Sept. 25/18

@ $O(0,0,0)$, no disp., $u = v = w = 0$

$\Rightarrow u_0 = v_0 = w_0 = 0$

$$\Rightarrow \begin{cases} u = c_1 x \\ v = c_2 y \\ w = c_3 z \end{cases}$$

For Assignment #2 Q1

Chapter 3: Linear Stress-strain Temp. Relations

3.1 First law of thermodynamics

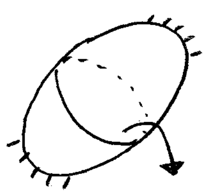
→ Conservation of energy

For adiabetic conditions – no net heat flow into the system, and static equilibrium

$$\boxed{\delta W = \delta U}$$

δW: work done by external forces

δU: increase the internal energy



U_0 = internal energy density

$\delta U = \iiint_V \delta U_0 \, V$ d(volume)

Displacement $\delta u, \delta v, \delta w$ (3 components)

↳ strains: $\delta \epsilon_{xx}, \delta \epsilon_{yy}, \delta \epsilon_{zz}, \delta \epsilon_{xy}, \delta \epsilon_{yz}, \delta \epsilon_{xz}$ (6 comp.)

Stress: $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}$

$$\Rightarrow \delta U_0 = \sigma_{xx}\delta\epsilon_{xx} + \sigma_{yy}\delta\epsilon_{yy} + \sigma_{zz}\delta\epsilon_{zz} + 2\sigma_{xy}\delta\epsilon_{xy} + \ldots$$

$$\ldots 2\sigma_{yz}\delta\epsilon_{yz} + 2\sigma_{xz}\delta\epsilon_{xz}$$

$$\boxed{U_0 = U_0(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})}$$

$$\boxed{\begin{array}{ll} \sigma_{xx} = \dfrac{\partial U_0}{\partial \epsilon_{xx}} & \sigma_{xy} = \left(\tfrac{1}{2}\right)\left(\dfrac{\partial U_0}{\partial \epsilon_{xy}}\right) \\ \sigma_{yy} = \dfrac{\partial U_0}{\partial \epsilon_{yy}} & \sigma_{yz} = \left(\tfrac{1}{2}\right)\left(\dfrac{\partial U_0}{\partial \epsilon_{yz}}\right) \\ \sigma_{zz} = \dfrac{\partial U_0}{\partial \epsilon_{zz}} & \sigma_{xz} = \left(\tfrac{1}{2}\right)\left(\dfrac{\partial U_0}{\partial \epsilon_{xz}}\right) \end{array}}$$

3.2 Hookes Law for anisotropic elasticity

A material is said to be elastically isotropic if its elastic coefficients are invariants under any rotation of coordinates

A material is said to be <u>homogeneous</u> if its properties are identical for any point in a member.

$$
\begin{array}{llll}
\sigma_{xx} \leftrightarrow \sigma_{11} \leftrightarrow s_1 & & \epsilon_{xx} \leftrightarrow \epsilon_{11} \leftrightarrow l_1 \\
\sigma_{yy} \leftrightarrow \sigma_{22} \leftrightarrow s_2 & & \epsilon_{yy} \leftrightarrow \epsilon_{22} \leftrightarrow l_2 \\
\sigma_{zz} \leftrightarrow \sigma_{33} \leftrightarrow s_3 & & \epsilon_{zz} \leftrightarrow \epsilon_{33} \leftrightarrow l_3 \\
\sigma_{yz} \leftrightarrow \sigma_{23} \leftrightarrow s_4 & & 2\epsilon_{yz} \leftrightarrow 2\epsilon_{23} \leftrightarrow l_4 \\
\sigma_{zx} \leftrightarrow \sigma_{31} \leftrightarrow s_5 & & 2\epsilon_{zx} \leftrightarrow 2\epsilon_{31} \leftrightarrow l_5 \\
\sigma_{xy} \leftrightarrow \sigma_{12} \leftrightarrow s_6 & & 2\epsilon_{xy} \leftrightarrow 2\epsilon_{12} \leftrightarrow l_6
\end{array}
$$

