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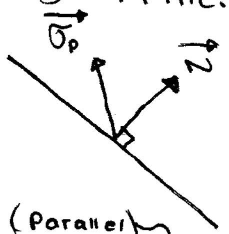
2.4.3 Principal Values and Directions



$\vec{N} = l\vec{i} + m\vec{j} + n\vec{k}$

$l^2 + m^2 + n^2 = 1$

$\vec{\sigma}_p = \sigma_{px}\vec{i} + \sigma_{py}\vec{j} + \sigma_{pz}\vec{k}$

(Parallel)

If $\vec{\sigma}_p \parallel \vec{N}$, there is a number σ such that $\vec{\sigma}_p = \sigma\vec{N}$

σ : normal stress

$\vec{N}$: the principal directions

$\Rightarrow \sigma_{px}\vec{i} + \sigma_{py}\vec{j} + \sigma_{pz}\vec{k} = \sigma(l\vec{i} + m\vec{j} + n\vec{k})$

$\Rightarrow \sigma_{px} = \sigma l \; ; \; \sigma_{py} = \sigma m \; ; \; \sigma_{pz} = \sigma n$

$$\Rightarrow \begin{cases} \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n = \sigma l \\ \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n = \sigma m \\ \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n = \sigma n \end{cases}$$

$l^2 + m^2 + n^2 = 1$

$$\begin{cases} (\sigma_{xx} - \sigma) l + \sigma_{xy} m + \sigma_{xz} n = 0 \\ \sigma_{xy} l + (\sigma_{yy} - \sigma) m + \sigma_{yz} n = 0 \\ \sigma_{xz} l + \sigma_{yz} m + (\sigma_{zz} - \sigma) n = 0 \end{cases}$$

A non-zero solution of (l, m, n)

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma \end{vmatrix} = 0$$

$\Rightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

Here : $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{xz}^2$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}$$

I_1, I_2, I_3 : <u>Invariants</u> of Stress

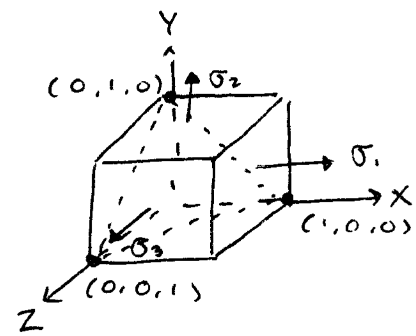
Three roots : $\sigma_1 \geq \sigma_2 \geq \sigma_3$

In the principal axes, the state of stress

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} I_1 = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = \sigma_1\sigma_2\sigma_3 \end{cases}$$

2.4.4 Octahedral Stress



XYZ : the principal directions

Consider the family of planes whose unit normal vectors satisfy :

$l^2 = m^2 = n^2 = 1/3$

The normal stress :

$$\sigma_{out} = l^2 \cancelto{\sigma_1}{\sigma_{xx}} + m^2 \cancelto{\sigma_2}{\sigma_{yy}} + n^2 \cancelto{\sigma_3}{\sigma_{zz}} + 2\cancelto{0}{lm\sigma_{xy}} + 2\cancelto{0}{mn\sigma_{yz}} + 2\cancelto{0}{nl\sigma_{zx}}$$

$$= 1/3\,(\sigma_1 + \sigma_2 + \sigma_3) = 1/3\, I_1$$

$$= 1/3\,(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$I_{oct} = \sqrt{2/9\, I_1^2 - 2/3\, I_2}$$

2.4.5 Mean and Deviator Stresses

The mean (normal) stress :

$$\sigma_m = 1/3\,(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = 1/3\, I_1$$

Define

$$[T_m] = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$ mean stress tensor

$$[T\Delta] = \begin{bmatrix} \sigma_{xx}-\sigma_m & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_m & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_m \end{bmatrix}$$ deviatoric stress tensor

$[T_m]$: hydro static stress

$[T_d]$: pure shear stress

Example : The state of stress at a point is given by

$$[T] = \begin{bmatrix} -19.0 & -4.70 & 6.45 \\ -4.70 & 4.60 & 11.8 \\ 6.45 & 11.8 & -8.30 \end{bmatrix} \text{MPa}$$

Determine the principal stress and their orientation w.r.t. the original coordinate system.

