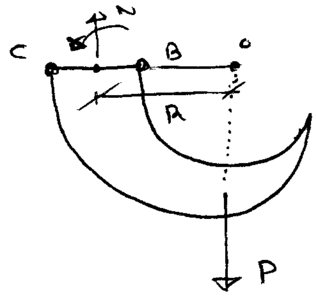
Statics



$N = P$

$M = PR$

$$\sigma_B = \frac{N}{A} + \frac{M}{A(RA_m - A)} \cdot \left(\frac{A}{r_B} - A_m \right) = 0.001309\,P$$

$$\sigma_c = \frac{N}{A} + \frac{M}{A(RA_m - A)} \cdot \left(\frac{A}{r_c} - A_m \right) = -0.000535\,P$$

$$\therefore \sigma_{max} = 0.001309\,P$$

$$\sigma_y / F_s = 0.001309\,P$$

$$500/2 = 0.001309\,P$$

$$P = 190900 \text{ N}$$

Chapter 11 : The thick-wall cylinder

Geometry : A thick wall cylinder

wall thickness is constant

* closed cylinder : with end caps
* open cylinder : w/o end caps

Loading : Internal pressure p_1

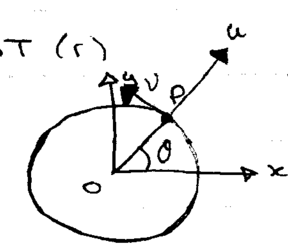
External pressure p_2

Axial load, P

Temperature Change $\Delta T(r)$

Deformation : axisymmetric :

$P(x,y,z) \Rightarrow P(r,\theta,z)$



Displacement:

$$\begin{cases} u(r,\theta,z) \\ v(r,\theta,z) \\ w(r,\theta,z) \end{cases}$$

$$\begin{cases} u = u(r,z) \\ v = 0 \\ w = w(r,z) \end{cases}$$

Consider the cross-section far away from the end caps :

$$\begin{cases} u = u(r) \\ v = 0 \end{cases}$$

$$\epsilon_{zz} = \partial w/\partial z = \text{const.}$$

Disp - strain :

$$\epsilon_{rr} = \partial u/\partial r = du/dr$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\,\partial u/\partial\theta + \frac{u}{r} = \frac{u}{r}$$

$$\epsilon_{zz} = \text{const.}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{\partial v}{\partial r}\right) = 0$$

$$\epsilon_{\theta z} = \frac{1}{2}\left(\partial u/\partial z + \frac{1}{r}\,\partial w/\partial z\right) = 0$$

$$\epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) = 0$$

Compatability equation

$$d/dr\,(r\,\epsilon_{\theta\theta}) = \epsilon_{rr}$$

Hooke's Law

$$\begin{cases} \epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right) + \alpha\Delta T \\ \epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{zz} + \sigma_{rr})\right) + \alpha\Delta T \\ \epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right) + \alpha\Delta T = \text{const.} \end{cases}$$

Equilibrium eq-n-s :

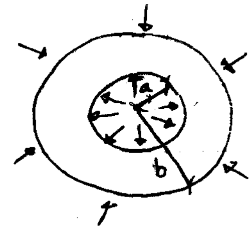
$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\cancel{\frac{\partial\sigma_{r\theta}}{\partial\theta}}^{0} + \cancel{\frac{\partial\sigma_{rz}}{\partial z}}^{0} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\rightarrow \frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0$$

$$\cancel{\frac{\partial \sigma_{rz}}{\partial r}} + \cancel{\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \cancel{\frac{\sigma_{rz}}{r}} = 0$$

Boundary Conditions :



At $r = a$, $\sigma_{rr} = -P_1$

At $r = b$, $\sigma_{rr} = -P_2$

$\Rightarrow$ Find σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, u and w

