

15 equations for 15 unknowns

boundary conditions :

## 6.1 Torsion of a prismatic bar of circular cross section




$\theta$ : Angle of twist per unit length

At position $z$, the angle of twist is $\beta = \theta z$

Displacement of a point on plane $z$ :

$P(x, y) \rightarrow P'(x', y')$ :

$$\therefore \begin{cases} u(x, y, z) = x' - x \\ v(x, y, z) = y' - y \\ w(x, y, z) = 0 \end{cases}$$

Define $OP = r$,

$x = r\cos\phi \qquad y = r\sin\phi$

$x' = r\cos(\phi + \beta) \qquad y' = r\sin(\phi + \beta)$

$$\therefore \; x' = r\cos\phi\cos\beta - r\sin\phi\sin\beta$$
$$= r\cos\theta z - y\sin\theta z$$
$$y' = r\sin\phi\cos\beta - r\cos\phi\sin\beta$$
$$= y\cos\theta z + x\sin\theta z$$

Small deformation, $\theta$ is very small,

$\theta z$ is small,

$\sin\theta z \doteq \theta z \qquad \cos\theta z \doteq 1$

$$\Rightarrow \; x' = x - \theta y z$$
$$y' = y + \theta x z$$

$$\therefore \begin{cases} u = x' - x = -\theta y z \\ v = y' - y = \theta x z \\ w = 0 \end{cases}$$

MOM

3

Strain $\epsilon_{xx} = \partial u/\partial x = 0$, $\epsilon_{yy} = \partial v/\partial y = 0$, $\epsilon_{zz} = 0$

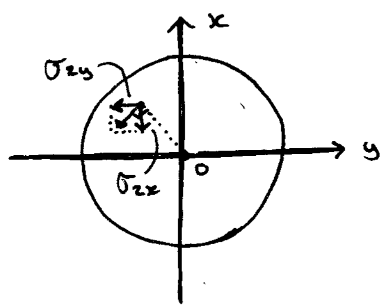
$$\epsilon_{xy} = (1/2)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 1/2(-\theta z + \theta z) = 0$$

$$\epsilon_{xz} = (1/2)(\partial u/\partial z) = -1/2\,\theta y$$

$$\epsilon_{yz} = (1/2)(\partial v/\partial z) = 1/2\,\theta x$$

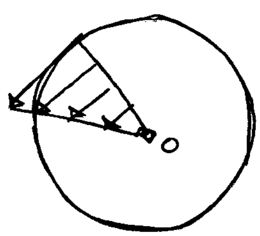
Hooke's Law:

$$\begin{cases} \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0 \\ \sigma_{xz} = 2G\epsilon_{xz} = -G\theta y \\ \sigma_{yz} = 2G\epsilon_{yz} = G\theta x \end{cases}$$




resultant of stress component:

$$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2}$$
$$= \sqrt{(-G\theta y)^2 + (G\theta x)^2}$$
$$= G\theta\sqrt{x^2+y^2}$$
$$\boxed{\tau = G\theta r}$$




From statics:

$$T = \iint_A \tau_r \, r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^r G\theta r \cdot r \cdot r \cdot dr$$
$$= 2\pi\, G\theta \cdot (1/4)\, b^4$$
$$T = G\theta \cdot \pi/2\, b^4$$

Define $J = \pi/2\, b^4$ (polar moment of inertia)

$$\Rightarrow \boxed{T = G\theta J = GJ\theta}$$

$\therefore$ Angle of twist per unit length

$$\boxed{\theta = T/GJ}$$

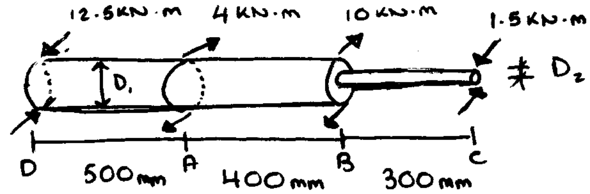
Stress:

$$\boxed{\tau = G\theta r = (T/J)\cdot r}$$

MOM

4

Example 6.4 - From textbook




$D_1 = 100$ mm

$D_2 = 50$ mm

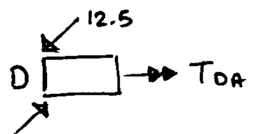
$G = 77.5$ GPa

Find 1° the max shear stress

2° the angle of twist of sections A, B, C, relative to section D.