Solution :

(units aren't necessary intermediate step)

$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

$= -19.0 + 4.60 - 8.30 = -22.7$ MPa

$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{xz}^2$

$= -170.8125$ MPa^2

$I_3 = |T| = 2647.522$ MPa^3

The cubic eq'n:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$\Rightarrow \sigma^3 + 22.7\sigma^2 - 170.8125\sigma - 2647.522 = 0$$

$\sigma_a = 2S\cos\alpha + \frac{1}{3}I_1$

$\sigma_b = 2S\cos(\alpha + 120°) + \frac{1}{3}I_2$

$\sigma_c = 2S\cos(\alpha + 240°) + \frac{1}{3}I_3$

Here $S = \sqrt{\frac{1}{9}I_1^2 - \frac{1}{3}I_2}$

$\alpha = \frac{1}{3}\cos^{-1}\left(-\frac{Q}{2T}\right)$

$Q = \frac{1}{3}I_1I_2 - I_3 - \frac{2}{27}I_1^3$

$T = S^3$

$S = \sqrt{\frac{1}{9}(-22.7)^2 - \frac{1}{3}(-170.8125)} = 10.6861$

$T = S^3 = 1220.26$

$Q = \frac{1}{3}(-22.7)(-170.8125) - (2647.522) - \frac{2}{27}(-22.7)$

$= -488.590$

$$\therefore \alpha = 1/3 \cos^{-1}\left(- \frac{-488.590}{2 \times 1220.26} \right)$$

$$= 0.456412 \text{ rad} = 26.151^\circ$$

$$\therefore \sigma_a = 2(10.6861)\cos 26.151^\circ + \left(\frac{-22.7}{3}\right) = 11.6178$$

$$\sigma_b = 2(10.6861)\cos(26.151^\circ + 120^\circ) + \left(\frac{-22.7}{3}\right) = -25.3163$$

$$\sigma_c = 2(10.6861)\cos(26.151^\circ + 240^\circ) + \left(\frac{-22.7}{3}\right) = -9.0015$$

$$\therefore \sigma_1 = 11.6178 \text{ MPa}$$

$$\sigma_2 = -9.0015 \text{ MPa}$$

$$\sigma_3 = -25.3103 \text{ MPa}$$

principal direction for σ_1:

$$(\sigma_{xx} - \sigma_1)l + \sigma_{xy} m + \sigma_{xz} n = 0$$

$$\sigma_{xy} l + (\sigma_{yy} - \sigma)m + \sigma_{yz} n = 0$$

$$\sigma_{xz} l + \sigma_{yz} m + (\sigma_{zz} - \sigma)n = 0$$

$$\Rightarrow \begin{cases} -30.6178\,l - 4.70\,m + 6.45\,n = 0 \\ -4.70\,l - 7.0178\,m + 11.8\,n = 0 \\ 6.45\,l + 11.8\,m - 19.9178\,n = 0 \end{cases}$$

$$a = \begin{vmatrix} -7.0178 & 11.8 \\ 11.8 & -19.9178 \end{vmatrix} = 0.53914$$

$$b = -\begin{vmatrix} -4.70 & 11.8 \\ 6.45 & -19.9178 \end{vmatrix} = -17.504$$

$$c = \begin{vmatrix} -4.70 & -7.0178 \\ 6.45 & 11.8 \end{vmatrix} = -10.195$$

$$k = \sqrt{a^2 + b^2 + c^2}$$

$$= \sqrt{(0.53914)^2 + (17.504)^2 + (10.195)^2} = 20.264$$

$$\therefore l = \frac{a}{k} = \frac{0.53914}{20.264} = 0.02661$$

$$m = \frac{b}{k} = -0.8638 \qquad \left(\text{verify } l^2 + m^2 + n^2 = 1\right)$$

$$n = \frac{c}{k} = -0.5031$$

For σ_2 : $(-0.6209,\ 0.3802,\ -0.6855)$

For σ_3 : $(-0.7834,\ -0.3306,\ 0.5262)$

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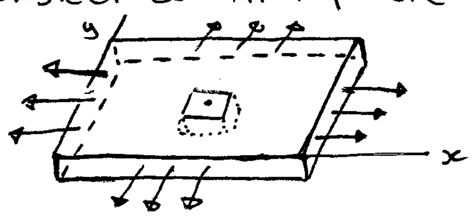
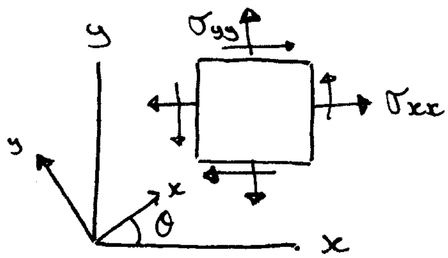
2.4.6 Plane stress

