cont'd from previous example:

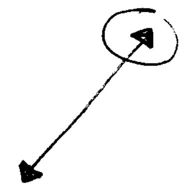
$\sigma_1 = 10 \qquad \vec{N}_1 = -0.6\vec{j} + 0.8\vec{k}$

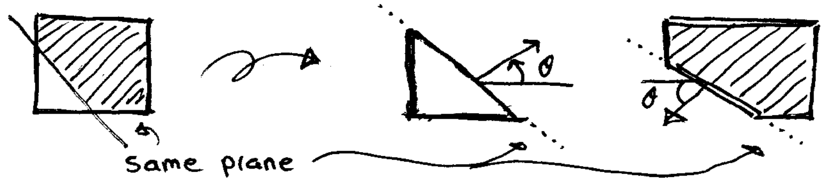
$\sigma_2 = 5 \qquad \vec{N}_2 = \vec{i}$

$\sigma_3 = -15 \qquad \vec{N}_3 = 0.8\vec{i} + 0.6\vec{j}$

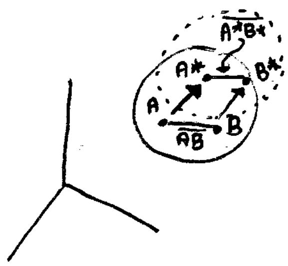
For $\sigma_1 = 10$, $\vec{N}_1 = 0.6\vec{j} - 0.8\vec{k}$

either solution works – from same line.





2.6 Deformation



$A(x, y, z)$

$\downarrow$

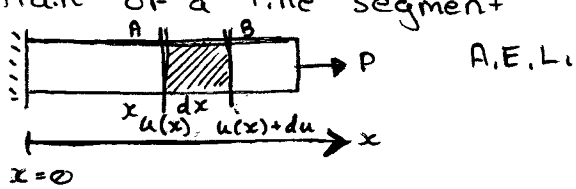
$A^*(x^*, y^*, z^*)$

disp: $\overrightarrow{AA^*} = (u, v, w)$

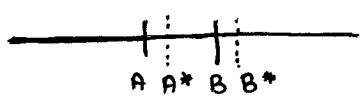
$$\Rightarrow \begin{cases} u = x^* - x \\ v = y^* - y \\ w = w^* - w \end{cases}$$

2.8 Small Displacement Theory

1° 1D: Strain of a line segment



A, E, L



line segment AB : dx

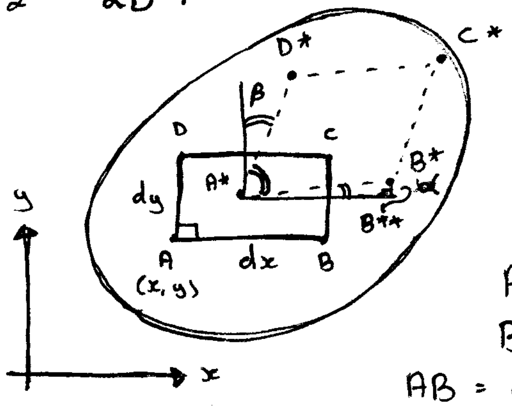
line segment A*B* : $x + dx + u(x) + du - (x+u) = du + dx$

The normal stream of line segment AB:

$$\epsilon = \frac{A^*B^* - AB}{AB} = \frac{du + \cancel{dx} - \cancel{dx}}{dx} = \frac{du}{dx}$$

$$\boxed{\epsilon(x) = \frac{du(x)}{dx}}$$

2° 2D :



$A \longrightarrow A^*$

$(x, y) \quad (x+u,\ y+v)$

$$\begin{cases} u = u(x; y) \\ v = v(x, y) \end{cases}$$

$A(x, y) \longrightarrow A^*(x+u,\ y+v)$

$B(x+dx, y) \longrightarrow B^*(x+dx+u_B,\ y+v_B)$

$AB = dx$

$A^*B^{**} = x + dx + u_B - (x+u) = u_B - u + dx$

normal strain of AB :

$$\epsilon_{xx} = \frac{A^*B^* - AB}{AB} = \frac{u_B - u + \cancel{dx} - \cancel{dx}}{dx}$$

$$= \frac{u_B - u}{dx}$$

Here, $u = u(x, y)$

$u_B = u(x+dx, y)$

$= u(x, y) + {}^{du}/_{dx} \cdot dx + \ldots$

$u_B - u = \dfrac{\partial u}{\partial x} dx \quad \Rightarrow \quad \dfrac{1}{\cancel{dx}} \cdot \left(\dfrac{\partial u}{\partial x}\right) \cancel{dx}$

thus

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \quad ; \quad \epsilon_{yy} = \frac{\partial v}{\partial y}$$

