Advanced Mechanics of Materials (6th edition) Sept. 4/18

↳ Boresi / Schmidt Mechanics

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→ Course outlines to be emailed out

→ Midterm Oct. 23rd (Tuesday) – 35%

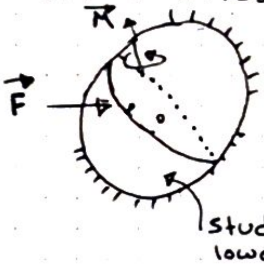
Final – 55%, Assignments – 10%

↳ Formula sheets provided

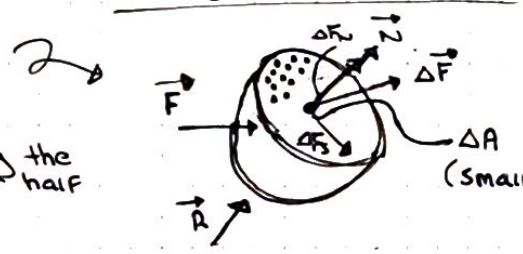
- Read Chapter 1 for review

→ Chapter 2 - Theories of Stress and Strain

2.1 → stress at a point



Method of sections



$\vec{N}$: unit normal vector

$\Delta\vec{F} = \Delta\vec{F}_N + \Delta\vec{F}_s$

normal, shear

Stress at the point:

$\vec{\sigma} = \lim_{\Delta A \to 0} \Delta\vec{F}/\Delta A$ (stress vector)

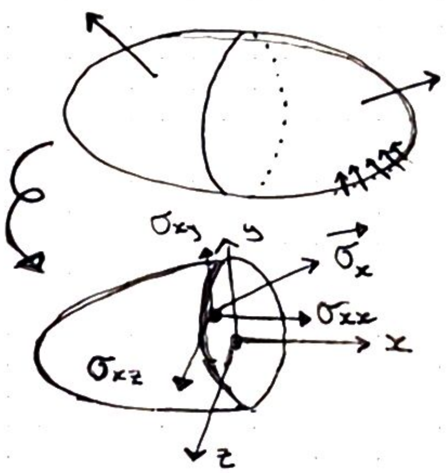
Normal stress:

$\vec{\sigma}_N = \lim_{\Delta A \to 0} \Delta F_N/\Delta A$

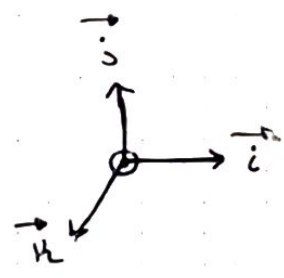
Shear stress:

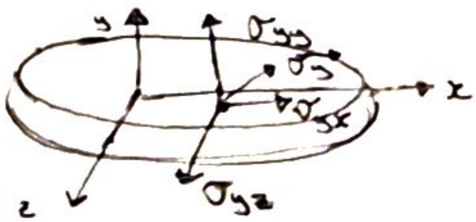
$\vec{\sigma}_s = \lim_{\Delta A \to 0} \Delta F_s/\Delta A$

$\Rightarrow \boxed{\vec{\sigma} = \vec{\sigma}_N + \vec{\sigma}_s}$



$\boxed{\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j} + \sigma_{xz}\vec{k}}$