For the linear material :

$$s_1 = C_{11} l_1 + C_{12} l_2 + C_{13} l_3 + C_{14} l_4 + C_{15} l_5 + C_{16} l_6$$

$$s_i = \sum_{j=1}^{6} C_{ij} e_j$$

$$s_2 = C_{21} l_1 + C_{22} l_2 + C_{13} l_3 + C_{14} l_4 + C_{15} l_5 + C_{16} l_6$$

$$\vdots \qquad \qquad \vdots$$

$$s_6 = C_{61} l_1 + C_{62} l_2 + C_{63} l_3 + C_{64} l_4 + C_{65} l_5 + C_{66} l_6$$

$$[C_{ij}]_{6\times6} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{16} \\ \vdots & \ddots & & & \vdots \\ C_{61} & \cdots & \cdots & \cdots & C_{66} \end{bmatrix}$$ real symmetric matrix 6x6

21 material constants

$$\begin{Bmatrix} s_1 \\ s_2 \\ \vdots \\ s_6 \end{Bmatrix} = [C_{ij}] \begin{Bmatrix} l_1 \\ l_2 \\ \vdots \\ l_6 \end{Bmatrix}$$

The internal energy density

$$U_0 = \tfrac{1}{2}(s_1 l_1 + s_2 l_2 + s_3 l_3 + s_4 l_4 + s_5 l_5 + s_6 l_6)$$
$$= (\tfrac{1}{2}) \sum_i s_i l_i$$
$$= (\tfrac{1}{2}) \sum_{i=1}^{6} \sum_{j=1}^{6} C_{ij} l_j l_i$$
$$= (\tfrac{1}{2})(l_1, l_2, \ldots, l_6)[C_{ij}]\begin{Bmatrix} l_1 \\ l_2 \\ \vdots \\ l_6 \end{Bmatrix}$$

3.3 Hookes Law For isotropic elasticity

$21 \to 2$ numbers : λ and G ← shear modulus

Lance's elastic coefficient

Define : $e = l_1 + l_2 + l_3 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$

$\sigma_{xx} = \lambda e + 2G\epsilon_{xx}$ $\qquad \sigma_{xy} = 2G\epsilon_{xy}$

$\sigma_{yy} = \lambda e + 2G\epsilon_{yy}$ $\qquad \sigma_{yz} = 2G\epsilon_{yz}$

$\sigma_{zz} = \lambda e + 2G\epsilon_{zz}$ $\qquad \sigma_{xz} = 2G\epsilon_{xz}$

The internal energy density

$$U_0 = (\tfrac{1}{2})\lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})^2 + G(\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2 + 2\epsilon_{xy}^2 + 2\epsilon_{yz}^2 + 2\epsilon_{xz}^2)$$

$$U_0 = (\tfrac{1}{2}\lambda + G)\bar{I}_1^2 - 2G\bar{I}_2$$

In the principal strain directions :

$\epsilon_1, \epsilon_2, \epsilon_3$

Stress : $\sigma_1 = \lambda e + 2G\epsilon_1 = (\lambda + 2G)\epsilon_1 + \lambda\epsilon_2 + \lambda\epsilon_3$

$\sigma_2 = \lambda e + 2G\epsilon_2$

$\sigma_3 = \lambda e + 2G\epsilon_3$

no shear stress.

Inverse Hookes Relationship

$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$

$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]$

$\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$

$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \quad \ldots \quad \ldots$

4 constants : λ, G, E, ν

$$E = \frac{G(3\lambda + 2G)}{\lambda + G} \qquad \nu = \frac{\lambda}{2(\lambda+G)}$$

Plane stress : $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$

$\epsilon_{xz} = \epsilon_{yz} = 0$

$\epsilon_{zz} = ?$

$$\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2G\epsilon_{zz} = 0$$

$$\epsilon_{zz} = -\frac{\lambda}{\lambda+2G}(\epsilon_{xx} + \epsilon_{yy}) \neq 0$$