Elastic Shear Strain

$$\epsilon_{xy} = \tfrac{1}{2}(\alpha + \beta) = \tfrac{1}{2}\left({}^{\partial u}/_{\partial y} + {}^{\partial v}/_{\partial x} \right)$$

3° 3D : $A(x, y, z) \longrightarrow A^*(x+u,\ y+v,\ z+\omega)$

$$\begin{cases} \epsilon_{xx} = \dfrac{\partial u}{\partial x} \quad ; \quad \epsilon_{yy} = \dfrac{\partial v}{\partial y} \quad ; \quad \epsilon_{zz} = \dfrac{\partial \omega}{\partial z} \\ \epsilon_{xy} = 1/2\,({}^{\partial u}/_{\partial y} + {}^{\partial v}/_{dx}) \quad ; \quad \epsilon_{yz} = 1/2\,({}^{\partial v}/_{\partial z} + {}^{\partial \omega}/_{\partial y}) \\ \epsilon_{zx} = 1/2\,({}^{\partial \omega}/_{\partial x} + {}^{\partial u}/_{\partial z}) \end{cases}$$

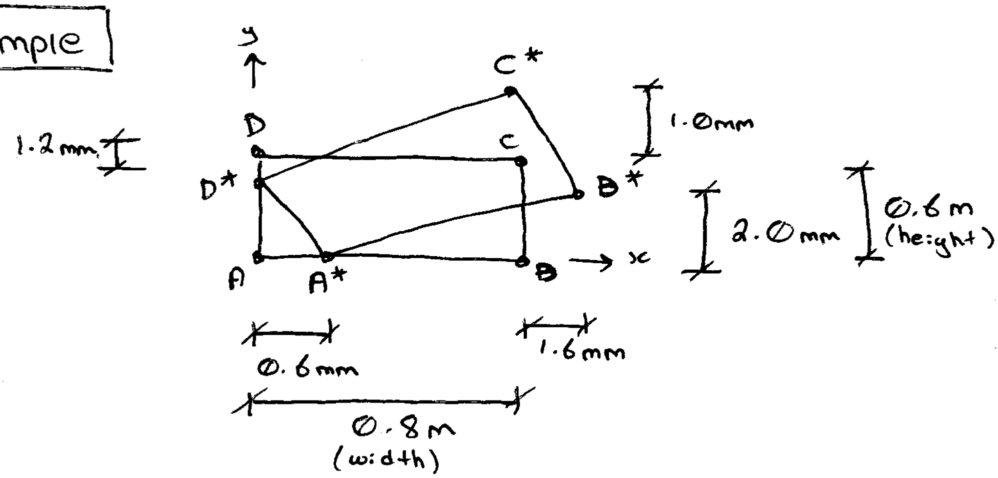
Strain compatability rotations

$$\frac{\partial^2 \epsilon_{yy}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = 2\,\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

$\vdots$

6 eq'ns
$$\left\{\begin{array}{c} \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \dfrac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \dfrac{\partial^2 \varepsilon_{xy}}{\partial z^2} = \dfrac{\partial^2 \varepsilon_{yz}}{\partial z \partial x} + \dfrac{\partial^2 \varepsilon_{zx}}{\partial z \partial y} \end{array}\right.$$

$\boxed{\text{Example}}$



Find strains at point A.

Solution : Assume

$$\begin{cases} u(x,y) = a_1 + b_1 x + c_1 y + d_1 xy \\ v(x,y) = a_2 + b_2 x + c_2 y + d_2 xy \end{cases}$$

From previous Example:

$u(x,y)$:

At A : $x = y = 0$, $u_A = (0.6)(10^{-3}) = a_1 + 0 + 0 + 0$

At B : $x = 0.8$, $y = 0$, $u_B = 1.6 \times 10^{-3} = a_1 + 0.8b_1 + 0 + 0$

At D : $x = 0$, $y = 0.6$, $u_D = 0 = a_1 + 0 + 0.6c_1 + 0$

At C : $x = 0.8$, $y = 0.6$, $u_C = 0 = a_1 + 0.8b_1 + 0.6c_1 + (0.8)(0.6)d_1$

$a_1 = 0.6 \times 10^{-3}$, $b_1 = 1.25 \times 10^{-3}$

$c_1 = -1.0 \times 10^{-3}$, $d_1 = -2.083 \times 10^{-3}$

$\therefore u(x,y) = (0.6 + 1.25x - y - 2.083xy) \times 10^{-3}$

From $\Rightarrow (a_1 + b_1x + c_1y + d_1xy)$

Using $V_a = 0$, $V_b = 2 \times 10^{-3}$, $V_c = 1.0 \times 10^{-3}$, $V_D = -1.2 \times 10^{-3}$