Solution :

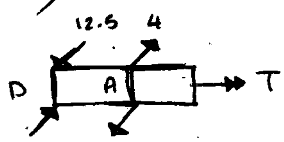

**DA**




$\Sigma T = 0, \quad T_{DA} + 12.5 = 0$

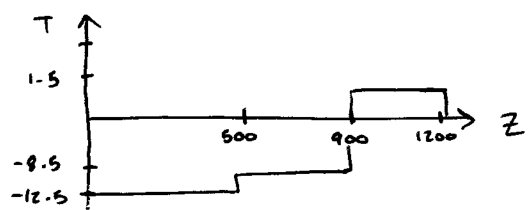
$T_{DA} = -12.5$ kN·m

**AB**




$\Sigma T = 0 \quad \therefore \quad T_{AB} = 8.5$ kN·m

$\therefore \quad T_{BC} = 1.5$ kN·m




$$J_{DA} = \frac{\pi}{2} b^4 = \frac{\pi}{2}\left(\frac{100}{2}\right)^4 = 9.8175\,(10^{-6})\ m^4$$

$$J_{AB} = J_{DA} = 9.8175\,(10^{-6})\ m^4$$

$$J_{BC} = \frac{\pi}{2} b^4 = \frac{\pi}{2}\left(\frac{50}{2}\right)^4 = 6.1359\,(10^{-8})\ m^4$$

$$\tau_{max_{AD}} = (T/J)\, r_{max} = \frac{12.5(10^3)}{9.8175(10^{-6})} \cdot \frac{(100)(10^{-3})}{2}$$

$$\Rightarrow 63.66\,(10^6)\ Pa$$

$$63.66\ MPa$$

$$\tau_{max_{BC}} = (T/J)\, r_{max} = \frac{1.5(10^3)}{6.1359(10^{-6})} \cdot \frac{50}{2}(10^{-3})$$

$$= 61.12\ MPa$$

$$\therefore \tau_{max} = 63.66\ MPa$$

MOM

5

$$\beta_{A/D} = \overset{m}{AD}\,\overset{rad/m}{\theta_{AD}}$$

$$= AD \cdot T/GJ$$

$$= (500)(10^{-3}) \cdot \frac{(-12.5)(10^{3})}{(77.5)(10^{9})(10^{-6})} \Rightarrow \beta_{A/D} = -0.00821 \text{ rad}$$

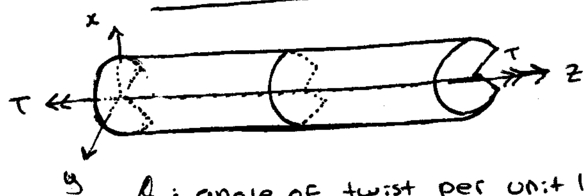
$$\beta_{B/A} = AB\,\theta_{AB}$$

$$= AB \cdot T/GJ$$

$$\Rightarrow -0.00447 \text{ rad}$$

$$\beta_{C/B} = BC\,\theta_{CB} = ?$$

## 6.2 Saint-Venant's Semi-inverse method




$\theta$ : angle of twist per unit length (constant)

Assumption: $u(x,y,z) = -\theta y z$

$v(x,y,z) = \theta x z$

$w(x,y,z) = \theta \psi(x,y)$ ← warping

Strains : $\epsilon_{xx} = \partial u/\partial x = 0$, $\epsilon_{yy} = \partial v/\partial y = 0$, $\epsilon_{zz} = \partial w/\partial z = 0$

$$\epsilon_{xy} = (1/2)\left(\frac{\partial u}{\partial y} + \partial v/\partial x\right) = (1/2)(-\theta z + \theta z) = 0$$

$$\epsilon_{xz} = (1/2)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = (1/2)\left(\theta \frac{\partial \psi}{\partial x} - \theta y\right)$$

$$\epsilon_{yz} = (1/2)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = (1/2)\left(\theta\, \partial\psi/\partial y + \theta x\right)$$

Stress : $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$

$$\begin{cases} \sigma_{xz} = 2G\epsilon_{xz} = G\theta(\partial\psi/\partial x - y) \\ \sigma_{yz} = 2G\epsilon_{yz} = G\theta(\partial\psi/\partial y + x) \end{cases}$$

Equilibrium eq's :

$$\cancel{\frac{\partial \sigma_{xx}}{\partial x}} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}} = 0$$

$$\cancel{\frac{\partial \sigma_{yx}}{\partial x}} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}} = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0}$$

$$\Rightarrow \sigma_{xz} = \frac{\partial \phi}{\partial y} \quad ; \quad \sigma_{yz} = \frac{-\partial \phi}{\partial x}$$

$\phi = \phi(x,y)$ : stress function

Since $\epsilon_{xz} = \frac{1}{2}\theta(\partial\psi/\partial x - y)$

$\epsilon_{yz} = \frac{1}{2}\theta(\partial\psi/\partial y + x)$

$$\Rightarrow \frac{\partial \epsilon_{xz}}{\partial y} = \frac{1}{2}\theta\left(\frac{\partial^2 \psi}{\partial y \partial x} - 1\right)$$

$$\frac{\partial \epsilon_{yz}}{\partial x} = \frac{1}{2}\theta\left(\frac{\partial^2 \psi}{\partial x \partial y} + 1\right)$$

$$\Rightarrow \boxed{\frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y} = \theta}$$

$$\Rightarrow \partial/\partial x\,(2G\epsilon_{yz}) - \partial/\partial y\,(2G\epsilon_{xz}) = 2G\theta$$

$$\Rightarrow \frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial y} = 2G\theta$$

