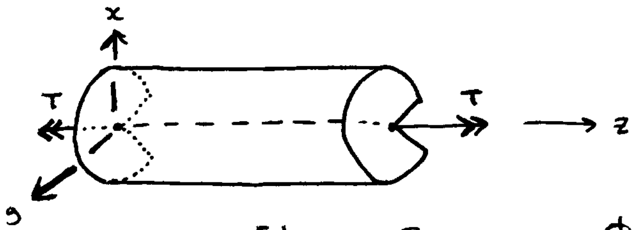
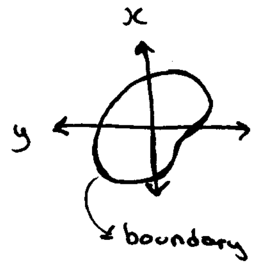


Stress Function $\phi(x,y)$

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = -2G\theta \\ \phi = 0 \quad \text{on the boundary} \end{cases}$$

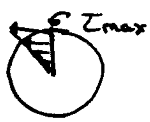


Stress
$$\begin{cases} \sigma_{xz} = \dfrac{\partial \phi}{\partial y} \\ \sigma_{yz} = \dfrac{-\partial \phi}{\partial x} \end{cases}$$

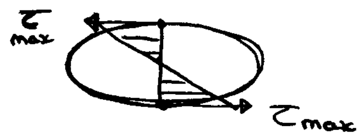
$$T = 2\iint_A \phi \, dx\,dy$$

$$\theta = T/GJ$$

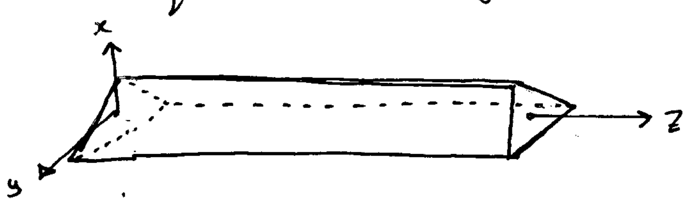
For a circle:

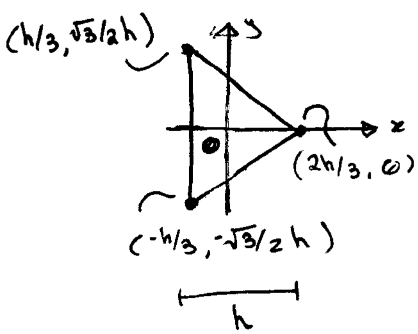


For an elipse:



6.32 - Equilateral Triangle Cross-Section



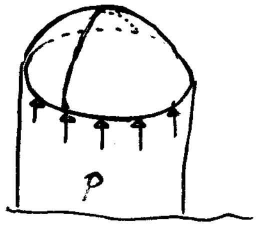


$$\phi = \frac{G\theta}{2h}\left(x - \sqrt{3}y - 2h/3\right) \cdot \left(\left(x + \sqrt{3}y - 2h/3\right)\right) \cdot \left(x + h/3\right)$$

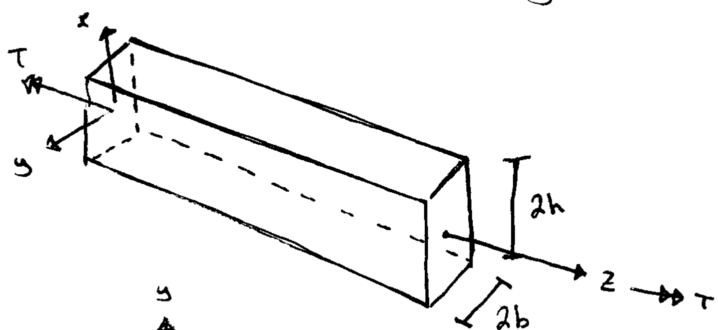
Define $J = \dfrac{h^4}{15\sqrt{3}}$

$$\tau_{max} = \frac{15\sqrt{3}}{2h^3}T \qquad \theta = \frac{T}{GJ}$$

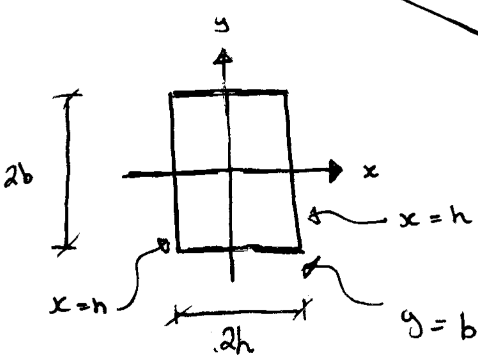
6.4 - The Prandtl elastic-membrane analogy



6.6 - Torsion of rectangular cross-section



$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = -2G\theta \\ \phi = 0 \text{ at } \begin{cases} x = h \\ x = -h \\ y = b \\ y = -b \end{cases} \end{cases}$$



$$\Rightarrow \quad \phi = G\theta(h^2 - x^2) - \frac{32G\theta h^2}{\pi^3} \cdot \frac{\sum_{n=1,3,\dots} (-1)^{\frac{n-1}{2}} \cos\left(\frac{n\pi x}{2h}\right) \cosh\left(\frac{n\pi y}{2b}\right)}{n^3 \cosh\left(\frac{n\pi b}{2h}\right)}$$

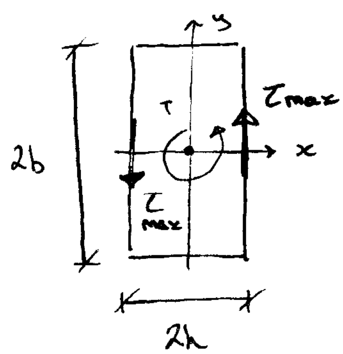
Define $J = k_1 (2h)^3 (2b)$

Then $\theta = T / GJ$

Max shear stress $(b > h)$

$\tau_{max} = G\theta \cdot (2h) \cdot (k_1 / k_2)$

The max. shear occurs at the location with the shortest distance to centre (at $x = \pm h$, $y = 0$)



b/h	1.0	2.0	... ∞
k_1	0.141	0.229	... 0.333
k_2	0.208	0.246	0.333

$b/h \geq 3$

$k_1 \approx k_2 = 1/3 - 0.210\, h/b$

6.5 Narrow rectangular cross-section

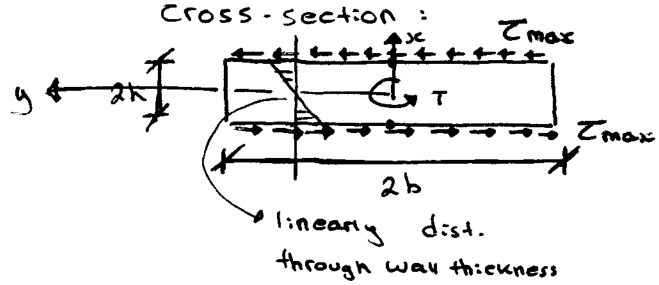
$b \geq 10h$ $\qquad\qquad k_1 = k_2 = 1/3$

$$J = 1/2\,(2h)^3(2b)$$

$$\theta = T/GJ$$

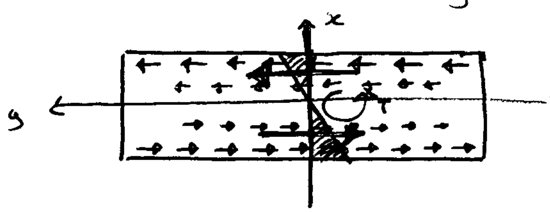
$$\tau_{max} = G\theta \cdot 2h \cdot k_1/k_2 = G\theta \cdot 2h$$
$$= 2G\theta h$$

Cross-section :



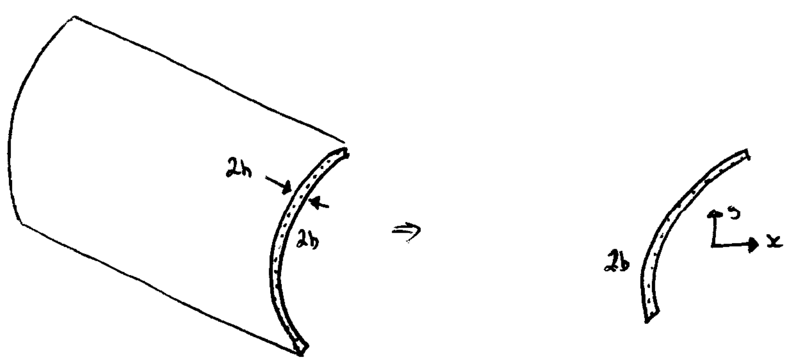
Shear stresses

$\sigma_{zx} = 0 \qquad \sigma_{zy} = 2G\theta x$

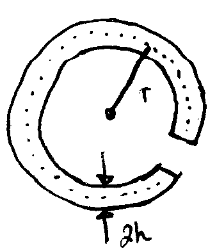


Resultant of the shear stress through the wall thickness is zero

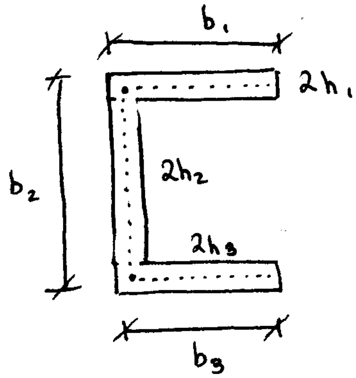
Summation of shear stress (moment) $= T/2$



$$J = 1/3\,(2h)^3(2b)$$

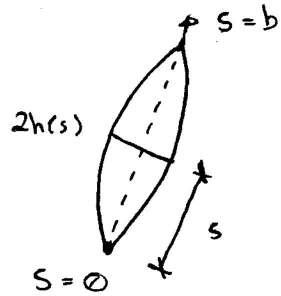


$$2b = 2\pi r$$
$$J = (1/3)(2\pi r)(2h)^3$$



$$J = \tfrac{1}{3}(2h_1)^3(b_1) + (\tfrac{1}{3})(2h_2)^3(b_2) + (\tfrac{1}{3})(2h_3)^3(b_3)$$

$$\tau_{max,1} = 2G\theta h_1$$
$$\tau_{max,2} = 2G\theta h_2$$
$$\tau_{max,3} = 2G\theta h_3$$
$$\therefore \quad \tau_{max} = 2G\theta \cdot h_{max}$$



$$J = \tfrac{1}{3}\int_0^b (2h)^3\, ds$$

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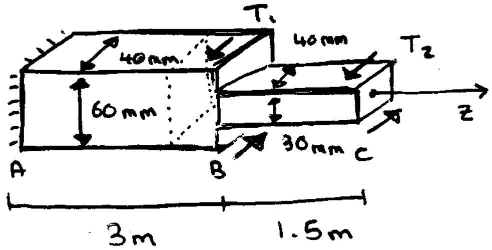
Example :