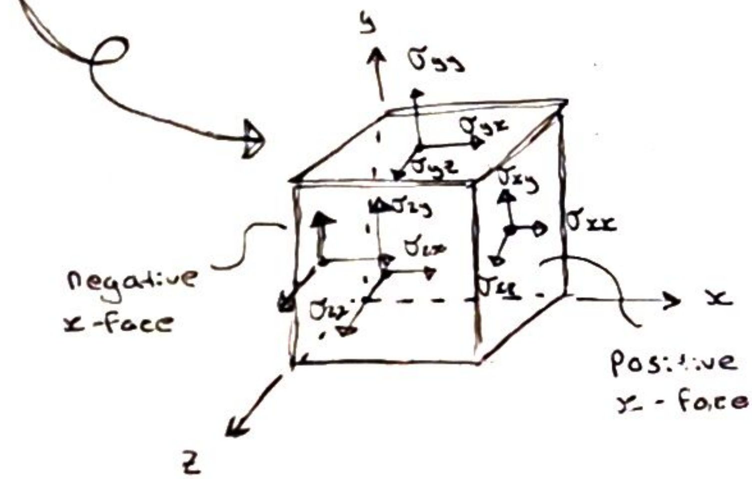




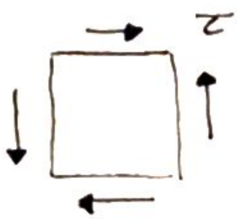
y : $\vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j} + \sigma_{yz}\vec{k}$

Z : $\vec{\sigma}_z = \sigma_{zx}\vec{i} + \sigma_{zy}\vec{j} + \sigma_{zz}\vec{k}$

(when subscripts are the same - normal stress
" different - shear stress)



→ 9 stress components at a point



Pure shear

$\sigma_{yx} = \sigma_{xy}$; $\sigma_{yz} = \sigma_{zy}$; $\sigma_{xz} = \sigma_{zx}$

Only 6 of these components are independent

$$[T] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Symmetric matrix

2.3 Symmetry of the stress array and stress on an arbitrarily oriented plane

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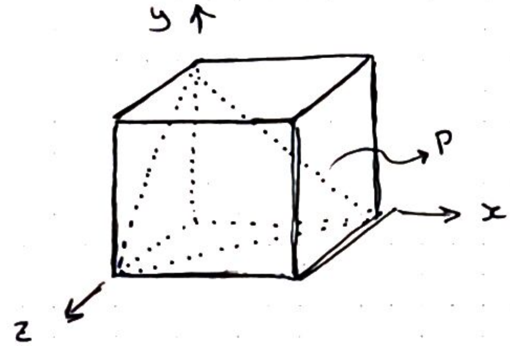

- Given $\vec{\sigma}_x, \vec{\sigma}_y, \vec{\sigma}_z$
- the normal of the inclined plane
  $\vec{N} = l\vec{i} + m\vec{j} + n\vec{k}$
- $l^2 + m^2 + n^2 = 1$

Find $\vec{\sigma}_p$ on the inclined plane

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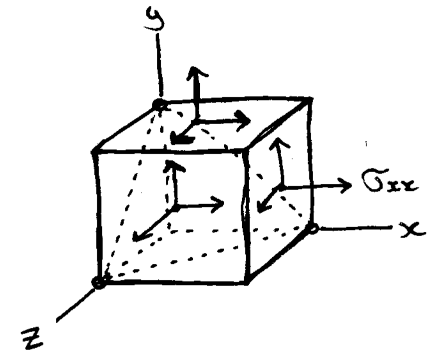
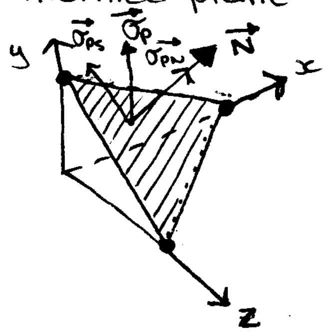


$$\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j} + \sigma_{xz}\vec{k}$$
$$\vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j} + \boxed{\sigma_{yz}\vec{k}}$$
$$\vec{\sigma}_z = \sigma_{zx}\vec{i} + \boxed{\sigma_{zy}\vec{j}} + \sigma_{zz}\vec{k}$$

Inclined plane :



unit normal vector

$$\vec{N} = l\vec{i} + m\vec{j} + n\vec{k}$$
$$l^2 + m^2 + n^2 = 1$$

stress vector $\vec{\sigma}_p$ = ?

$$\vec{\sigma}_p = \sigma_{px}\vec{i} + \sigma_{py}\vec{j} + \sigma_{pz}\vec{k}$$