Solution for constant temperature :

$$\sigma_{rr} = \frac{P_1 a^2 - P_2 b^2}{b^2 - a^2} - \frac{a^2 b^2}{r^2(b^2-a^2)}(P_1 - P_2)$$

$$\sigma_{\theta\theta} = \frac{P_1 a^2 - P_2 b^2}{b^2 - a^2} - \frac{a^2 b^2}{r^2(b^2-a^2)}(P_1 - P_2)$$

$$\sigma_{zz} = \frac{P_1 a^2 - P_2 b^2}{b^2 - a^2} + \frac{P}{\pi(b^2-a^2)}$$

$$u = \frac{r}{E(b^2-a^2)}\left[(1-2\nu)(P_1 a^2 - P_2 b^2) + (1+\nu)\left(\frac{a^2 b^2}{r}\right)(P_1 - P_2) - \frac{\nu P}{\pi}\right]$$

<u>Example</u> : Cylinder with internal pressure P_1 only.

Find the max shear stress.

Solution: Since $P_2 = 0$,

$$\sigma_{rr} = \frac{P_1 a^2}{b^2-a^2} - \frac{a^2 b^2}{r^2(b^2-a^2)} P_1$$

$$= \frac{P_1 a^2}{b^2-a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{P_1 a^2}{b^2-a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_{zz} = \frac{P_1 a^2}{b^2-a^2}$$

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$: the principal stresses ($\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$)

$$\therefore \tau_{max}(r) = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{2} = \frac{p_1 a^2 b^2}{(b^2-a^2) r^2}$$

$\therefore$ max shear occurs at the inner surface where $r = a$

$$\therefore \tau_{max} = \frac{p_1 b^2}{b^2 - a^2}$$

$$\tau_{max} = \frac{p_1 (b/a)^2}{(b/a)^2 - 1}$$

Case study: $b/a = 3$

p_1 produces the allowable τ_{max}

$\Rightarrow \tau_{max} = (p_1)\cdot(9/8)$

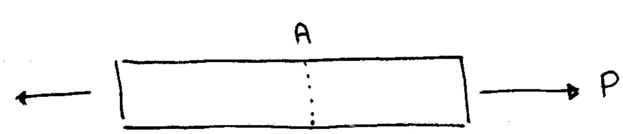
To maintain the max shear τ_{max} under the new internal pressure $1.1\,p_1$, the new cylinder should have the ratio b/a:

$$\tau_{max} = \frac{1.1\, p_1 (b/a)^2}{(b/a)^2 - 1} = (p_1)\cdot(9/8)$$

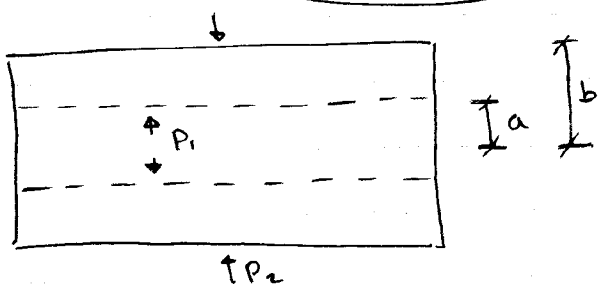
$$\rightarrow \frac{1.1\,(b/a)^2}{(b/a)^2 - 1} = 9/8$$

$\hookrightarrow b/a = 6.7$

Case:



$\sigma = P/A$



$$\sigma_{\theta\theta} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} \ldots$$

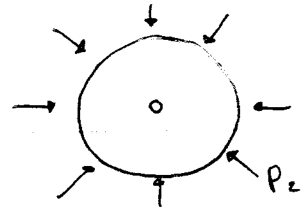
$$\ldots + \frac{a^2 b^2}{r^2 (b^2 - a^2)} (p_1 - p_2)$$

$b \to \infty \qquad (b/a \to \infty)$

$\sigma_{\theta\theta} = -P_2$ At the inner surface, $r = a$

$\sigma_{\theta\theta} = -P_2 + P_1 - P_2$

$$\boxed{\sigma_{\theta\theta} = P_1 - 2P_2}$$



IF $P_1 = 0$, $\sigma_{\theta\theta} = -2P_2$

Nov. 29/18

11.7 Rotating Disc of Constant Thickness

Geometry : t = const., $t << b$ (outer radius)