$T_1 = 750 \text{ N}\cdot\text{m}$

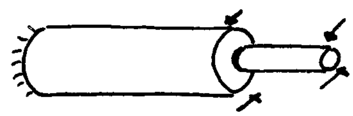
$T_2 = 400 \text{ N}\cdot\text{m}$

$G = 77.5 \text{ GPa}$

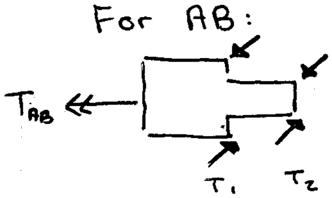
$= 77500 \text{ N/mm}^2$

Find τ_{max} and angle of twist of the free end.

Solution :



For AB:

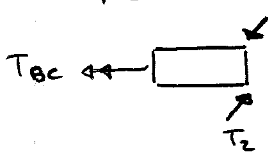


$\sum M_z = 0$

$T_1 + T_2 - T_{AB} = 0$

$T_{AB} = T_1 + T_2 \Rightarrow T_{AB} = 1150 \text{ N}\cdot\text{m}$

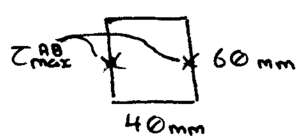
For BC:



$\sum M_z = 0$

$T_{BC} = T_2 \Rightarrow T_{BC} = 400 \text{ N}\cdot\text{m}$

For AB:



$$\left.\begin{array}{l} 2b = 60 \\ 2h = 40 \end{array}\right\} \frac{b}{h} = 1.5$$

$$\left.\begin{array}{l} 1\text{m} = 10^3 \text{mm} \\ 1\text{N/m}^2 = 1 \text{ Pa} \end{array}\right\} 1 \text{ N/mm}^2 = 1 \text{MPa}$$

From table $K_1 = 0.196$

$K_2 = 0.231$

$$\theta_{AB} = \frac{T_{AB}}{GJ_{AB}} = \frac{1150 \times 10^3}{(77.5)(\times 10^3)(752640)}$$

$$= 1.9716(10^{-5})$$

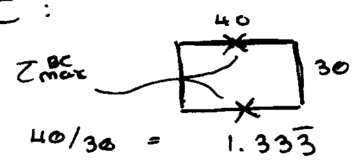
$$J_{AB} = K_1(2b)(2h)^3$$

$$= (0.196)(60)(40)^3$$

$$= 752640 \text{ mm}^4$$

$$\tau_{max}^{AB} = \frac{T_{AB}}{K_2(2b)(2h)^3} = \frac{1150 \times 10^3}{(0.231)(60)(40)^2} = 51.9 \text{ MPa}$$

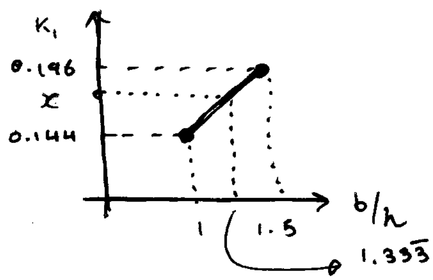
For BC :



$$2b = 40$$
$$2h = 30$$

$$b/h = 40/30 = 1.33\bar{3}$$

$$K_1 = k_1(b/h)$$



Assuming linearly distributed.

$$\frac{0.196 - 0.144}{1.5 - 1} = \frac{x - 0.141}{1.333 - 1}$$

$$x = 0.1776 = K_1$$

$$K_2 = 0.2233$$

$$J_{BC} = K_1(2b)(2h)^3$$
$$= (0.1776)(40)(30)^3$$
$$= 19180.8 \text{ mm}^4$$

$$\therefore \theta_{BC} = \frac{T_{BC}}{GJ_{BC}} = \frac{400(10^3)}{(77.5\times10^3)(19180.8)}$$
$$= 2.6909(10^{-5}) \text{ rad/mm}$$

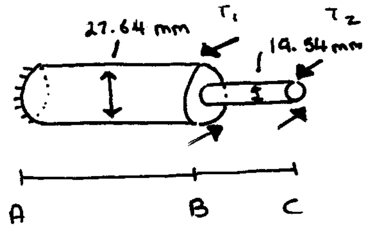
$$\tau^{BC}_{max} = \frac{T_{BC}}{K_2(2b)(2h)^2} = \ldots = 49.8 \text{ MPa}$$

$$\therefore \tau_{max} = 51.9 \text{ MPa}$$

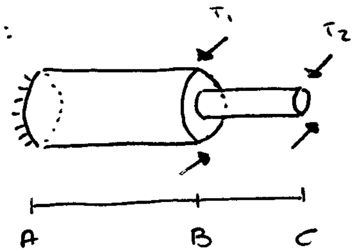
it occurs in the AB segment.

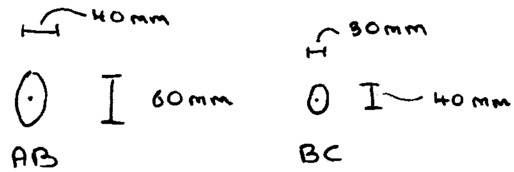
$$\beta_{C/A} = \beta_{C/B} + \beta_{B/A}$$
$$\beta_{C/A} = (BC \cdot \theta_{BC}) + (AB \cdot \theta_{AB})$$
$$\Rightarrow (1500)(2.6909\times10^{-5}) + (3000)(9.9512\times10^{-6}) \text{ rad}$$

Case 1 :



Case 2 :





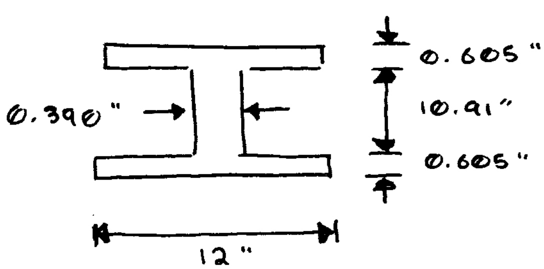
Example : For the given I - beam :

a) Find the torsional constant J

b) Find the maximum torque that the beam can take if the yield shear stress is $\tau_y = 36$ ksi

Given $G = 12 \times 10^3$ ksi

→ a)



$$J = \left(\tfrac{1}{3}\right)(2b)(2h)^3$$

$$\rightarrow \; J = \left(\tfrac{1}{3}\right)(12)(0.605)^3 \times 2 \; \cdots$$

$$\cdots + \left(\tfrac{1}{3}\right)(10.91)(0.390)^3$$

$$J = 1.99 \text{ in}^4$$

→ b) $$\tau_{max} = \frac{T}{J}(2h)_{max}$$

$$\Rightarrow \tau_{max} = \frac{T}{J}(0.605)$$

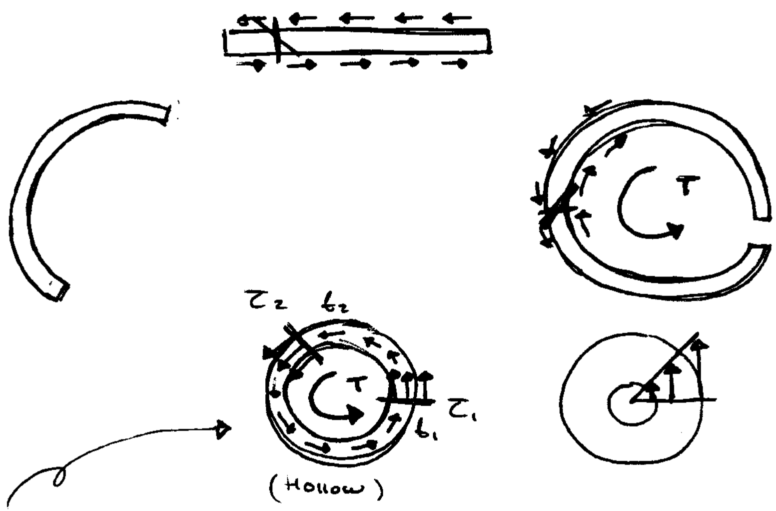
Since $\tau_{max} \leq \tau_y$

$$\Rightarrow \frac{\tau_{max}}{J}(0.605) = \tau_y$$

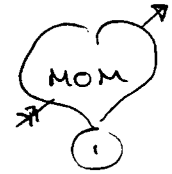
$$\Rightarrow \tau_{max} = \left(\frac{36 \times 1.99}{0.605}\right) \text{ kip-in}$$

6.7 Hollow thin-wall torsion members and multiply connected cross-section.

Narrow rectangular cross-section.



- Shear stress is practically constant through the wall thickness.
- Shear stress is parallel to the boundary of the section.
- $q = \tau t$ = Shear Flow
- q = const.