Equilibrium of the small volume :

$$\begin{cases} \sigma_{px} = l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz} \\ \sigma_{py} = l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz} \\ \sigma_{pz} = l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz} \end{cases}$$

$$\begin{Bmatrix} \sigma_{px} \\ \sigma_{py} \\ \sigma_{pz} \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{Bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

Normal and shear stresses

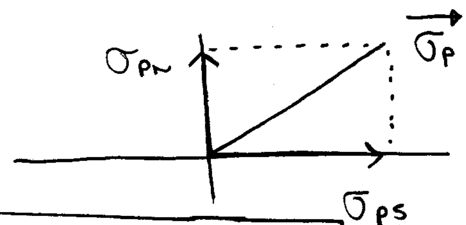
$$\sigma_{pN} = \vec{\sigma}_p \cdot \vec{N}$$
$$= (\sigma_{px}\vec{i} + \sigma_{py}\vec{j} + \sigma_{pz}\vec{k}) \cdot (l\vec{i} + m\vec{j} + n\vec{k})$$
$$= l\sigma_{px} + m\sigma_{py} + n\sigma_{pz} = \dots$$

$$\boxed{\sigma_{pN} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\sigma_{xy} + 2mn\sigma_{yz} + 2nl\sigma_{zx}}$$

OR $\sigma_{pN} = (l \;\; m \;\; n) \begin{Bmatrix} \sigma_{px} \\ \sigma_{py} \\ \sigma_{pz} \end{Bmatrix}$

$$\left(\sigma_{PN} = (l \;\; m \;\; n) \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} \right)$$

Shear Stress :



$$\boxed{\begin{aligned} \sigma_{PS} &= \sqrt{\sigma_P^2 - \sigma_{PN}^2} \\ &= \sqrt{\sigma_{Px}^2 - \sigma_{Py}^2 + \sigma_{Pz}^2 - \sigma_{PN}^2} \end{aligned}}$$

Direction of Shear stress :

$$\vec{\sigma_P} = \sigma_{PN}\vec{N} + \sigma_{PS}\vec{S}$$

2.4 Transformation of Stress

Principal Stresses

xyz - old $\qquad$ XYZ - new

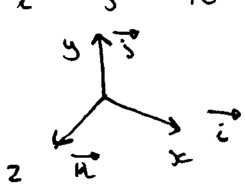
one point

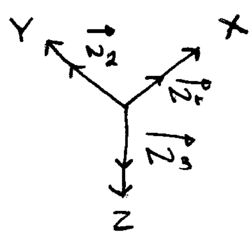
σ_{xx} σ_{yy} σ_{zz} $\qquad$ σ_{XX} σ_{YY} σ_{ZZ}

σ_{xy} σ_{yz} σ_{zx} $\qquad$ σ_{XY} σ_{YZ} σ_{ZX}

$\Downarrow$

$\vec{i}$ $\vec{j}$ $\vec{k}$





Define :
$$\begin{cases} \vec{N_1} = l_1\vec{i} + m_1\vec{j} + n_1\vec{k} \\ \vec{N_2} = l_2\vec{i} + m_2\vec{j} + n_2\vec{k} \\ \vec{N_3} = l_3\vec{i} + m_3\vec{j} + n_3\vec{k} \end{cases}$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_2^2 + m_2^2 + n_2^2 = 1 \qquad l_3^2 + m_3^2 + n_3^2 = 1$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$$

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

$$\therefore \quad \sigma_{XX} = l_1^2 \sigma_{xx} + m_1^2 \sigma_{yy} + n_1^2 \sigma_{zz} \cdots$$

$$\cdots + 2 l_1 m_1 \sigma_{xy} + 2 m_1 n_1 \sigma_{yz} + 2 n_1 l_1 \sigma_{zx}$$

$$\sigma_{YY} = l_2^2 \sigma_{xx} + m_2^2 \sigma_{yy} + n_2^2 \sigma_{zz} \cdots$$

$$\cdots + 2 l_2 m_2 \sigma_{xy} + 2 m_2 n_2 \sigma_{yz} + 2 n_2 l_2 \sigma_{zx}$$