Consider a thin plate :



non zero stress : $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ } plane stress

others : $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$



along the x-axis :

$\vec{N}_1 = l_1 \vec{i} + m_1 \vec{j} + n_1 \vec{k}$

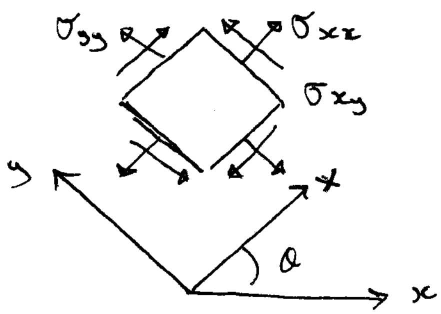
$l_1 = \cos\theta$; $m_1 = \sin\theta$; $n_1 = 0$

$\vec{N}_2 = l_2 \vec{i} + m_2 \vec{j} + n_2 \vec{k}$

$l_2 = -\sin\theta$; $m_2 = \cos\theta$; $n_2 = 0$

$\vec{N}_3 = l_3 \vec{i} + m_3 \vec{j} + n_3 \vec{k}$

$l_3 = 0, \quad m_3 = 0, \quad n_3 = 1$



$$\sigma_{xx} = l^2 \sigma_{xx} + m^2 \sigma_{yy} + n^2 \cancel{\sigma_{zz}}^{0} \cdots$$
$$\cdots + 2 l_1 m_1 \sigma_{xy} + 2 m_1 n_1 \cancel{\sigma_{yz}}^{0} \cdots$$
$$\cdots + 2 n_1 l_1 \cancel{\sigma_{zx}}^{0}$$

$$\Rightarrow \quad \sigma_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta$$

$$\Rightarrow (*) \quad \sigma_{xx} = \sigma_{avg} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta$$

$$\sigma_{avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{I_1}{2}$$

$$\sigma_{yy} = \sigma_{avg} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta$$

$$(*) \quad \sigma_{xy} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta$$

Max or min. normal stress occurs when

$\boxed{\sigma_{xy} = 0}$ ← principal stresses

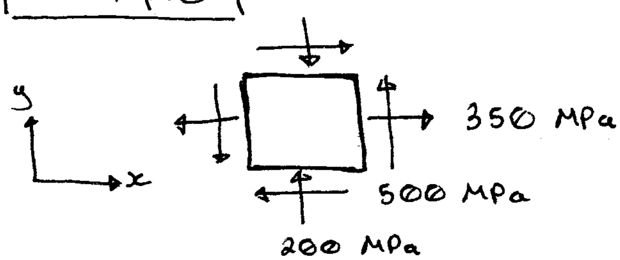
$$\Rightarrow \tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

the principal stresses :

$$\sigma_1 = \sigma_{avg} + R \ , \quad \sigma_2 = \sigma_{avg} - R$$

Here, $R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$

Example



Find the principal stresses and the orientations.

Solution : $\sigma_{xx} = 350 \quad \sigma_{yy} = -200$

$\sigma_{xy} = 500$

$$\sigma_{avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{350 - 200}{2} = 75$$

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\Rightarrow \sqrt{\left(\frac{350 + 200}{2}\right)^2 + 500^2} \quad \Rightarrow 570.64$$

$$\therefore \sigma_1 = \sigma_{avg} + R = 75 + 570.64 = 645.64$$

$$\sigma_2 = \sigma_{avg} - R = 75 - 570.64 = -496.36$$

$$\tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2(500)}{350 - 200} = 1.82$$

$$2\theta_p = 61.19^\circ, \quad 61.19^\circ + 180^\circ$$

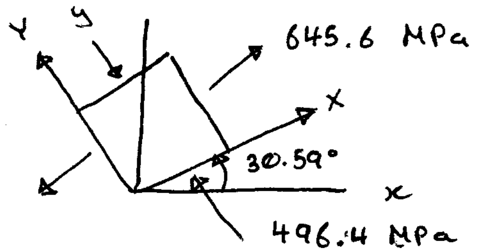
$$\theta_p = 30.59^\circ, \quad 120.59^\circ$$