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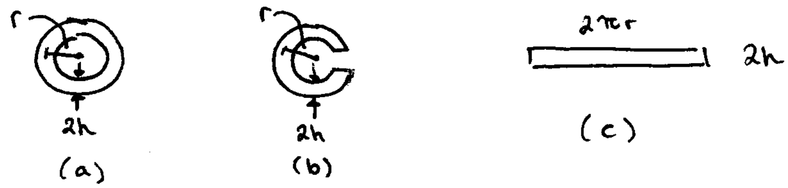
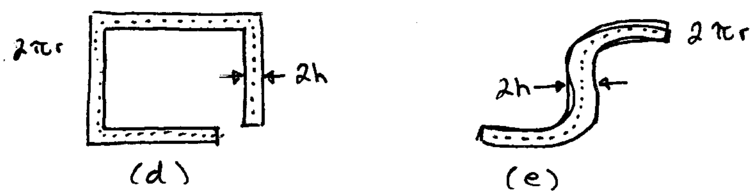
Example:



→ three thin wall members, Find the ratio of the largest Shear stress and the ratio of the angle of twist per unit length.

Solution: b and c are the same

$$I_b = \tfrac{1}{3}(2\pi r)(2h)^3$$



For (a) outer radius: $r+h$

inner radius: $r-h$

$$J_a = \frac{\pi}{2}\left[(r+h)^4 - (r-h)^4\right]$$

$$\Rightarrow \pi/2\left[(r^4+4r^3+6r^2h^2+4rh^3+h^4) - (r^4-4r^3h+6r^2h^2-4rh^3+h^4)\right]$$

$$\Rightarrow (\pi/2)\left[8r^3h + 8rh^3\right]$$

$$\approx 4\pi r^3 h$$

$$\tau^a_{max} = \frac{T}{J_a}\cdot(r+\cancel{h}) \;\; \text{0 neglect} \;\; = \left(\frac{T}{4\pi r^3 h}\cdot r\right) = \frac{T}{4\pi r^2 h}$$

$$\tau^b_{max} = T/J_b\cdot(2h) = \left(\frac{T}{(1/3)2\pi r(2h)^3}\right) = \frac{3T}{8\pi r h^2}$$

$$\frac{\tau^a_{max}}{\tau^b_{max}} = \left(\frac{T}{4\pi r^2 h}\right)\left(\frac{8\pi r h^2}{3T}\right) \Rightarrow \left(\frac{2}{3}\cdot\frac{h}{r}\right)$$

$\theta = \frac{T}{GJ}$ $\rightarrow$ where T, G don't change.

$$\therefore \frac{\theta_a}{\theta_b} = \frac{J_b}{J_a} = \frac{(\frac{1}{3})\, 2\pi r\,(2h)^3}{4\pi r^3 h} = \frac{4}{3}\left(\frac{h}{r}\right)^2$$

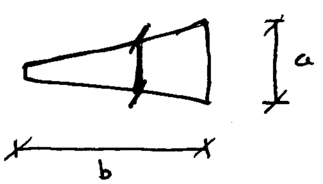
A special case:

$r = 400$ mm $\qquad$ $2h = 30$ mm

$$\frac{\tau^a_{max}}{\tau^b_{max}} = \frac{2}{3}\cdot\left(\frac{15}{400}\right) = \frac{1}{40}$$

$$\frac{\theta_a}{\theta_b} = \frac{4}{3}\left(\frac{15}{400}\right)^2 = \frac{2}{1067}$$

Example

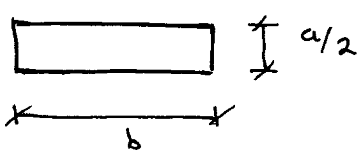


$b \gg a$

a) Find the max shear stress in terms of:

T, a, b, G

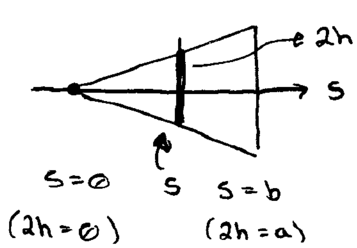
b)



what are the percentage errors of τ_{max} and θ if using $J = (1/3)(b)(a/2)^3 = (\frac{1}{24})a^3 b$

Solution: a)

$$J = 1/3 \int_0^b (2h)^3\, ds$$



$$\frac{s}{b} = \frac{2h}{a} \Rightarrow 2h = \frac{a}{b}\cdot s$$

$$\Rightarrow J = 1/3 \int_0^b \left(\frac{a}{b} s\right)^3 ds$$

$$= \frac{1}{3}\, a^3/b^3 \int_0^b s^3\, ds$$

$$= (1/3)(a^3/b^3)(1/4)(b^4) = 1/12\, a^3 b$$

$$\Rightarrow \tau_{max} = \frac{T}{J}(2h)_{max}$$

$$\Rightarrow \tau_{max} = T/J\,(2h)_{max}$$

$$= \frac{T}{(\frac{1}{12})a^3 b} \cdot a = \frac{12T}{a^2 b}$$

$$\theta = \frac{T}{GJ} = \frac{T}{G(\frac{1}{12})a^3 b} = \frac{12T}{Ga^3 b}$$

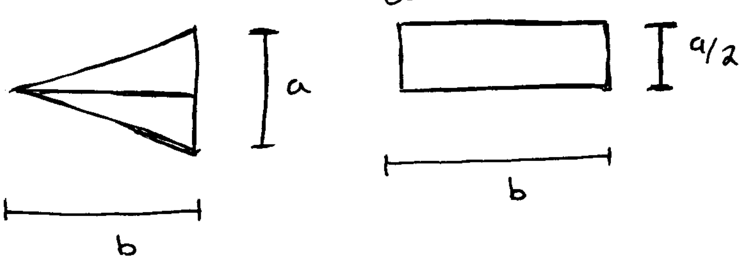
b) $J = (\frac{1}{12})a^3 b$

$$J_{rec} = (\tfrac{1}{24})a^3 b$$

$$\tau^{rec}_{max} = \frac{T}{J_{rec}} \cdot (2h)_{max}$$

$$= \frac{T}{(1/24)a^3 b} \cdot \left(\frac{a}{2}\right)$$

$$= \frac{12T}{a^2 b}$$



$\theta = ?$

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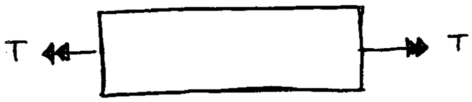
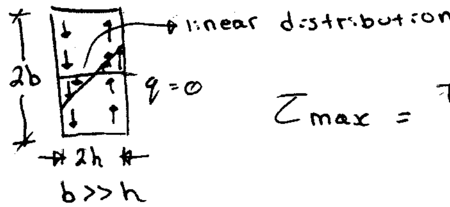
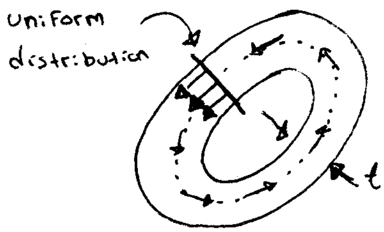




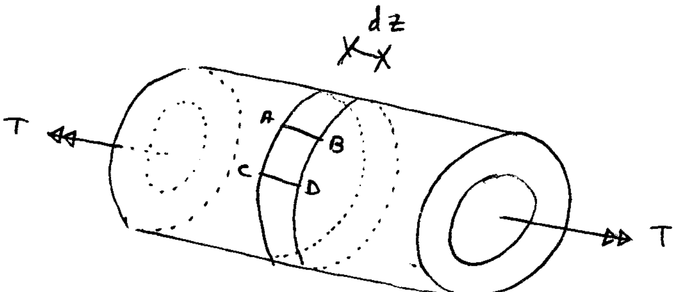
$\tau_{max} = T/J\,(2H)_{max}$

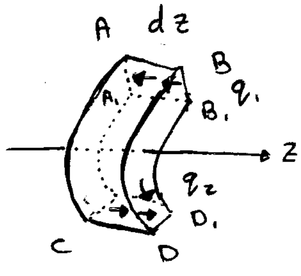


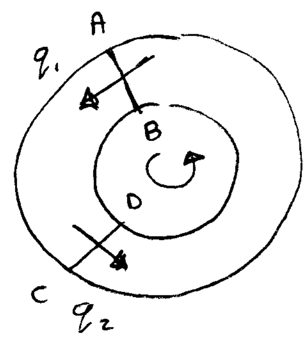
thin-wall cylinder
(closed)

$q = \tau t$ (shear flow)

q = const. (along the cross section)







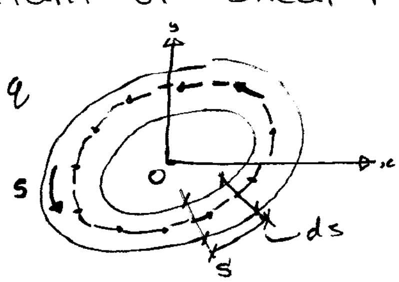
$\Sigma F_z = 0$:

$-q_1 dz + q_2 dz \Rightarrow q_1 = q_2$

Resultant of Shear Flow over the cross-section

Internal Force :

T

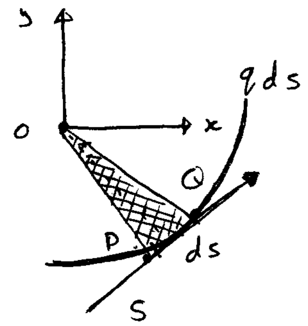


Internal stress

τ

$\Sigma F_x = 0$, $\Sigma F_y = 0$

$\Sigma M_O = T$

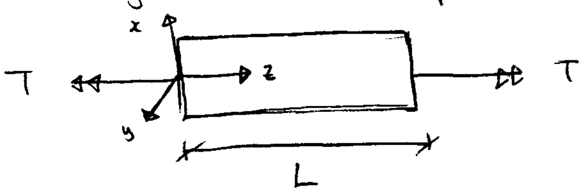


$$d\,q\,ds = q\cdot d\cdot ds$$
$$= q\cdot 2S_{\triangle OPQ}$$
$$\Sigma M_o = q\cdot 2A = T$$