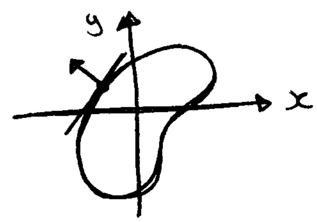
Using stress function:

$$-\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 2G\theta$$

$$\Rightarrow \boxed{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2G\theta}$$

Poisson's Equation

Boundary conditions




traction-free

$$\vec{n} = (l, m, 0)$$

$$\begin{cases} \cancel{\sigma_{xx} l} + \sigma_{\cancel{xy}} m + \cancel{\sigma_{xz} n} = 0 \\ \cancel{\sigma_{yx} l} + \cancel{\sigma_{yy} m} + \cancel{\sigma_{yz} n} = 0 \\ \sigma_{zx} l + \sigma_{zy} m + \cancel{\sigma_{zz} n} = 0 \end{cases}$$

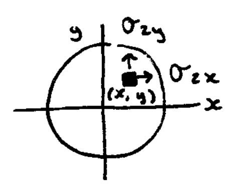
$$\Rightarrow \sigma_{xz} l + \sigma_{yz} m = 0$$

$$\Rightarrow \frac{d\Phi}{ds} = 0$$

$\Rightarrow$ $\phi$ = const along the boundary

Solid cross-section : $\phi = 0$

Resultant of stress on xy-plane :




$$\iint_A \sigma_{zx}\, dx\,dy = F_x = 0$$

$$\iint_A \sigma_{zy}\, dx\,dy = F_y = 0$$

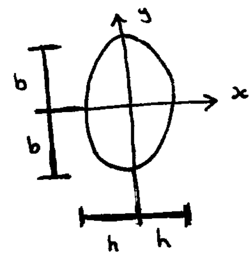
$$\boxed{\iint_A (\sigma_{zy}\, x\, dx\,dy - \sigma_{zx}\, y\, dx\,dy) = T}$$

$$\Rightarrow \iint_A \left( \frac{-\partial\phi}{\partial x} \cdot x - \frac{\partial\phi}{\partial y} \cdot y \right) dx\,dy = T$$

$$\Rightarrow \boxed{T = 2\iint_A \phi\, dx\,dy}$$

## 6.3 Linear Elastic Solution

### 6.3.1 Elliptical cross section




Boundary curve:

$$\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1 = 0$$

Assume:

$$\phi(x,y) = B\left( x^2/h^2 + y^2/b^2 - 1 \right)$$

Then $\phi = 0$ on the boundary

and $$\frac{\partial^2 \phi}{\partial x^2} = \frac{2B}{h^2} \qquad \frac{\partial^2 \phi}{\partial y^2} = \frac{2B}{b^2}$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2B\left( \frac{1}{h^2} + \frac{1}{b^2} \right) = -2G\theta$$

Let $$B = -\frac{G\theta}{1/h^2 + 1/b^2} = \frac{-G\theta b^2 h^2}{b^2 + h^2}$$

then $\phi(x,y)$ is the stress function

Stress: $$\begin{cases} \sigma_{zx} = \dfrac{\partial \phi}{\partial y} = \dfrac{2B}{b^2} \cdot y \\ \sigma_{zy} = \dfrac{-\partial \phi}{\partial x} = \dfrac{-2B}{h^2} \cdot x \end{cases}$$

Find $\theta$:

$$T = 2\iint_A \phi \, dx dy \Rightarrow 2\iint_A B\left( x^2/h^2 + y^2/b^2 - 1 \right) dx dy$$

$$\Rightarrow 2B\left[ \frac{1}{h^2} \iint_A x^2 \, dx dy + \frac{1}{b^2} \iint_A y^2 \, dx dy - \iint_A 1 \cdot dx dy \right]$$

Define $I_x = \iint_A y^2 \, dx dy$, $I_y = \iint_A x^2 \, dx dy$

$$\Rightarrow T = 2B\left( \frac{I_y}{h^2} + \frac{I_x}{b^2} - A \right)$$

Since $I_x = \dfrac{\pi b^3 h}{4}$ $\quad I_y = \dfrac{\pi b h^3}{4}$

$A = \pi b h$

$$\Rightarrow T = -\pi B b h = (\pi b^3 h^3)/(b^2 + h^2) \cdot G\theta$$

Define $J = \dfrac{\pi b^3 h^3}{b^2 + h^2}$

$$\Rightarrow T = GJ\theta \qquad \theta = T/GJ$$

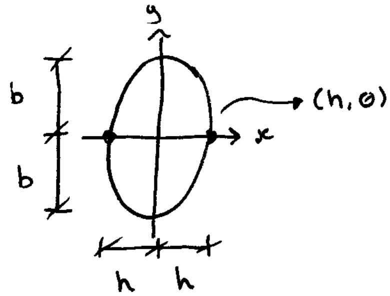
Shear stress

$$I = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}$$

$$= \sqrt{\left(\frac{2B}{b^2} \cdot y\right)^2 + \left(-\frac{2B}{h^2} \cdot x^2\right)^2}$$