Let $\theta = 30.59^\circ$

$$\sigma_{xx} = \sigma_{avg} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta$$

$$= 75 + \frac{350 + 200}{2} \cos 61.19^\circ + 500 \sin 61.19^\circ$$

$$= 645.64$$



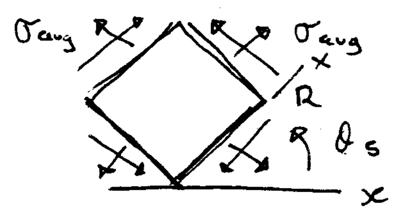
Max <ins>in-plane</ins> shear stress

$$\frac{d\sigma_{xy}}{d\theta} = 0 \quad \Rightarrow \quad -\frac{\sigma_{xx} - \sigma_{yy}}{2}(2\cos 2\theta) - \sigma_{xy}(2\sin 2\theta) = 0$$

$$\Rightarrow \boxed{\tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}}}$$

and $\sigma_{xy,\ \underset{\text{in-plane}}{\max}} = R$

In the directions of the max in-plane shear stress, the normal stress is σ_{avg}



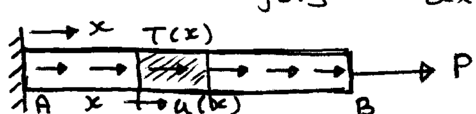
2.4.7

READING

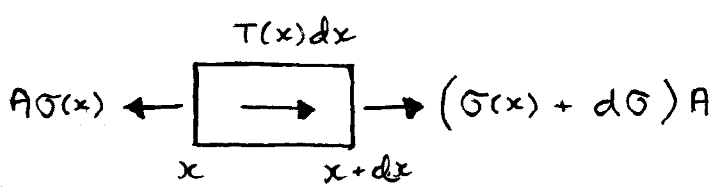
2.5 Differential equations of motion of a deformable body

1° 1D stress analysis - axial member

A.L.E



the displacement at B ? $[\sigma(x)]$



$\Sigma F_x = 0$:

$$T(x)\,dx + (\sigma(x) + d\sigma)A - A\sigma(x) = 0$$

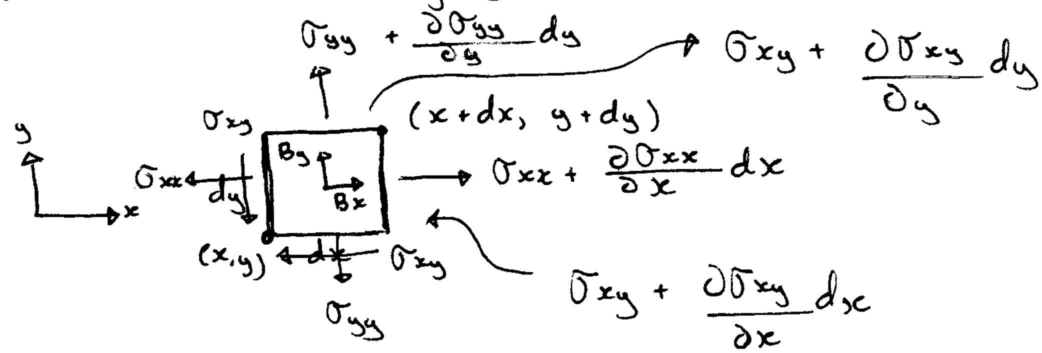
$$\Rightarrow A\,d\sigma + T(x)\,dx = 0$$

$$\Rightarrow \boxed{d\sigma/dx + T(x)/A = 0}$$

Define: $B_x = \dfrac{T(x)}{A}$ (force / volume)

$$\boxed{\frac{d\sigma}{dx} + B_x = 0}$$

2° 2D stress analysis



$\Sigma F_x = 0$:

$$-\sigma_{xx}dy + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dy \;\cdots$$

$$\cdots\; -\sigma_{xy}dx + \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y}dy\right)dx \;\cdots$$

$$\cdots\; + B_x\,dx\,dy = 0$$

$$\Rightarrow \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + B_x = 0$$

ΣF_y :

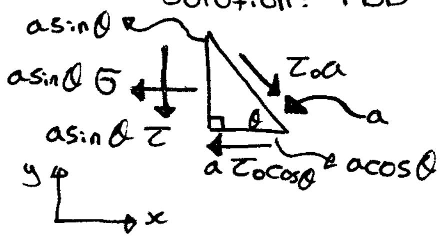
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0$$

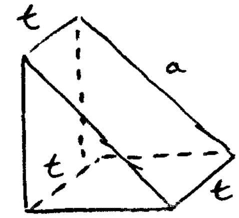
3° <u>3D</u>

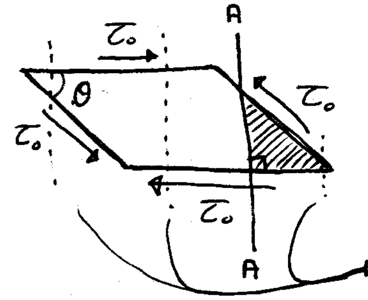
$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} + B_x = 0 \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + B_y = 0 \\[2ex] \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + B_z = 0 \end{cases}$$

Example: A thin plate is subjected to uniform shear stress. Find the stress vector on the cutting plane A-A

Solution: FBD







Same results for diff. cutting plane.