A: the area enclosed by the mean perimeter of the cross-section

$$\boxed{q = \tau t = \frac{T}{2A}}$$

Angle of twist per unit length θ



rotation between end sections

$$\beta = L\theta$$

work done is: $(\frac{1}{2})T\beta = (\frac{1}{2})TL\theta$

stress components : $\sigma_{zx} \neq 0$, $\sigma_{zy} \neq 0$

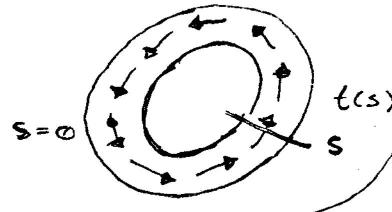
Strain energy density :

$$U_0 = \frac{1}{2E}\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2\right) - \frac{\nu}{E}\left(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right) \cdots$$
$$\cdots + \frac{1}{2G}\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2\right)$$
$$\Rightarrow \frac{1}{2G}\left(\sigma_{xz}^2 + \sigma_{yz}^2\right)$$
$$\Rightarrow \frac{1}{2G}\tau^2$$

Strain energy of the torsional member

$$U = \iiint_V U_0\,dV = \iint_A L\,U_0\,dA$$

$$= L\int\int U_0\,dt$$
$$= L\oint\left(\int_0^{t(s)} U_0\,dt\right)ds$$
$$= L\oint\left(\int_0^{t} \frac{\tau^2}{2G}\,dt\right)ds$$



$$= L\oint \frac{\tau^2}{2G}\,t\,ds$$
$$= L\oint \frac{\tau q}{2G}\,ds$$
$$= \frac{Lq}{2G}\oint \tau\,ds$$

$$\frac{1}{2} T \cancel{L} \theta = \frac{\cancel{L} q}{2G} \oint \tau \, ds$$

Since $q = \frac{T}{2A}$ $\Rightarrow$ $\theta = \frac{q}{GT} \oint \tau \, ds$

$$\boxed{\theta = \frac{1}{2GA} \oint \tau \, ds}$$ (Angle of twist)

Since $\tau = \frac{q}{t}$

$$\Rightarrow \theta = \frac{1}{2GA} \oint q/t \, ds = \frac{q}{2GA} \oint {}^{1}/_{t} \, ds$$

$$\theta = \frac{T}{4GA} \oint \frac{1}{t} \, ds$$

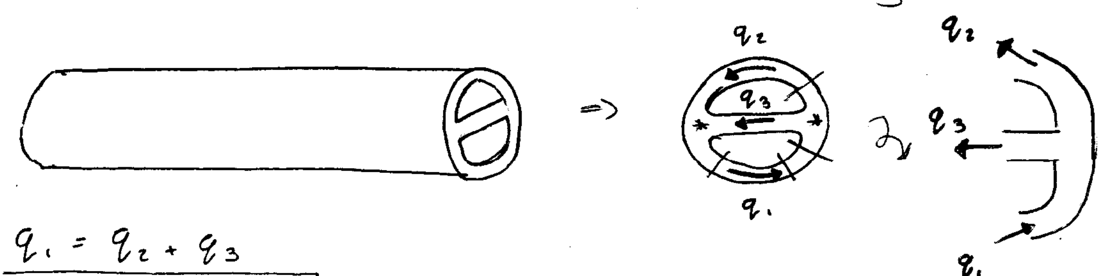
Define $J = \dfrac{4A^2}{\oint {}^{1}/_{t} \, ds}$

$$\Rightarrow \theta = \frac{T}{GJ}$$

Special Case: t = const.

$\oint {}^{1}/_{t} \, ds = {}^{s}/_{t}$ $\quad$ s: the wall mean perimeter length

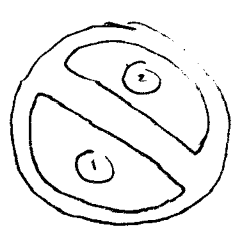
6.7.1 Hollow thin-wall member having several compartments



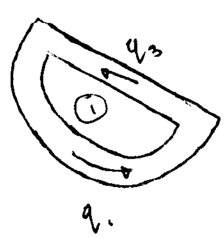
$q_1 = q_2 + q_3$

$$\boxed{\therefore q_3 = q_1 - q_2}$$

$$\theta = \frac{1}{2GA} \oint \tau \, ds$$



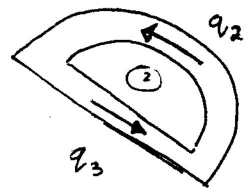
Compartment 1



Compartment 2:

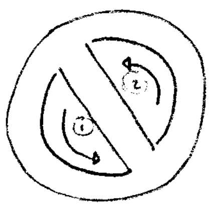
$\theta = \frac{1}{2GA} \oint \tau \, ds$

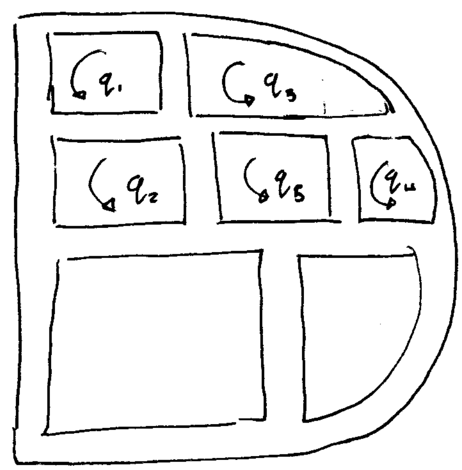
θ must be the same for either compartment.



Internal resultant:

$T = 2A_1 q_1 + 2A_2 q_2$





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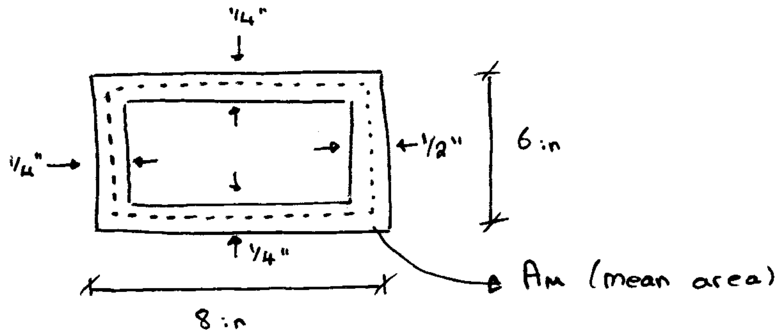
Example:



$T = 50$ kip-in

Find the max shear stress developed in the cross-section

Also find the effective polar moment of inertia.

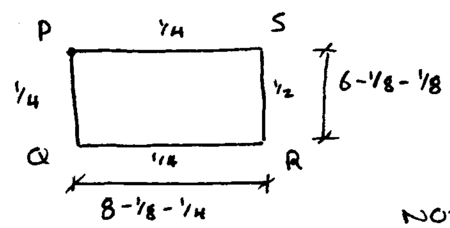
Solution: $q = \frac{T}{2A}$ (mean area)

$A = (8 - 1/8 - 1/4) \times (6 - 1/8 - 1/8) = 43 \text{ in}^2$

$\therefore q = \dfrac{50 \text{ kip-in}}{(2)(43 \text{ in}^2)} \rightarrow q = \tau \cdot t \quad \therefore \tau_{max} = \dfrac{q}{t_{min}}$

$\therefore \tau_{max} = \dfrac{50}{(2)(43)}\left(\dfrac{1}{1/4}\right) = 2.28 \text{ ksi}$

$J = \dfrac{4A^2}{\oint \frac{1}{t} ds}$



$\oint 1/t \, ds = \int_{SPQR} 1/t \, ds + \int_{RS} 1/t \, ds$

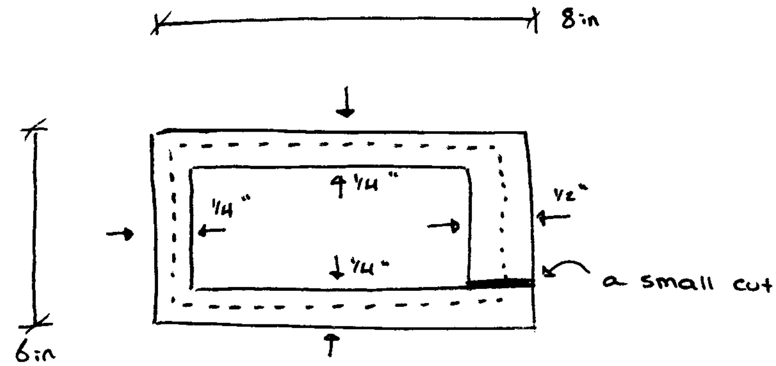
NOTE:
$\int_{SP} 1/t \, ds = 1/t \int_{SP} ds = SP/t$

$= 1/(1/4)\,(SP + PQ + QR) + 1/(1/2)\,(RS)$

t

$\Rightarrow 4((8 - 1/8 - 1/4) + (6 - 1/8 - 1/8) + (8 - 1/8 - 1/4)) + 2(6 - 1/8 - 1/8) = 95.5$

and $J = \dfrac{4A^2}{\oint \frac{1}{t} ds} = \dfrac{4 \times 43.8^2}{95.5} = 80.354 \text{ in}^2$



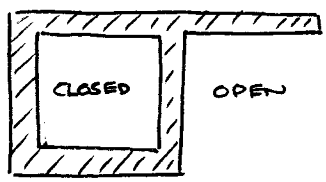
For the open section, find J

$$J = \Sigma\, \tfrac{1}{3}(2b)(2h)^3$$

$$= \left(\tfrac{1}{3}\right)\left[\left(8 - \tfrac{1}{8} - \tfrac{1}{4}\right) + \left(6 - \tfrac{1}{8} - \tfrac{1}{8}\right) + \left(8 - \tfrac{1}{8} - \tfrac{1}{4}\right)\right] \times \left(\tfrac{1}{4}\right)^3 \cdots$$