$$\sigma_{ZZ} = l_3^2 \sigma_{xx} + m_3^2 \sigma_{yy} + n_3^2 \sigma_{zz} \cdots$$

$$\cdots + \circ \circ \circ$$

Stress vector on x-face

$$\vec{\sigma_X} = \sigma_{XX} \vec{N_1} + \sigma_{XY} \vec{N_2} + \sigma_{XZ} \vec{N_3}$$

$$\Rightarrow \begin{cases} \sigma_{XY} = \vec{\sigma_X} \cdot \vec{N_2} \\ \sigma_{XZ} = \vec{\sigma_X} \cdot \vec{N_3} \end{cases}$$

$$\therefore \quad \sigma_{XY} = l_1 l_2 \sigma_{xx} + m_1 m_2 \sigma_{yy} + n_1 n_2 \sigma_{zz} \cdots$$

$$\cdots + (l_1 m_2 + l_2 m_1) \sigma_{xy} + (m_2 n_1 + m_1 n_2) \sigma_{yz} \cdots$$

$$\cdots + (n_1 l_2 + n_2 l_1) \sigma_{zx}$$

$$\sigma_{YZ} = l_2 l_3 \sigma_{xx} + m_2 m_3 \sigma_{yy} + n_2 n_3 \sigma_{zz} \cdots$$

$$\cdots + (l_2 m_3 + l_3 m_2) \sigma_{xy} + (m_3 n_2 + m_2 n_3) \sigma_{yz} \cdots$$

$$\cdots + ((n_2 l_3 + n_3 l_2) \sigma_{zx}$$

$$\sigma_{ZX} = \circ \circ \circ$$

Principal stresses :

There exist three mutually perpendicular planes at the point on which the shear stresses <u>vanish</u>

The remaining three normal stresses components on these three planes are called principal stresses.

Here the principal plane has a unit normal vector:

$$\vec{N} = l\vec{i} + m\vec{j} + n\vec{k}$$

$$\Rightarrow \boxed{\vec{\sigma}_{p} = \sigma_{pN}\vec{N}} \boxed{= \sigma\vec{N}}$$

principal stress

$$\Rightarrow \sigma_{px}\vec{i} + \sigma_{py}\vec{j} + \sigma_{pz}\vec{k}$$

$$= \sigma(l\vec{i} + m\vec{j} + n\vec{k})$$

$$\begin{cases} \sigma_{px} = \sigma l \\ \sigma_{py} = \sigma m \\ \sigma_{pz} = \sigma n \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_{px} = l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz} = \sigma l \\ \sigma_{py} = l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz} = \sigma m \\ \sigma_{pz} = l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz} = \sigma n \end{cases}$$

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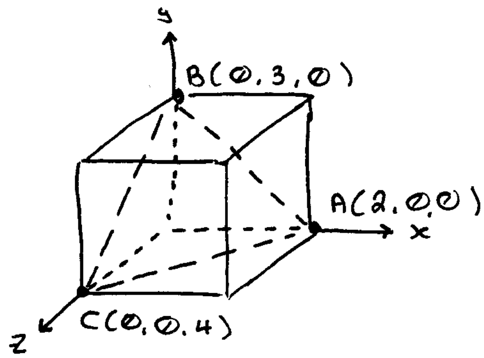
Example 1 -



Find the unit normal vector of the plane through A, B, and C.