To Find a_2, b_2, c_2, d_2 :

$v(x,y) = (2.5x - 2y + 0.4167xy) \times 10^{-3}$

Strains :

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = (1.25 - 2.083y) \times 10^{-3}$$

$$\epsilon_{yy} = \frac{\partial v}{dy} = (-2 + 0.4167x) \times 10^{-3}$$

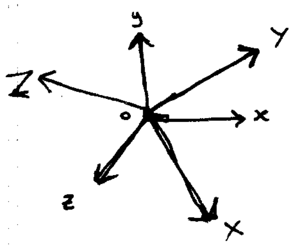
$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(-1 - 2.083x + 2.5 + 0.4167y) \times 10^{-3}$$

$$= (0.75 - 1.0415x + 0.20835y) \times 10^{-3}$$

At A : $x = y = 0$

$$\begin{cases} \epsilon_{xx} = 1.25 \times 10^{-3} \\ \epsilon_{yy} = -2.0 \times 10^{-3} \\ \epsilon_{xy} = 0.75 \times 10^{-3} \end{cases}$$

Stress Transformation



ϵ_{xx}, ϵ_{yy}, ϵ_{zz}, ϵ_{xy}, ϵ_{xz}, ϵ_{yz}

$$[S] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}_{3\times 3}$$

$$X : \vec{N_1} = l_1 \vec{i} + m_1 \vec{j} + n_1 \vec{k}$$
$$Y : \vec{N_2} = l_2 \vec{i} + m_2 \vec{j} + n_2 \vec{k}$$
$$Z : \vec{N_3} = l_3 \vec{i} + m_3 \vec{j} + n_3 \vec{k}$$

$$\begin{cases} \dot{N}_1 \cdot \dot{N}_1 = \dot{N}_2 \cdot \dot{N}_2 = \dot{N}_3 \cdot \dot{N}_3 = 1 \\ \dot{N}_1 \cdot \dot{N}_2 = \dot{N}_2 \cdot \dot{N}_3 = \dot{N}_1 \cdot \dot{N}_3 = 0 \end{cases}$$

$$\Rightarrow \vec{N_p} \cdot \vec{N_q} = \delta_{pq} = \begin{cases} 1 & p = q \\ 0 & p \neq q \end{cases}$$

$$p \cdot q = 1, 2, 3$$

in the "New" XYZ :

$$\begin{bmatrix} \epsilon_{XX} & \epsilon_{XY} & \epsilon_{XZ} \\ \epsilon_{YX} & \epsilon_{YY} & \epsilon_{YZ} \\ \epsilon_{ZX} & \epsilon_{ZY} & \epsilon_{ZZ} \end{bmatrix}$$

Normal Strain

$$\epsilon_{XX} = l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + n_1^2 \epsilon_{zz} + 2 l_1 m_1 \epsilon_{xy} + 2 l_1 n_1 \epsilon_{xz} + 2 m_1 n_1 \epsilon_{yz}$$

Shear strain

$$\epsilon_{XY} = l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} + n_1 n_2 \epsilon_{zz} + (l_1 m_2 + l_2 m_1) \epsilon_{xy} + \cdots$$
$$\cdots + (m_1 n_2 + m_2 n_1) \epsilon_{yz} + (l_1 n_2 + l_2 n_1) \epsilon_{xz}$$

Principal Strains

* Three mutually perpendicular line segments at one point will remain perpendicular after deformation

The three normal strains — principal strains

Principal strains :

$$\det \begin{bmatrix} \epsilon_{xx} - \epsilon & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} - \epsilon & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} - \epsilon \end{bmatrix} = 0$$

$$\epsilon^3 - \bar{I}_1 \epsilon^2 + \bar{I}_2 \epsilon - \bar{I}_3 = 0$$

Here $\bar{I}_1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$

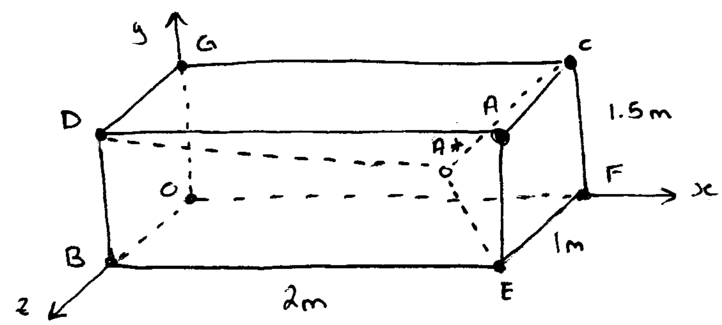
$\bar{I}_2 = \epsilon_{xx}\epsilon_{yy} + \epsilon_{xx}\epsilon_{zz} + \epsilon_{yy}\epsilon_{zz} - \epsilon_{xy}^2 - \epsilon_{zy}^2 - \epsilon_{xz}^2$