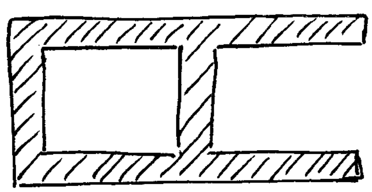
$$\cdots + \left(\tfrac{1}{3}\right)\left(6 - \tfrac{1}{8} - \tfrac{1}{8}\right)\left(\tfrac{1}{2}\right)^3$$

$$J = 0.34896 \text{ in}^4$$



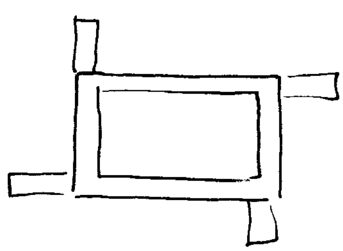
$$J = J_{close} + J_{open}$$

$$= \frac{4A^2}{\oint \frac{1}{t}\, ds} + \tfrac{1}{3}(2b)(2h)^3$$



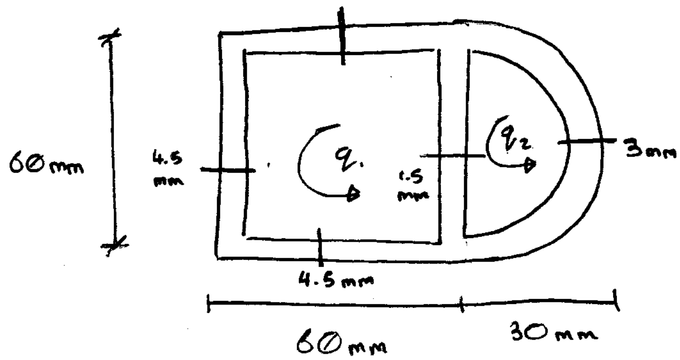
$$J = J_{close} + J_{open_1} + J_{open_2}$$

$$= \frac{4A^2}{\oint \frac{1}{t}\, ds} + \tfrac{1}{3}(2b)(2h)^3 + \tfrac{1}{3}(2b)(2h)^3$$



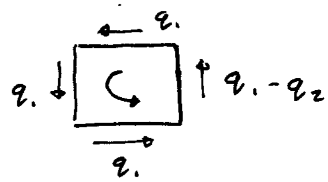
Same thing.

Example :



Given $G = 26.0$ GPa the max shear stress is 40 MPa, Find the max T the member can support.

Cell 1 :

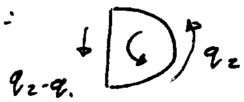


$$\theta_1 = \frac{1}{2GA_1} \oint \tau\, ds = \frac{1}{2GA_1} \oint \frac{q}{t}\, ds$$

Since $A_1 = 60^2 = 3600$

$$\theta = \frac{1}{2G(3600)} \left[\left(\frac{q_1}{4.5} \times 3 \times 60 \right) + \left(\frac{q_1 - q_2}{1.5} \times 60 \right) \right]$$

Cell 2 :



$$A_2 = \tfrac{1}{2} \pi r^2 = (\tfrac{1}{2}) \pi (30)^2 = 450\pi$$

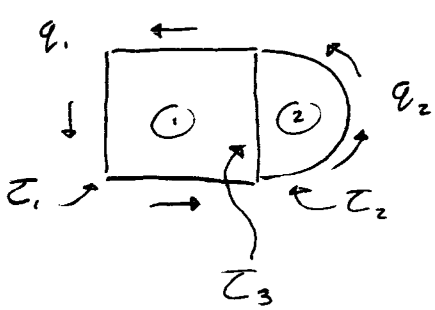
$$\theta_2 = \frac{1}{2GA_2} \oint \frac{q}{t}\, ds = \frac{1}{2G(450\pi)} \left[\frac{q_2}{3} \pi (30) + \frac{q_2 - q_1}{1.5} (60) \right]$$

Since its a rigid body, $\theta_1 = \theta_2$

$$\frac{1}{2G(3600)} \left[\frac{q_1}{4.5} \times 180 + \frac{q_1 - q_2}{1.5} \times 60 \right]$$

$$\Rightarrow \frac{1}{2G(450\pi)} \left[\frac{q_2}{3} \times 30\pi + \frac{q_1 - q_2}{1.5} (60) \right]$$

$$\Rightarrow q_1 = 1.220\, q_2$$



$$\tau_1 = \frac{q_1}{4.5} \qquad \tau_2 = \frac{q_2}{3.0}$$

$$\tau_3 = \frac{q_1 - q_2}{1.5}$$

$$\tau_{max} = \max(\tau_1, \tau_2, \tau_3) = 40$$

$$T = 2A_1 q_1 + 2A_2 q_2$$

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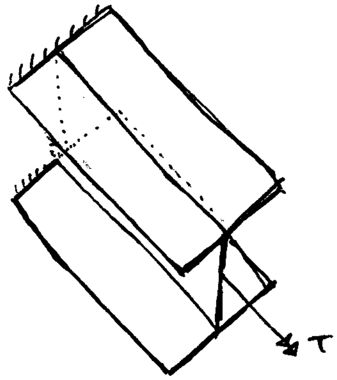
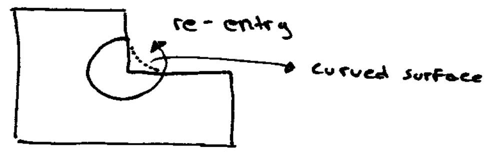
Torsion member with restrained ends.

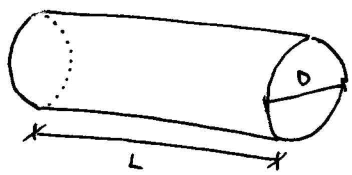


Shear concentration:



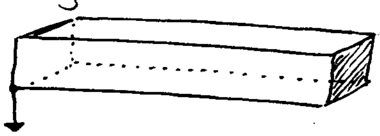
Ch. 7 - Bending of Straight Beams

7.1 - Fundamentals of beam bending



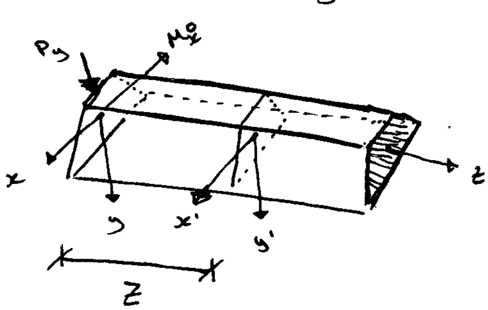
$L/D \geq 5$

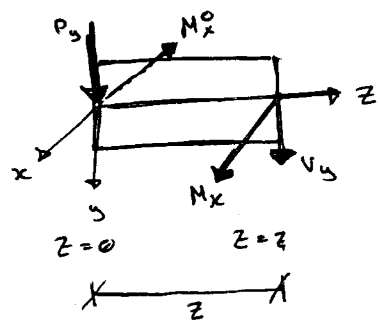
Homogeneous and isotropic (material assumption)



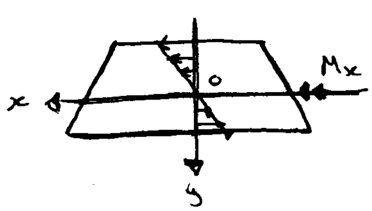
no twisting (only bending deformation)

Beam has a symmetrical plane

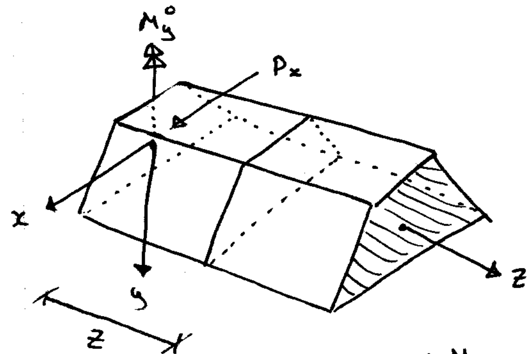


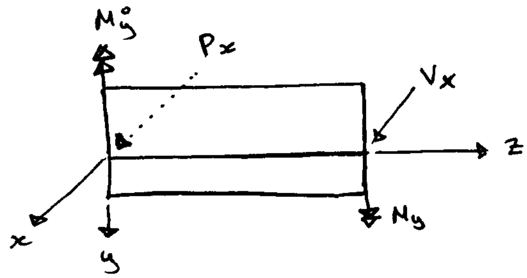


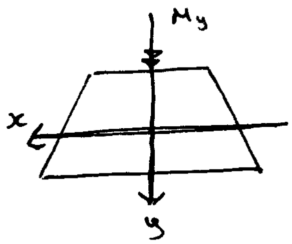
$$\sigma_{zz} = \frac{M_x y}{I_x}$$



$$I_x = \iint_A y^2 \, dx\, dy$$







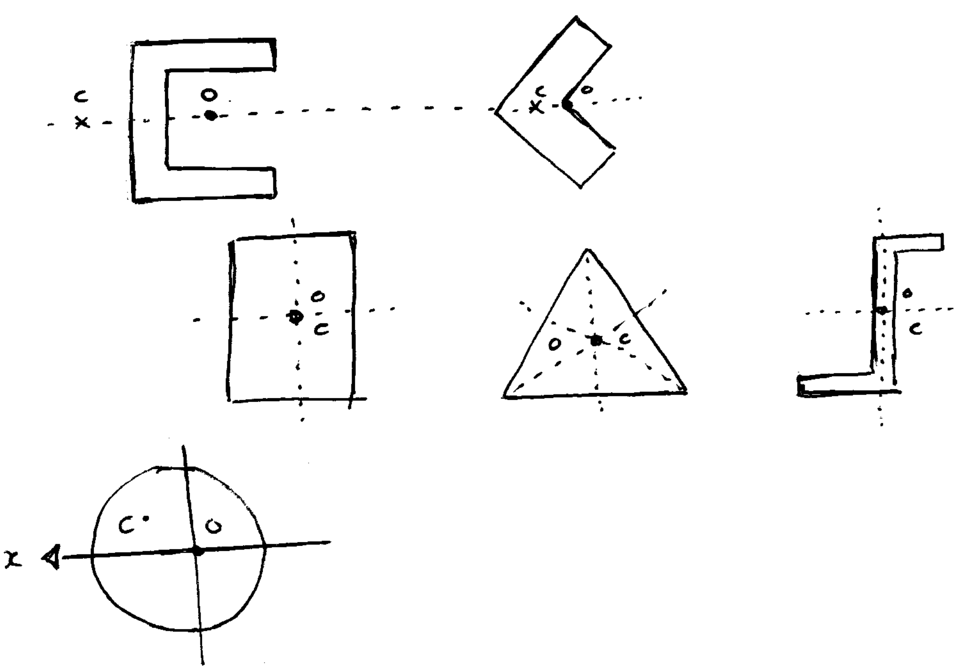
$$\sigma_{zz} = -\frac{M_y x}{I_y}$$

$$\Rightarrow \boxed{\sigma_{zz} = \frac{M_x y}{I_x} - \frac{M_y x}{I_y}}$$

(no twisting)