Solution : $ax + by + cz = d$

The normal vector is (a, b, c)

A $(2,0,0)$ $\quad 2a + 0 + 0 = d$; $\quad a = d/2$

B $(0,3,0)$ $\quad 0 + 3b + 0 = d$; $\quad b = d/3$

C $(0,0,4)$ $\quad 0 + 0 + 4c + d$; $\quad c = d/4$

$\Rightarrow \quad \cancel{d}/2\, x + \cancel{d}/3\, y + \cancel{d}/4\, z = \cancel{d}$

$\Rightarrow \quad x/2 + y/3 + z/4 = 1$

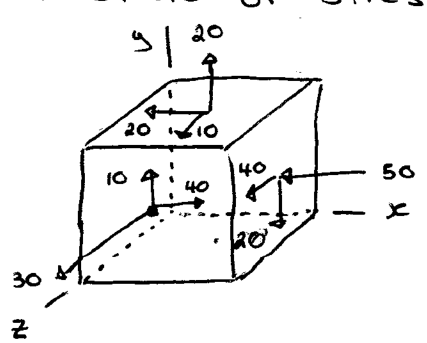
The normal vector $(1/2, 1/3, 1/4)$

Length $= \sqrt{(1/2)^2 + (1/3)^2 + (1/4)^2}$

$\therefore$ the unit normal vector is : $\left(\frac{1}{2L}, \frac{1}{3L}, \frac{1}{4L}\right)$

Example - The state of stress at a point is given by

unit : kPa



Determine the stress vector on the plane with unit normal vectors :

$\vec{N} = 1/\sqrt{2}\,\vec{i} + 1/2\,\vec{j} + 1/2\,\vec{k}$

Also find the normal stress and shear stress

Solution : $\sigma_{xx} = -50 \quad \sigma_{xy} = -20 \quad \sigma_{xz} = 40$

$\sigma_{yx} = -20 \quad \sigma_{yy} = 20 \quad \sigma_{yz} = 10$

$\sigma_{zx} = 40 \quad \sigma_{zy} = 10 \quad \sigma_{zz} = 30$

and : $l = 1/\sqrt{2} \quad m = 1/2 \quad n = 1/2$

Stress vector : $\vec{\sigma_p} = \sigma_{px}\vec{i} + \sigma_{py}\vec{j} + \sigma_{pz}\vec{k}$

Here

$$\sigma_{px} = l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz}$$
$$= (1/\sqrt{2})(-50) + (1/2)(-20) + (1/2)(40)$$
$$= -25.36$$

$$\sigma_{py} = l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz}$$
$$= (1/\sqrt{2})(-20) + (1/2)(20) + (1/2)(10)$$
$$= 0.8579$$

$$\sigma_{pz} = l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz}$$
$$= (1/\sqrt{2})(40) + (1/2)(10) + (1/2)(30)$$
$$= 48.78$$

$$\therefore \vec{\sigma_p} = -25.36\vec{i} + 0.8579\vec{j} + 48.78\vec{k}$$

Normal stress

$$\sigma_{pn} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} \cdots$$
$$\cdots + 2lm\sigma_{xy} + 2mn\sigma_{yz} + 2nl\sigma_{zx}$$
$$= (1/\sqrt{2})^2(-50) + (1/2)^2(20) + (1/2)^2(30) \cdots$$
$$\cdots + (2)(1/\sqrt{2})(1/2)(-20) + (2)(1/2)(1/2)(10) \cdots$$
$$\cdots (2)(1/2)(1/\sqrt{2})(40)$$
$$= \cdots$$

Shear stress :

$$\sigma_{ps} = \sqrt{\sigma_{px}^2 + \sigma_{py}^2 + \sigma_{pz}^2 - \sigma_{pn}^2}$$
$$= \cdots$$

$$\vec{\sigma}_p = \sigma_{px}\,\vec{i} + \sigma_{py}\,\vec{j} + \sigma_{pz}\,\vec{k}$$

$$\Updownarrow$$

$$\vec{\sigma}_p = \sigma_{pN}\,\vec{N} + \sigma_{ps}\,\vec{S}$$

✓ ✓ ✓ ✓

$$\Rightarrow \quad \vec{S} = 1/\sigma_{ps}\left(\vec{\sigma}_p - \sigma_{pN}\vec{N}\right)$$

$\hookrightarrow$ direction of stress vector in the plane

Verify $\left.\begin{array}{l} |\vec{S}| = 1 \\ \vec{S}\cdot\vec{N} = 0 \end{array}\right\}$