$\sum F_x = 0$:

$$-\sigma a \sin\theta - \tau_0 a \cos\theta - \tau_0 a \cos\theta = 0$$

$$\Rightarrow \quad \sigma = -2\tau_0 \frac{\cos\theta}{\sin\theta} = -2\tau_0 \cot\theta$$ (normal stress)

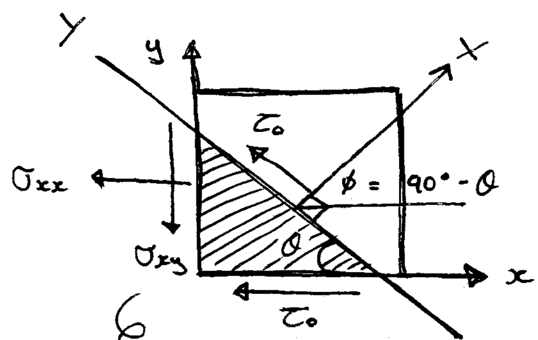
$\sum F_y = 0$:

$$-\tau a \sin\theta + \tau_0 a \sin\theta = 0$$

$$\Rightarrow \quad \tau = \tau_0$$ (shear stress)

Method 2:

(Putting small triangle into differential element)



(by inspection, you can see $\sigma_{xy} = \tau_0$ right away)

State of Stress

$$\begin{cases} \sigma_{xx} = \text{unknown} \\ \sigma_{yy} = 0 \\ \sigma_{xy} = \text{unknown} \end{cases}$$

Stress transformation

$$\sigma_{XX} = 0 = \sigma_{xx}\cos^2(90^\circ-\theta) + \cancel{\sigma_{yy}}^{\,0}\sin^2(90^\circ-\theta) + \sigma_{xy}\sin(90^\circ-\theta)\cos(90^\circ-\theta)$$

$$\Rightarrow \sigma_{xx}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta = 0 \quad (1)$$

$$\sigma_{XY} = \tau_0 = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2(90^\circ-\theta) + \sigma_{xy}\cos 2(90^\circ-\theta)$$

$$\tau_0 = -\left(\frac{\sigma_{xx}}{2}\right)\sin 2\theta - \sigma_{xy}\cos 2\theta \quad (2)$$

$$\Rightarrow \sigma_{xx} = -2\tau_0\cot\theta \qquad \Rightarrow \sigma_{xy} = \tau_0$$

Example :

$$[T] = \begin{bmatrix} 5 & \emptyset & \emptyset \\ \emptyset & -6 & -12 \\ \emptyset & -12 & 1 \end{bmatrix} \quad (MPa)$$

Determine

a) the principal stress (10, 5, -15)

b) the principal directions

Solution : $\sigma_{xx} = 5 \quad \sigma_{xy} = \sigma_{xz} = \emptyset$

$\sigma_{yx} = \emptyset \quad \sigma_{yy} = -6 \quad \sigma_{yz} = -12$

$\sigma_{zx} = \emptyset \quad \sigma_{zy} = -12 \quad \sigma_{zz} = 1$

and $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 5-6+1 = \emptyset$

$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2$

$\Rightarrow -175$

$$I_3 = \begin{vmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{vmatrix} = 5\begin{vmatrix} -6 & -12 \\ -12 & 1 \end{vmatrix} = -75\emptyset$$

$\Rightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3$

$\sigma^3 - 175\sigma + 750 = \emptyset$

$\sigma = 5$ is a root (by inspection)

$\sigma = 1\emptyset$ and $\sigma = -15$ are the rest of the roots

$\therefore \sigma_1 = 1\emptyset, \quad \sigma = 5, \quad \sigma_3 = -15$

$$\frac{\sigma^3 - 175\sigma + 75\emptyset}{\sigma - 5} = \sigma^2 + 5\sigma - 15\emptyset = \emptyset$$