Plane strain : $\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0$

$\sigma_{xz} = \sigma_{yz} = 0$

but

$\sigma_{zz} \neq 0$

Plane strain $\neq$ Plane stress

$\boxed{\text{Example}}$ Given the principal strains at a point

$$\epsilon_1 ; \epsilon_2 ; \epsilon_3 = 5 ; 4 ; 3$$

The largest principal stress $\sigma_1 = 140$ MPa. Find σ_2 and σ_3 at the point. Given

$$E = 200 \text{ GPa}, \quad \nu = 0.3$$

Solution :
$$\begin{cases} \sigma_{xx} = \lambda e + 2G\epsilon_{xx} \\ \sigma_{yy} = \lambda e + 2G\epsilon_{yy} \\ \sigma_{zz} = \lambda e + 2G\epsilon_{zz} \end{cases}$$
General case

In the principal directions,

$$\begin{cases} \sigma_1 = \lambda e + 2G\epsilon_1 \\ \sigma_2 = \lambda e + 2G\epsilon_2 \\ \sigma_3 = \lambda e + 2G\epsilon_3 \end{cases} \qquad e = \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$\lambda = \frac{\nu E}{(1+\nu)(1+2\nu)} = \frac{0.3(200)}{(1+0.3)(1+0.6)} = 111.11 \text{ GPa}$$

$$G = \frac{E}{2(1+\nu)} = \frac{200}{2(1+0.3)} = 76.923 \text{ GPa}$$

Assume $\epsilon_1 = 5C, \quad \epsilon_2 = 4C, \quad \epsilon_3 = 3C$

$$\Rightarrow e = \epsilon_1 + \epsilon_2 + \epsilon_3 = 12C$$

Using $\sigma_1 = \lambda e + 2G\epsilon_1$

$$\Rightarrow (10^{-3})140 = 111.11(12C) + 2(76.923)5C$$

$$\Rightarrow C = 66.586(10^{-6})$$

$$\Rightarrow \epsilon_1 = 5C \Rightarrow 332.93(10^{-6})$$

$$\epsilon_2 = 4C \Rightarrow 266.35(10^{-6})$$

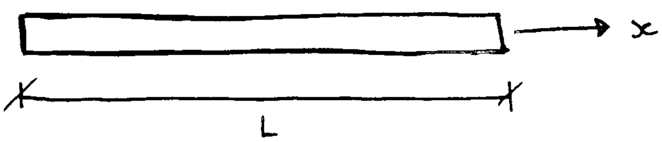
$$\epsilon_3 = 3C \Rightarrow 199.76(10^{-6})$$

$$\Rightarrow \sigma_2 = \lambda e + 2G\epsilon_2 = 129.76 \text{ MPa}$$

$$\Rightarrow \sigma_3 = \lambda e + 2G\epsilon_3 = 119.53 \text{ MPa}$$

3.4 Thermoelasticity for isotropic materials

temperature T_0



$\downarrow$

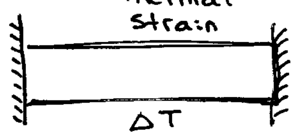
$T_1 = T_0 + \Delta T \rightarrow$ change in length ΔL

The coefficient of thermal expansion

$$\alpha = \frac{\Delta L}{L \cdot \Delta T} \Rightarrow \Delta L = \alpha \cdot L \cdot \Delta T$$

$\Rightarrow$ Strains $\epsilon = \frac{\Delta L}{L} = \alpha \Delta T$

thermal strain



(case where there is thermal stress)

In 3D, change temperature from $T_0 \rightarrow T_1 = T_0 \Delta T$

$\Rightarrow$ equal expansion

$$\rightarrow \epsilon'_{xx} = \epsilon'_{yy} = \epsilon'_{zz} = \alpha \Delta T$$

$$\epsilon'_{xy} = \epsilon'_{xz} = \epsilon'_{yz} = \emptyset$$

Strains due to the loadings

$$\epsilon''_{xx},\ \epsilon''_{yy},\ \epsilon''_{zz},\ \epsilon''_{xy},\ \epsilon''_{xz},\ \epsilon''_{yz}$$