State of stress : plane stress & axisymmetry

$$\sigma_{zr} = \sigma_{z\theta} = \sigma_{zz} = \sigma_{r\theta} = 0$$

and σ_{rr} and $\sigma_{\theta\theta}$ are functions of r

Equation of motion :

$$\begin{cases} \partial\sigma_{rr}/\partial r + \frac{1}{r}\cancel{\frac{\partial\sigma_{r\theta}}{\partial\theta}}^{0} + \cancel{\frac{\partial\sigma_{rz}}{\partial z}}^{0} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \rho r\omega^2 = 0 \\ \cancel{\frac{\partial\sigma_{r\theta}}{\partial r}}^{0} + \frac{1}{r}\cancel{\frac{\partial\sigma_{\theta\theta}}{r\,\partial\theta}}^{0} + \cancel{\frac{2\sigma_{r\theta}}{r}}^{0} = 0 \end{cases}$$

$$\Rightarrow \quad d\sigma_{rr}/dr + (\sigma_{rr} - \sigma_{\theta\theta})/r + \rho r\omega^2 = 0$$

Stress-strain - temperature :

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}(\epsilon_{rr} - \nu\epsilon_{\theta\theta}) - \frac{E\alpha T}{1-\nu} \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\epsilon_{\theta\theta} - \nu\epsilon_{rr}) - \frac{E\alpha T}{1-\nu} \end{cases}$$

Strain - disp :

$$\epsilon_{rr} = \partial u/\partial r = du/dr$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\partial u/\partial\theta + u/r = u/r$$

The solution of the displacement : $u = u(r)$

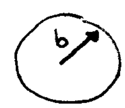
$$u(r) = \frac{1-\nu^2}{8E}\rho\omega^2 r^3 + \frac{\alpha(1+\nu)}{r}\int_a^r rT\,dr + C_1 r + C_2/r$$

C_1 and C_2 are unknown constants.

The stress $\swarrow$ temp. change

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\frac{u}{r}\right] - \frac{E\alpha T}{1-\nu} \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[\nu\frac{du}{dr} + \frac{u}{r}\right] - \frac{E\alpha T}{1-\nu} \end{cases}$$

Case 1: Solid Disk



(w/ const. temperature : $T = \Delta T = 0$)

Traction free at $r=b$

Boundary condition :

At $r=b$, $\sigma_{rr} = 0$

$$
\begin{aligned}
\sigma_{px} &= \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n \\
\sigma_{py} &= \sigma_{xy} + \sigma_{yy} m + \sigma_{yz} n \\
\sigma_{pz} &= \sigma_{xz} + \sigma_{yz} m + \sigma_{zz} n \\
\hline
\sigma_{pr} &= \sigma_{rr} l + \cancel{\sigma_{r\theta} m}^{0} + \cancel{\sigma_{rz} n}^{0} \\
\sigma_{p\theta} &= \cancel{\sigma_{r\theta} l}^{0} + \cancel{\sigma_{\theta\theta} m}^{0} + \cancel{\sigma_{\theta z} n}^{0} \\
\sigma_{pz} &=
\end{aligned}
$$

At $r = b$, $\sigma_{rr} = 0$

At $r = 0$, $|u| < \infty$

$C_2 = 0$

After solving for C_1, we have:

$$\sigma_{rr} = [(3+\nu)/8]\, \rho\omega^2 (b^2 - r^2)$$

$$\sigma_{\theta\theta} = [(3+\nu)/8]\, \rho b^2 \omega^2 - \frac{1+3\nu}{8} \rho \omega^2 r^2$$