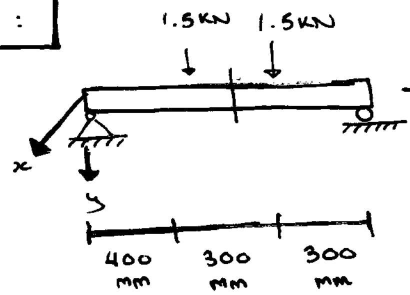
Forces P_x and P_y are passing through the shear center of the beam.

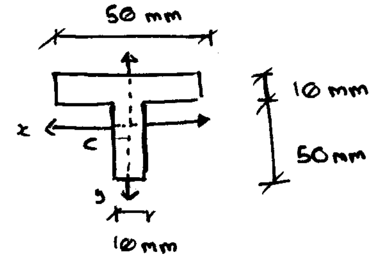
* Shear centre



(anti-symmetric)

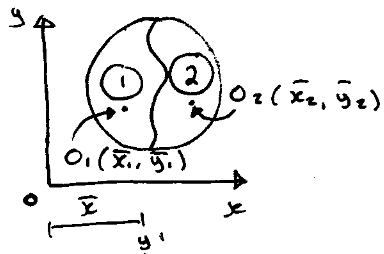
Example :





Find the max tensile and compressive normal stress at the middle span of the beam.

Solution :

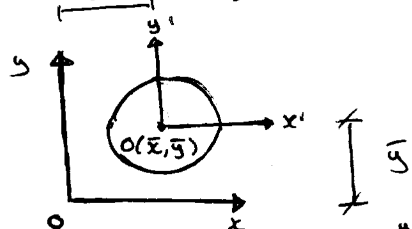


$A = A_1 + A_2$

$$\begin{cases} \bar{x} = \dfrac{A_1\bar{x}_1 + A_2\bar{x}_2}{A} \\ \bar{y} = \dfrac{A_1\bar{y}_1 + A_2\bar{y}_2}{A} \end{cases}$$

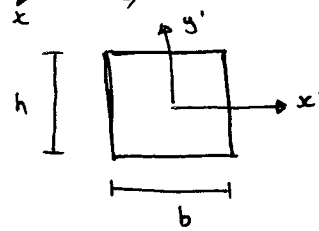
$I_{x'}, I_{y'}, I_{xy'}$

$\Rightarrow$ $I_x \quad I_y \quad I_{xy}$



Parallel axis theorem:

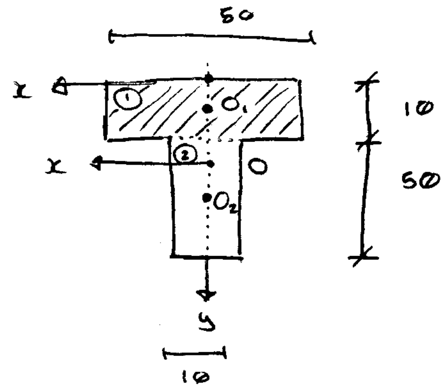
$$\begin{cases} I_x = I_{x'} + A\bar{y}^2 \\ I_y = I_{y'} + A\bar{x}^2 \\ I_{xy} = I_{x'y'} + A\bar{x}\bar{y} \end{cases}$$



$I_{x'} = \frac{1}{12} bh^3$

$I_{y'} = \frac{1}{12} hb^3$

$I_{x'y'} = 0$



$O_1(\bar{x}_1, \bar{y}_1)$

$O_2(\bar{x}_2, \bar{y}_2)$

$\bar{y}_1 = (10/2) = 5$

$\bar{y}_2 = (\frac{50}{2}) + 10 = 35.$

$A_1 = (50) \times (10) = 500$

$A_2 = (50)(10) = 500$

$$\therefore \bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = \frac{500(5) + (500)(35)}{(500+500)} = 20$$

$I_x = I_x^{①} + I_x^{②}$

$I_x^{①} = I_{x_1}^{①} + Ad^2$

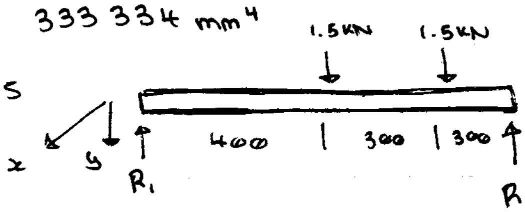
$= (1/12)(50)(10)^3 + (500)(20-5)^2$

$I_x^{②} = I_{x_2}^{②} + Ad^2$

$= (1/12)(10)(50)^3 + (500)(20-35)^2$

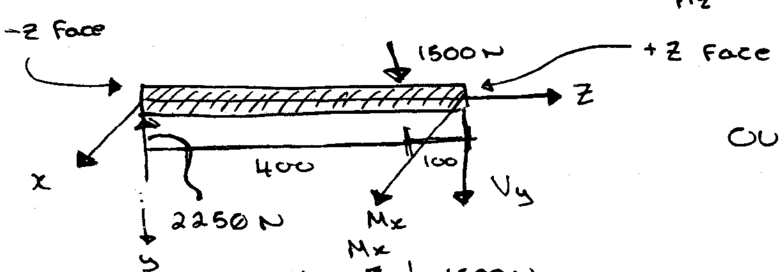
$I_x = 333\,334 \text{ mm}^4$

Statics

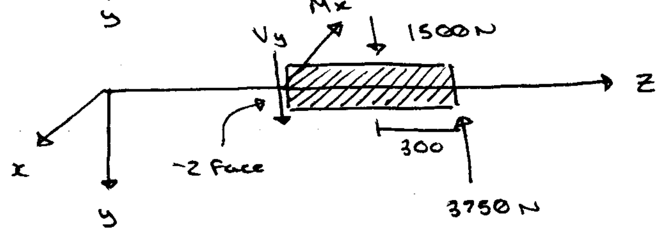


$\rightarrow R_1 = 2250 \text{ N}$

$R_2 = 3750 \text{ N}$



Outer normal



$\Sigma M_{x_1} = 0$: $\quad M_x - (2250)(500) + (1500)(100)$

$M_x = 975000$ N-mm

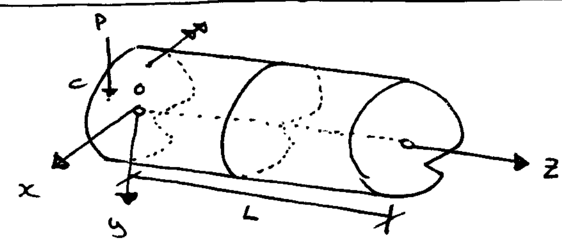
Normal Stress

$$\boxed{\sigma_{zz} = \frac{M_x}{I_x} y}$$

At the top, $y = -20$

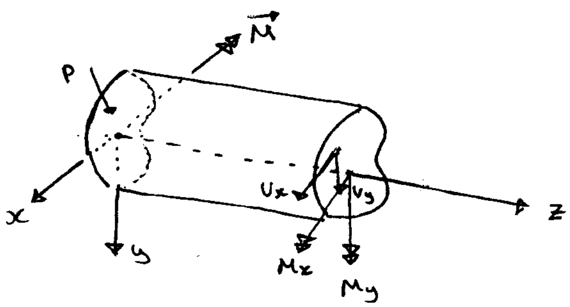
$$\sigma_{zz} = \frac{975000}{333334} \times (-20) = -58.8 \text{ MPa}$$

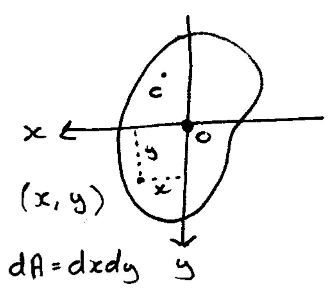
7.2 Bending Stress in Beams Subjected to Non-symmetric bending



P : through shear center C
(no twisting)
(Plane cross-section remain Plane)

Method of Section





$dF_z = \sigma_{zz}\, dx\,dy$

Resutant

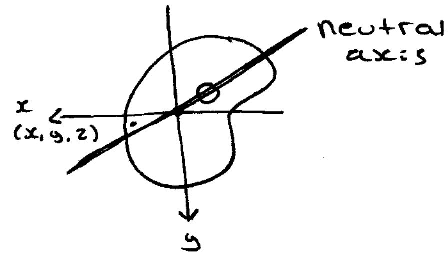
$\iint_A \sigma_{zz}\, dx\,dy = 0$ (no axial force)

$\rightarrow \iint_A y\,\sigma_{zz}\, dx\,dy = M_x$

$\iint_A x\,\sigma_{zz}\, dx\,dy = -M_y$

Cross-section has a rigid body rotation

The disp. at point (x, y, z)



$$\begin{cases} u = 0 \\ v = 0 \\ w = a''_{(z)} + x b''_{(z)} + y c''_{(z)} \end{cases}$$