$$\bar{I}_3 = \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{vmatrix}$$

Principal direction :

$$\begin{cases} (\epsilon_{xx} - \epsilon)l + \epsilon_{xy} m + \epsilon_{xz} n = 0 \\ \epsilon_{yx} l + (\epsilon_{yy} - \epsilon) m + \epsilon_{yz} n = 0 \\ \epsilon_{zx} l + \epsilon_{zy} m + (\epsilon_{zz} - \epsilon) n = 0 \end{cases}$$

Example :



$A(2, 1.5, 1) \longrightarrow A^*(1.9985, 1.4988, 1.0009)$

Find 1° the strain components at **A**

2° the normal strain in the direction of line **AB**

3° the shear strain for lines **AB** and **AC**

4° the principal strains at **A**

Solution :

$$u(x,y,z) = \cancel{a_1} + \cancel{b_1 x} + \cancel{c_1 y} + \cancel{d_1 z} + \cancel{e_1 xy} + \cancel{f_1 yz} + \cancel{g_1 zx} + h_1 xyz$$

$$\rightarrow \begin{cases} u(x,y,z) = h_1 \, xyz \\ v(x,y,z) = h_2 \, xyz \\ w(x,y,z) = h_3 \, xyz \end{cases}$$

At A : $u_A = 1.9985 - 2 = -0.0015$

$\Rightarrow -0.0015 = h_1 (2)(1.5)(1) \Rightarrow h_1 = -0.005$

$v_A = 1.4988 - 1.5 = -0.0012$

$\Rightarrow -0.0012 = h_2 (2)(1.5)(1)$

$h_2 = -0.0004$

$w_A = 1.0009 - 1 = 0.0009 = h_3 (2)(1.5)(1)$

$h_3 = 0.0003$

$$\Rightarrow \begin{cases} u = -0.0005\, xyz = -500\mu\, xyz \\ v = -0.0004\, xyz = -400\mu\, xyz \\ w = 0.0003\, xyz = 300\mu\, xyz \end{cases}$$

$(1\mu = 10^{-6})$

1° Strains at A :

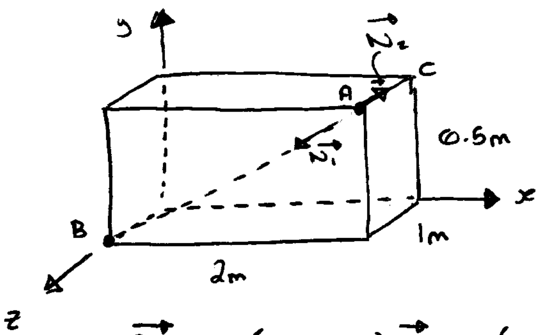
$\epsilon_{xx} = \partial u/\partial x = -500\mu\, yz = -500\mu(1.5)(1)$
$= -750\mu$

$\epsilon_{yy} = \partial v/\partial y = -400\mu\, xz = -400\mu(2)(1)$
$= -800\mu$

$\epsilon_{zz} = \partial w/\partial z = 300\, xy = 300\mu\,(2)(1.5)$
$= 900\mu$

$\epsilon_{xy} = \frac{1}{2}\left(\partial u/\partial y + \partial v/\partial x\right) = \frac{1}{2}(-500\mu\, xz - 400\mu\, yz)$
$= -800\mu$

$\epsilon_{xz} = -525\mu \qquad \epsilon_{yz} = -300\mu$



$A(2, 1.5, 1)$

$B(0, 0, 1)$

$$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$\vec{AB} = -2\vec{i} - 1.5\vec{j}$$

$$|\vec{AB}| = \sqrt{(-2)^2 + (-1.5)^2 + 0^2} = 2.5$$

the unit vector in AB:

$$\vec{N}_1 = \frac{\vec{AB}}{|\vec{AB}|} = \frac{-2\vec{i} - 1.5\vec{j}}{2.5} = -0.8\vec{j} - 0.6\vec{j}$$