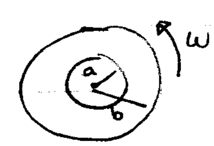
The displacement:

$$u(r) = \frac{1}{8E} \rho\omega^2 [(1-\nu)(3+\nu) b^2 r - (1-\nu^2) r^3]$$

The max normal stress occurs at the center of the solid disk:

$$\sigma_{rr,\max} = \sigma_{\theta\theta,\max} = \frac{3+\nu}{8} \rho b^2 \omega^2$$

Case 2: A disk with a hole and $T = 0$



At $r = a$, $\sigma_{rr} = 0$

At $r = b$, $\sigma_{rr} = 0$

$$\Rightarrow \sigma_{rr} = \frac{3+\nu}{8} \rho\omega^2 \left[b^2 + a^2 - \frac{a^2 b^2}{r^2} - r^2 \right]$$

$$\sigma_{\theta\theta} = \frac{3+\nu}{8} \rho\omega^2 \left[b^2 + a^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right]$$

$(\sigma_{\theta\theta} > \sigma_{rr})$

The maximum normal stress

For σ_{rr}: $\dfrac{d\sigma_{rr}}{dr} = 0$

$r = \sqrt{ab}$ and $\sigma_{rr,\max} = \frac{3+\nu}{8} \rho\omega^2 (b-a)^2$

For $\sigma_{\theta\theta}$:

Max $\sigma_{\theta\theta}$ occurs at $r = a$

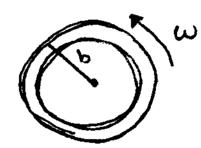
and $\sigma_{\theta\theta,max} = \left(\frac{3+\nu}{4}\right)\rho\omega^2\left(b^2 + \frac{r\nu}{3+\nu}a^2\right)$

Consider when $a \rightarrow 0$

$\sigma_{\theta\theta,max} \rightarrow \left(\frac{3+\nu}{4}\right)\rho\omega^2 b^2$

Special case : $a \xrightarrow[\text{approaches}]{} b$

$\sigma_{\theta\theta,max} \longrightarrow \rho(b\omega)^2$



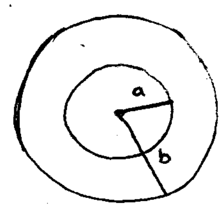
Example :

$a = 0.1\,m$

$b = 0.3\,m$

$E = 200\,GPa$

$\nu = 0.29$



$\rho = 7850\ kg/m^3$

$Y = 620\ MPa$

* max shear stress criterion

The disc is traction free at $r = a$, and $r = b$

$T = 0$

Find a) the max angular velocity ω

b) at the yield velocity, what is the change in thickness in the radial direction.

Solution : $\tau_{max} = \frac{\sigma_{\theta\theta,max}}{2} = \frac{Y}{2}$

$\Rightarrow \sigma_{\theta\theta,max} = Y$

$\Rightarrow \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right)\ Y$

$\Rightarrow \omega = 1020.77\ rad/s$

b) Since $u(r) = \frac{\rho\omega^2}{E}\left[\frac{(1-\nu)(3+\nu)}{8}(b^2+a^2)r + \frac{(1+\nu)(3+\nu)}{8}\frac{a^2b^2}{r} - \frac{r\nu^2}{8}r^3\right]$

At $r = a$

$u_a = u(a) = \frac{\rho\omega^2}{E}\cdot a\cdot\left[(1-\nu)a^2 + (3+\nu)b^2\right]$

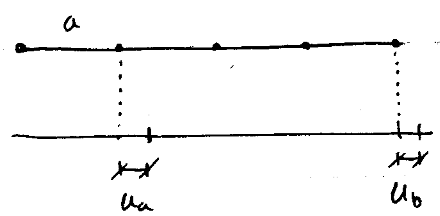
$= 0.0003100$

At $r = b$:

$$u_b = u(b)$$

$$= \frac{\rho \omega^2}{4E} \cdot b \cdot [(3+\nu)a^2 + (1-\nu)b^2)$$

$$= 0.0002969 \text{ m}$$



New inner radius $a' = a + u_a$

New outer radius $b' = b + u_b$

$\therefore$ New thickness in the radial direction

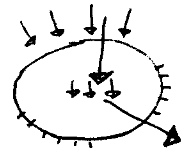
$$h' = b' - a' = (b + u_b) - (a + u_a)$$

$$= 0.2 + 0.0002969 - 0.0003100$$

$$= 0.1999999869 \text{ m}$$

Theory of Elasticity

Stress, strain



Disp: $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$

Stress: $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz}$ } 15 unknowns

Strain: $\epsilon_{xx}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yy}, \epsilon_{yz}, \epsilon_{zz}$

Strain Disp:

$$\begin{cases} \epsilon_{xx} = \partial u / \partial x \quad \cdots \\ \epsilon_{xy} = \tfrac{1}{2}\left(\partial u / \partial y + \partial v / \partial x \right) \quad \cdots \end{cases}$$

Hooke's Law:

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) \right] \quad \cdots$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \quad \cdots$$

Equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$

$$\vdots$$

Boundary conditions:

∂uP : $u = \bar{u}$, $v = \bar{v}$, $w = \bar{w}$

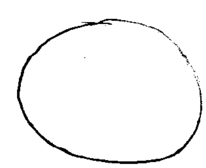
$\partial \sigma P$: $\sigma_{px} = \bar{f}_x$, $\sigma_{py} = \bar{f}_y$, $\sigma_{yz} = \bar{f}_z$

Compatability Conditions (6)

* Torsion of a general cross-section:

— Semi-inverse method

$u = -\theta yz$, $v = \theta xz$



CROSS-SECTION

Stress Function $\phi(x,y)$:

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = -2G\theta \end{cases} \qquad \begin{cases} \phi = \text{const along boundary} \\ {*}\, T = 2\iint_A \phi \, dA \end{cases}$$

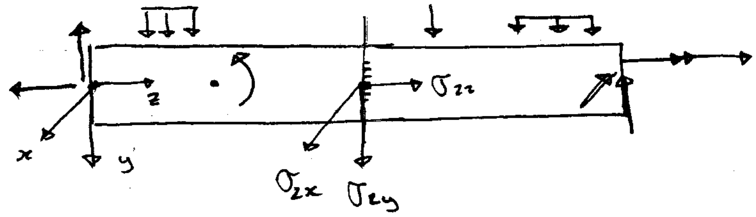
$\sigma_{xz} = \partial\phi / \partial y$

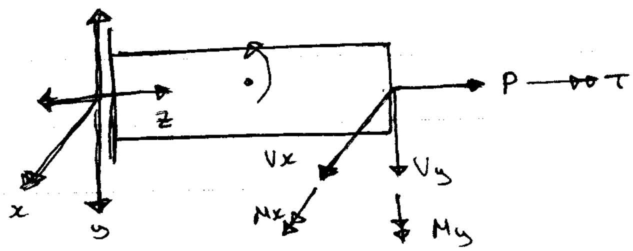
$\sigma_{yz} = -\partial\phi / \partial x$

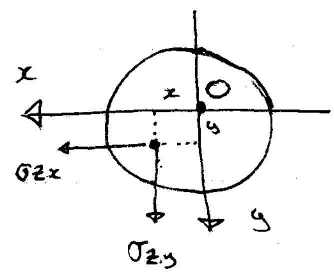
* Thick cylinder and rotating disk

- * σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz}
- * $u(r)$

Mechanics of Materials Method :







$\iint_A \sigma_{zz}\, dA = P$

$\iint_A \sigma_{zx}\, dA = V_x$

$\iint_A \sigma_{zy}\, dA = V_y$

$\iint_A y\,\sigma_{zz}\, dA = M_x$

$\iint_A x\,\sigma_{zz}\, dA = M_y$

$\iint_A (x\sigma_{zy} - y\sigma_{zx})\, dA = T$

* Thin-wall member

1° torsion → open
→ closed

2° bending