The total deformation is

$$\begin{cases} \epsilon_{xx} = \epsilon'_{xx} + \epsilon''_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] + \alpha \Delta T \\ \epsilon_{yy} = \epsilon'_{yy} + \epsilon''_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] + \alpha \Delta T \\ \epsilon_{zz} = \epsilon'_{zz} + \epsilon''_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] + \alpha \Delta T \\ \epsilon_{xy} = \epsilon''_{xy} = \frac{1}{2G}\sigma_{xy} \\ \epsilon_{yz} = \epsilon''_{yz} = \frac{1}{2G}\sigma_{yz} \\ \epsilon_{xz} = \epsilon''_{xz} = \frac{1}{2G}\sigma_{xz} \end{cases}$$

3.5 Hookes Law : orthotropic materials

3 orthogonal planes of symmetry and 3 Corresponding orthogonal axes — called orthotropic axes.

9 independent material constants

$$
\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{xy} \end{bmatrix}
=
\left[\begin{array}{ccc:ccc}
\frac{1}{E_x} & \frac{-\nu_{yx}}{E_y} & \frac{-\nu_{zx}}{E_z} & \emptyset & \emptyset & \emptyset \\
\frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{-\nu_{yz}}{E_z} & \emptyset & \emptyset & \emptyset \\
\frac{-\nu_{xz}}{E_x} & \frac{-\nu_{zy}}{E_y} & \frac{1}{E_z} & \emptyset & \emptyset & \emptyset \\
\hdashline
\emptyset & \emptyset & \emptyset & \frac{1}{2G_{yz}} & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset & \frac{1}{2G_{zx}} & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \frac{1}{2G_{xy}}
\end{array}\right]
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}
$$

where $\dfrac{\nu_{yz}}{E_y} = \dfrac{\nu_{zy}}{E_z}$, $\dfrac{\nu_{zx}}{E_z} = \dfrac{\nu_{xz}}{E_x}$, $\dfrac{\nu_{yx}}{E_y} = \dfrac{\nu_{xy}}{E_x}$

$\boxed{\text{Example}}$ For Rochelle Salt the stress components at a point is

$\sigma_{xx} = 7\ MPa$ $\quad \sigma_{yy} = 2.1\ MPa$ $\quad \sigma_{zz} = -2.8\ MPa$

$\sigma_{xy} = 1.4\ MPa$ $\quad \sigma_{xz} = \sigma_{yz} = \emptyset$

Find 1° Strain components

2° the principal stresses and directions

3° the principal strains and directions

Solution 1° :

$$
\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{xz} \end{Bmatrix}
= 10^{-12}
\left[\begin{array}{ccc:ccc}
52 & -16.3 & -11.6 & & & \\
-16.3 & 36.8 & -12.2 & & & \\
-11.6 & -12.2 & 35.9 & & & \\
\hdashline
& & & 150.2 & & \\
& & & & 350.3 & \\
& & & & & 104.2
\end{array}\right]
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}
$$

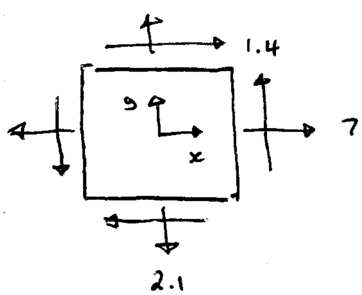
$$\Rightarrow \begin{Bmatrix} 362.25\,\mu \\ -2.66\,\mu \\ -207.34\,\mu \\ \emptyset \\ \emptyset \\ 145.88\,\mu \end{Bmatrix} \quad \mu = 10^{-6}$$

2°: Stress Analysis

Since $\sigma_{xz} = \sigma_{yz} = \emptyset$, $\sigma_{zz} = -2.8\,MPa$

$\therefore$ Z-axis is a principal direction, σ_{zz} is a principal stress

The other two principal directions in x-y plane



$$\tan\theta_p = \frac{2\theta_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{2(1.4)}{(7-2.1)} = 0.5714$$