Strains : $\epsilon_{xx} = 0 \ , \ \epsilon_{yy} = 0$

$\epsilon_{zz} = \dfrac{\partial w}{\partial z} = a'(z) + x b'(z) + y c'(z)$

Stress : $\sigma_{zz} = E \epsilon_{zz} = a(z) + x b(z) + y c(z)$

$$\iint_A \sigma_{zz}\, dx dy = \iint_A (a(z) + b(z) x + c(z) y)\, dx\, dy$$

$$= a(z) \iint_A dx dy + b(z) \iint_A x dx dy + c(z) \int y dx dy = 0$$

$\Rightarrow \quad a(z) \cdot A = 0 \quad \Rightarrow \quad a(z) = 0$

$\Rightarrow \quad \sigma_{zz} = b(z) x + c(z) y$

$\Rightarrow \quad \iint_A y \sigma_{zz}\, dx dy = \iint_A (b(z) xy + c(z) y^2)\, dx dy$

$$= b(z) \iint_A xy dx dy + (c(z) \iint_A y^2 dx dy \quad \ldots$$

$$= b(z) \iint_A x^2 dx dy + (c(z) \iint_A xy\, dx dy = -M_x$$

Define $I_x = \iint_A y^2\, dx dy$

$I_y = \iint_A x^2\, dx dy$

$I_{xy} = \iint_A xy\, dx dy$

$$\rightarrow \begin{cases} b(z) I_{xy} + c(z) I_x = M_y \\ b(z) I_y + c(z) I_{xy} = -M_x \end{cases}$$

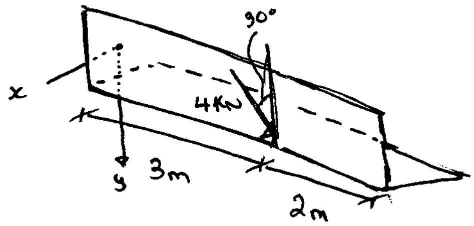
$\rightarrow \quad b(z) = \dfrac{-M_y I_x + M_x I_{xy}}{\Delta} \qquad \Big\{ \ \Delta = I_x I_y - I_{xy}^2$

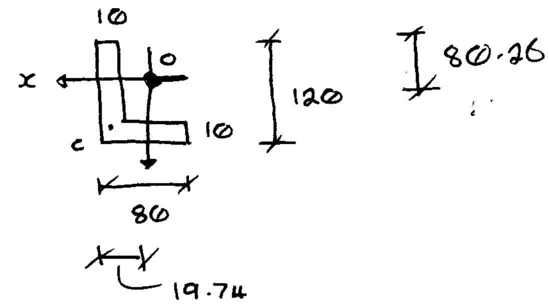
$$\sigma_{zz} = \frac{M_x I_y + M_y I_{xy}}{\Delta} y - \frac{M_y I_x + M_x I_{xy}}{\Delta} x$$

If $I_{xy} = 0$, then $\Delta = I_x I_y$

$$\Rightarrow \boxed{\sigma_{zz} = \frac{M_x}{I_x} y - \frac{M_y}{I_y} x}$$

Nov. 6/18

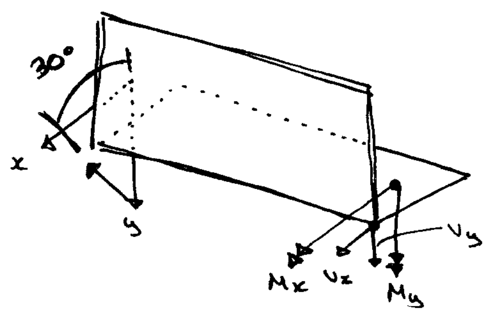




$I_x = 2.783 \times 10^6 \ mm^4$

$I_y = 1.003 \times 10^6 \ mm^4$

$I_{xy} = 0.9726 \times 10^6 mm^4$



Normal Stress

$$\sigma_{zz} = \frac{M_x I_y + M_y I_{xy}}{\Delta} y - \frac{M_y I_x + M_x I_{xy}}{\Delta} x$$

Here, $\Delta = I_x I_y - I_{xy}^2$

$\Delta = 1.8454 \times 10^{12} \ (mm^8)$

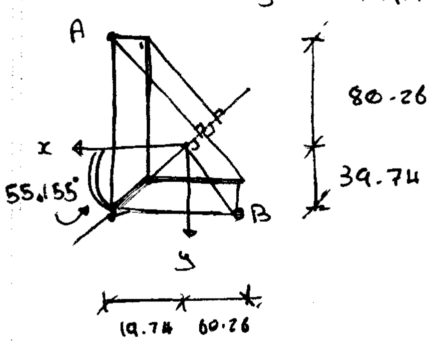
$M_x I_y + M_y I_{xy} = 1.8352 \times 10^{12}$

$M_y I_x + M_x I_{xy} = 2.6361 \times 10^{12}$

$$\Rightarrow \quad \sigma_{zz} = \left(\frac{1.8352 \times 10^{12}}{1.8454 \times 10^{12}}\right) y - \left(\frac{2.6361 \times 10^{12}}{1.8454 \times 10^{12}}\right) x$$

$$\sigma_{zz} = 0.99449 (y - 1.4364 x)$$

x, y : mm $\qquad \sigma_{zz}$: MPa



$y - 1.4364 x = 0$

Let: $\tan\alpha = (1.4364)$

$\alpha = 55.155°$

$A(19.74, -80.26)$
$B(-60.26, 39.74)$

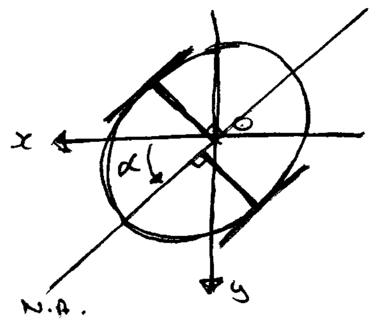
$\sigma_A = 0.99449(-80.26 - 1.4364 \times 19.74)$

$= -108.0$ MPa

$\sigma_B = 0.99449(39.74 - 1.4364 \times (-60.26))$

$\sigma_B = 125.6$ MPa

$$\left[\sigma_{zz} = \frac{M_x I_y + M_y I_{xy}}{\Delta} y \cdots \right.$$
$$\left. \cdots - \frac{M_y I_x + M_x I_{xy}}{\Delta} x \right] \quad (*)$$



Neutral axis:

$\sigma_{zz} = 0$

$$\Rightarrow \frac{M_x I_y + M_y I_{xy}}{\Delta} y - \frac{M_y I_x + M_x I_{xy}}{\Delta} x = 0$$

$$\Rightarrow \quad y = x \tan\alpha = 0$$

and $\quad \tan\alpha = \dfrac{M_y I_x + M_x I_{xy}}{M_x I_y + M_y I_{xy}}$

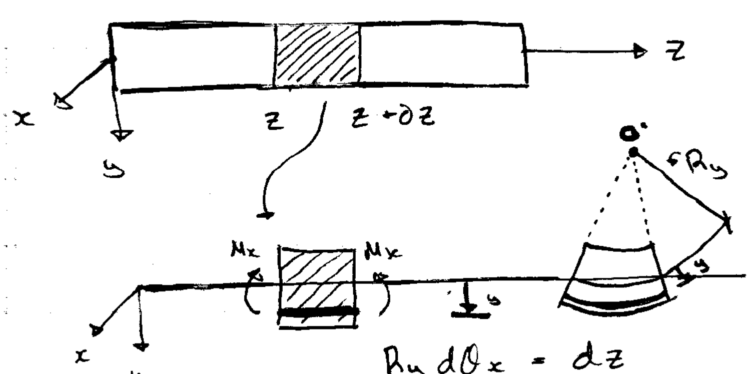
The normal stress

$$\sigma_{zz} = \frac{M_x I_y + M_y I_{xy}}{\Delta}(y - x\tan\alpha)$$

IF $M_x \neq 0$

$$\sigma_{zz} = \frac{M_x}{I_x - I_{xy}\tan\alpha}(y - x\tan\alpha)$$

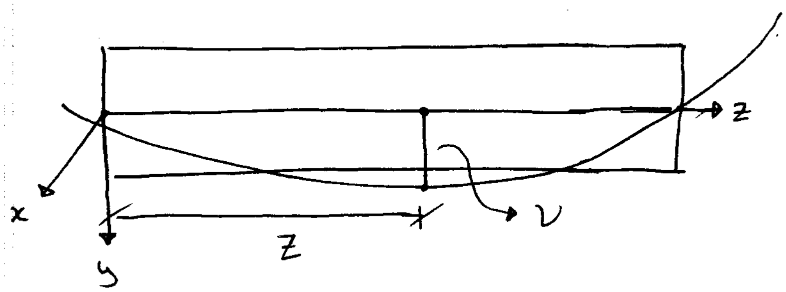
7.3 Deflections of Straight Beams Subjected to nonsymmetrical bending.



$$R_y \, d\theta_x = dz$$
$$dz \longrightarrow (R_y + y)\, d\theta_x$$

Normal Strain

$$\epsilon_{zz} = \frac{(R_y + y)\, d\theta_x - dz}{dz} = \frac{y}{R_y}$$



$$\frac{1}{R_y} = \frac{\left|\frac{\partial^2 v}{\partial z^2}\right|}{\left(\sqrt{1 + \left(\frac{\partial v}{\partial z}\right)^2}\right)^3}$$

Small deformation

$$\left|\frac{\partial v}{\partial z}\right| \ll 1$$

$$\Rightarrow \frac{1}{R_y} = \left|\frac{\partial^2 v}{\partial z^2}\right| = -\frac{\partial^2 v}{\partial z^2}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial z^2} = \frac{\epsilon_{zz}}{y} = \frac{\sigma_{zz}}{E y}$$