1° max shear will occur at the boundary of the cross-section

2° max shear occurs at the boundary nearest the centroid of the cross-section




$h < b$

$\sigma_{xz} = 0$ ; $\sigma_{zy} = -\frac{2B}{h^2} \cdot h$

$$\sigma_{zy} = -2/h \cdot \left(- \frac{G\theta b^2 h^2}{b^2 + h^2}\right)$$

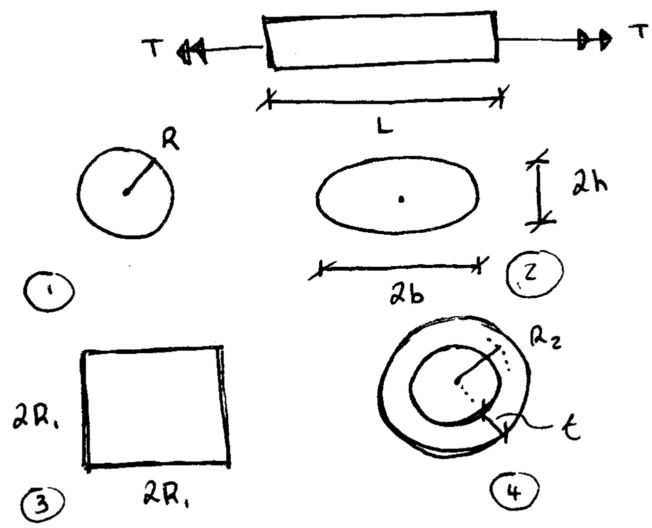
$$= G\theta \cdot \frac{b^2(2h)}{b^2+h^2}$$

$$\Rightarrow \tau_{max} = T/J \cdot b^2(2h)/_{b^2+h^2}$$

$$= \frac{T}{\frac{\pi b^3 h^3}{b^2+h^2}} \cdot \frac{b^2(2h)}{(b^2+h^2)}$$

$$\Rightarrow \boxed{\tau_{max} = \frac{2T}{\pi b h^2} \quad (h < b)}$$

Example:




$h = 0.707R$
$b = 1.414R$

$R_2 = 5R$
$t = 0.02R_2$

$R_1 = 0.866R$

For section 3

$$J = 0.141\,(2R_1)^4$$

$$\tau_{max} = \frac{T}{(0.208)(2R_1)^3}$$

$$\theta = T/GJ$$ for all members

→ Find: 1° $\theta$ for each section
2° $\tau_{max}$ for each section

1°) J for each section

1. $J_1 = \pi R^4/2$
2. $J_2 = \dfrac{\pi b^3 h^3}{b^2+h^2} \Rightarrow J = \dfrac{\pi(1.414R)^3(0.707R)^3}{(1.414R)^2+(0.707R)^2}$
3. $J_3 = 0.141\,(2R_1)^4$
4. where $R_2 = 5R$, $t = 0.02R_2$

$$t = 0.1R$$

then $r_1 = 4.95R,\ 5.05R$

$$J_4 = \frac{\pi(5.05R^4 - 4.95R^4)}{2}$$

Comparing:

1 : $J_1 = 1.571\,R^4$
2 : $J_2 = 1.257\,R^4$
3 : $J_3 = 1.390\,R^4$
4 : $J_4 = 78.54\,R^4$

Treat as constant

$\theta = (T/G)(1/J) \Rightarrow$ 4 has the largest $\theta$

2°) $\tau_{max}$ :

1 : $\tau_{max,1} = \frac{T}{J_1} R = \frac{T}{1.571 R^3}$

2 : $\tau_{max,2} = \frac{2T}{\pi b h^2} = \frac{2T}{\pi (1.414R)(0.707R)^2}$

$= \frac{T}{1.110 R^3}$

3 : $\tau_{max,3} = \frac{T}{(0.208)(2R_1)^3} = \frac{T}{(0.208)(2 \times .886 R)^3}$

$= \frac{T}{(1.15) R^3}$

4 : $\tau_{max,4} = \frac{T}{J_4} R_2$

$\Rightarrow \frac{T}{78.54 R^4} (5R) \Rightarrow \frac{T}{15.71 R^3}$

$\boxed{\text{(*) Midterm covers Ch. 1, 2, 3}}$