$2\theta_p = 29.74°,\ 180° + 29.74°$

$\theta_p = 14.87,\ 104.87°$

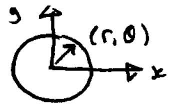
$$\sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{7+2.1}{2} = 4.55$$

$$R = \sqrt{\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = 2.82$$

$\therefore\ \sigma = \sigma_{AVG} \pm R = 7.37,\ 1.73$

Check $\theta_p = 14.87°$ corresponds to $\sigma = 7.37\,MPa$

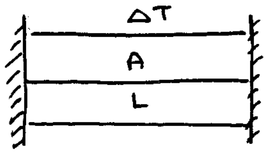
Example 3.7, Page 98



radial, circumferential

σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz} ← axial

$\sigma_{r\theta}$, σ_{rz} , $\sigma_{\theta z}$ = ∅



Unknowns: $\sigma_{\theta\theta}$ σ_{zz} in two cylinders

Stress in the aluminum (inner) cylinder: $\sigma_{\theta A}$, σ_{zA}

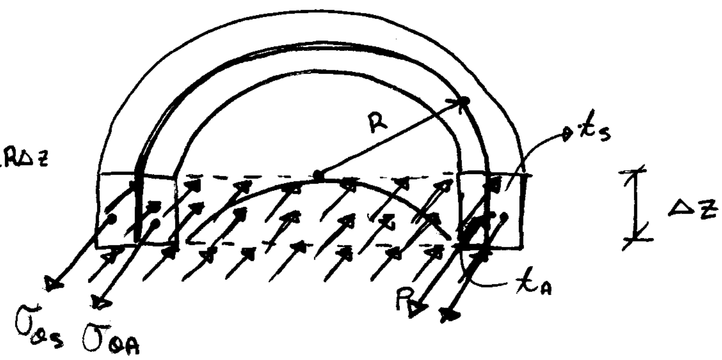
Stress in the steel (outer) cylinder: $\sigma_{\theta s}$, σ_{zs}

Equilibrium :

$\Sigma F_\theta = \emptyset$:

$2\sigma_{\theta s} t_s \Delta z + 2\sigma_{\theta A} t_A \Delta z - p 2R\Delta z$

① $\Rightarrow \sigma_{\theta s} t_s + \sigma_{\theta A} t_A = PR$



Deformation :

Inner cylinder : $\epsilon_{\theta A}$, ϵ_{zA} , ϵ_{rA}

Outer cylinder : $\epsilon_{\theta s}$, ϵ_{zs} , ϵ_{rs}

Constraints :

$$\text{Axial} \begin{cases} \epsilon_{zA} = \emptyset & ② \\ \epsilon_{zs} = \emptyset & ③ \end{cases}$$

Circumferential :

@ r = R : $\epsilon_{\theta A} = \epsilon_{\theta s}$ ④ .

(since $\epsilon_\theta = u/r + 1/r \, \cancel{\partial v / \partial \theta}$)

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_\theta) + \alpha\Delta T$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_z) + \alpha\Delta T$$

Aluminum : $\epsilon_{zA} = \frac{1}{E}(\sigma_{zA} - \nu\sigma_{\theta A}) + \alpha\Delta T$

$\epsilon_{\theta A} = \frac{1}{E}(\sigma_{\theta A} - \nu\sigma_{zA}) + \alpha\Delta T$

Steel : $\epsilon_{zs} = \frac{1}{E}(\sigma_{zs} - \nu\sigma_{\theta s}) + \alpha\Delta T$

$\epsilon_{\theta s} = \frac{1}{E}(\sigma_{\theta s} - \nu\sigma_{zs}) + \alpha\Delta T$

Given : $E_S = 207$ GPa $\qquad \nu_S = 0.280$

$E_A = 69$ GPa $\qquad \nu_A = 0.333$

$\alpha_A = 21.6 \times 10^{-6}/°C$

$\alpha_S = 10.8 \times 10^{-6}/°C$

$t_S = t_A = t = 0.02R$

$P = 689.4$ kPa , $\Delta T = 100°C$

$$\Rightarrow \begin{cases} \sigma_{\theta A} = 12.840P - 0.86337(10^6)\Delta T \\ \sigma_{\theta S} = 37.160P + 0.86337(10^6)\Delta T \\ \sigma_{zA} = 4.2757P - 1.7779(10^6)\Delta T \\ \sigma_{zS} = 10.405P - 1.9939(10^6)\Delta T \end{cases}$$