$\therefore\ l_1 = -0.8,\quad m_1 = -0.6,\quad n_1 = 0$

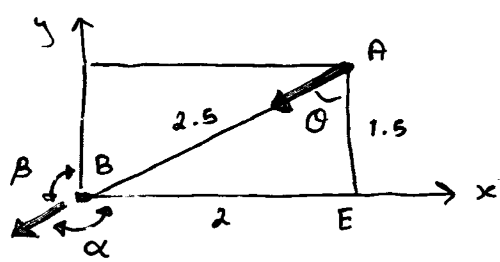
Normal strain along the line segment AB:

$$\epsilon_{xx} = l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + \cancel{n_1^2 \epsilon_{zz}}^{\,0} + 2 l_1 m_1 \epsilon_{xy} + \cancel{2 l_1 n_1 \epsilon_{xz}}^{\,0} + \cancel{2 m_1 n_1 \epsilon_{yz}}^{\,0}$$

$$= (-0.8)^2(-750\mu) + (-0.6)^2(-800\mu) + 2(-0.8)(-0.6)(-800\mu)$$

$$= -1536\mu$$

OTHER METHOD



$l_1 = \cos\alpha$

$m_1 = \cos\beta$

$n_1 = \cos\gamma$

$\triangle ABE$; $\tan\theta = \frac{BE}{AE} = \frac{2}{1.5}$

$\theta = 53.13°$

$\therefore\ \alpha = 90° + \theta = 143.13°$

$\beta = 180° - \theta = 136.87°$

$\gamma = 90°$

$\Rightarrow\ l_1 = \cos\alpha = -0.8$

$m_1 = \cos\beta = -0.6$

$n_1 = \cos\gamma = 0$

3° Shear Strain of the angle BAC (= 90°)

$\vec{N}_2 = l_2\vec{i} + m_2\vec{j} + n_2\vec{k}$

is a unit vector in AC

$\vec{N}_2 = -\vec{k}$

$l_2 = m_2 = 0, \quad n_2 = -1 \qquad$ (where $l_1 = -0.8,\ m_1 = -0.6,\ n_1 = 0$)

$\therefore\ \epsilon_{xx'} = l_1\cancel{l_2}\epsilon_{xx} + m_1\cancel{m_2}\epsilon_{yy} + \cancel{n_1}n_2\epsilon_{zz} + (l_1\cancel{m_2} + l_2\cancel{m_1})\epsilon_{xy} + (l_1 n_2 + \cancel{l_2}n_1)\epsilon_{xz}$

$\cdots + (n_1\cancel{m_2} + n_2 m_1)\epsilon_{yz}$

$\Rightarrow \epsilon_{xx'} = l_1 n_2 \epsilon_{zx} + m_1 n_2 \epsilon_{yz}$

$\Rightarrow (-0.8)(-1)(-525\mu) + (-0.6)(-1)(-300\mu)$

$= -600\mu$

the angle change

$\gamma_{xx'} = 2\epsilon_{xx'} = -1200\mu$

4° Principal Strains

$\bar{I}_1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$

$= (-750\mu) + (-800\mu) + (900\mu)$

$= -650\mu$

$\bar{I}_2 = \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{xx}\epsilon_{zz} - \epsilon_{xy}^2 - \epsilon_{yz}^2 - \epsilon_{xz}^2$

$= (\cdots)$

$= -1.800625 \times 10^{-6}\mu^2$

$$\bar{I}_3 = \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{vmatrix} \Rightarrow \begin{vmatrix} -750\mu & -800\mu & -525\mu \\ -800\mu & -800\mu & -300\mu \\ -525\mu & -300\mu & 900\mu \end{vmatrix}$$

$= 0$

$\Rightarrow \epsilon^3 - \bar{I}_1\epsilon^2 + \bar{I}_2\epsilon - \bar{I}_3 = 0$

$\Rightarrow \epsilon^3 + 650\mu\,\epsilon^2 - 1.800625(10^{-6})\mu^2\epsilon = 0$

Let $\epsilon = \mu \bar{\epsilon}$

$\Rightarrow \mu^3 \bar{\epsilon}^3 + 650 \mu \cdot \mu^2 \bar{\epsilon}^2 - 1.800625(10^6) \mu^2 \cdot \mu \bar{\epsilon} = 0$

$\Rightarrow \bar{\epsilon}^3 + 650 \bar{\epsilon}^2 - 1.800625(10^6) \bar{\epsilon} = 0$

$\Rightarrow \bar{\epsilon}_1 = 1055.67$

$\bar{\epsilon}_2 = 0$

$\bar{\epsilon}_3 = -1705.67$

$\Rightarrow \epsilon_1 = 1055.67\mu \; , \; \epsilon_2 = 0 \; , \; \epsilon_3 = -1705.67\mu$