Since $\sigma_{zz} = \frac{M_x I_y + M_y I_{xy}}{\Delta} y - \cancel{\frac{M_y I_x + M_x I_{xy}}{\Delta}} x$

Let $x = \emptyset$:

Then : $\boxed{-\frac{\partial v^2}{\partial z^2} = \frac{M_x I_y + M_y I_{xy}}{E\Delta}}$

Special case : $I_{xy} = \emptyset$

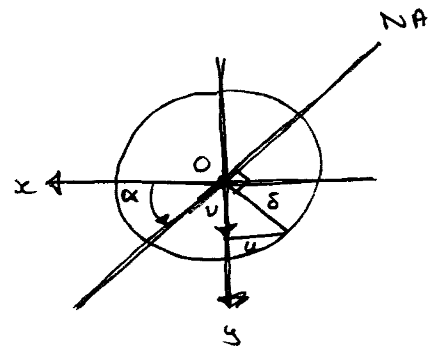
then $\Delta = I_x I_y$

$\Rightarrow \boxed{\frac{-\partial^2 v}{\partial z^2} = \frac{M_x}{E I_x}}$ $\qquad - E I v'' = M$

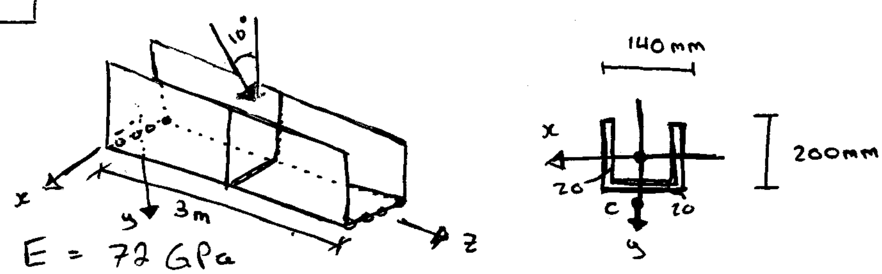
The total deflection:

$u = -v \tan\alpha$

$\delta = \sqrt{u^2 + v^2}$



Example



Given $E = 72$ GPa

Find the max deflection of the beam

Solution $\quad y_o = 82.0$ mm

$I_x = 39.69 \times 10^6$ mm^4

$I_y = 30.73 \times 10^6$ mm^4

$I_{xy} = 0$

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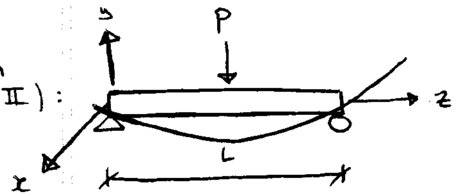
From (MM II):



EI = const.

$$\boxed{\frac{-\partial^2 v}{\partial z^2} = \frac{M_x}{EI_x}}$$

For unsymmetrical beam bending

$$\frac{-\partial^2 v}{\partial z^2} = \frac{M_x I_y + M_y I_x}{E\Delta}$$

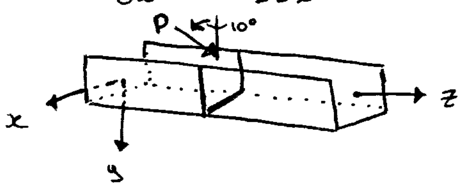
$$\boxed{\frac{-\partial^2 v}{\partial x^2} = \frac{M_x}{E(I_x - I_y \tan\alpha)}}$$

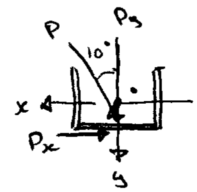
M_x is a function of Z

Since $I_{xy} = \emptyset$:

(And $P = 35$ kN)

$$\frac{-\partial^2 v}{\partial x^2} = \frac{M_x}{EI_x}$$





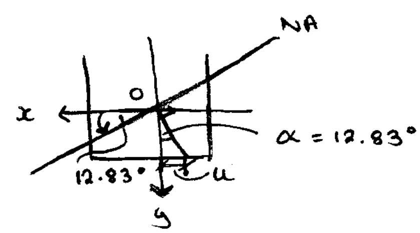
Since $P_y = P\cos(10°)$

$$v_{max} = \frac{P_y L^3}{48 EI_x} \quad \text{From table}$$

$$v_{max} = \frac{PL^3\cos(10°)}{48EI_x}$$

$$v_{max} = \frac{(35\times10^3)(3\times10^4)(\cos 10°)}{(48)(72\times10^3)(39.69\times10^6)}$$

$$= \mathbf{6.78}\text{ mm}$$



$$\tan\alpha = \frac{M_y I_x + M_x I_{xy}}{M_x I_y + M_y I_{xy}}$$

$$M_y = \left(\tfrac{1}{2}\right)P_x \cdot \left(\tfrac{L}{2}\right) = \tfrac{1}{4}P(\sin 10°)L$$

$$M_x = \left(\tfrac{1}{2}\right)P_y \cdot \left(\tfrac{L}{2}\right) = \tfrac{1}{4}P(\cos 10°)L$$

$$\tan\alpha = \frac{M_y I_x}{M_x I_y} = \frac{\cancel{(1/4)L}\sin(10°)\times 39.69\cancel{\times10^6}}{\cancel{(1/4)L}\cos(10°)\times 30.73\cancel{\times10^6}} = 0.227$$

$$\alpha = 12.83°$$

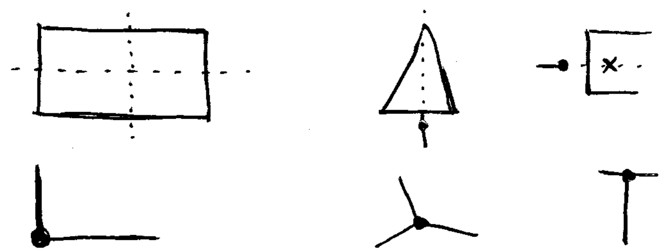
∴ max deflection

$$\delta_{max} = \frac{v_{max}}{\cos\alpha} = \frac{6.78}{\cos(12.83°)} = 6.95\text{ mm}$$

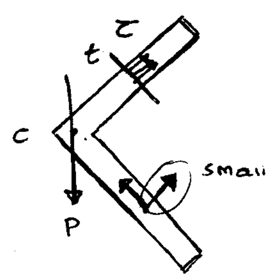
Ch. 8 - Shear Center For Thin-wall beam Cross-section

8.1 Approximation For Shear in thin-wall cross section.

Sheer center : A point in the cross-section of a beam through which the loads must pass For the beam to be subjected to only bending deformation. No torsion is caused by the transverse loads that act through the shear center.



Shear stress in the thin wall :



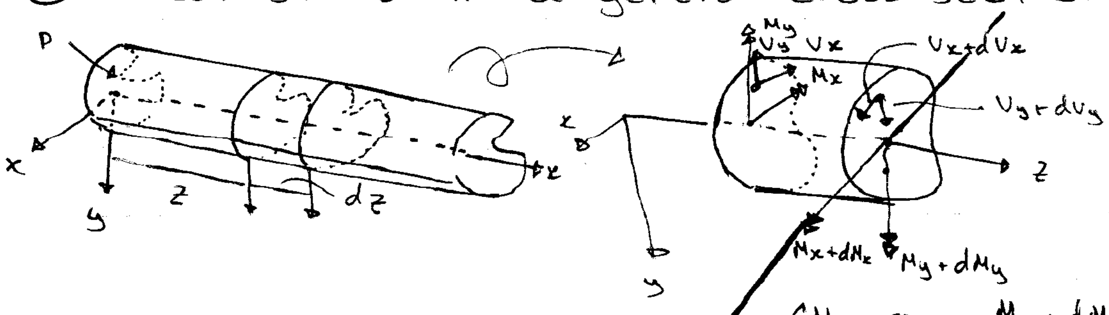
1° Shear stress is parallel to the boundary

2° Shear stress is uniform through the wall thickness

The resultant of the shear stress through the wall thickness : $q = \tau t$ (shear flow)

8.2 Shear Flow in Thin-wall beam Cross-section

① Shear stress in a general cross-section

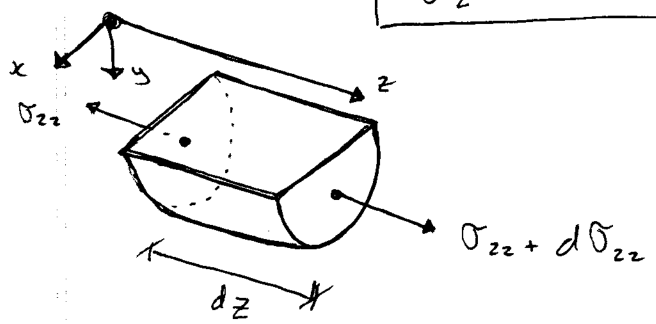


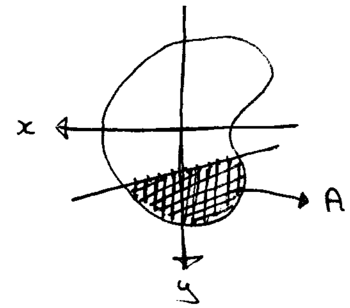
$\Sigma M_x = 0$: $M_x + dM_x - M_x - V_y dz = 0$

$$\frac{dM_x}{dz} = V_y$$

$$\Sigma M_y = 0 : \quad M_y + dM_y - M_y + V_x dz = 0$$

$$\boxed{\frac{dM_y}{dz} = -V_x}$$

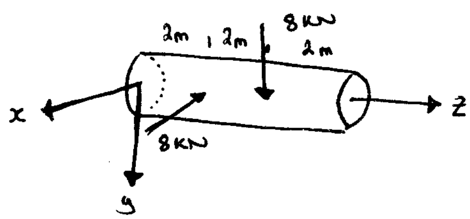


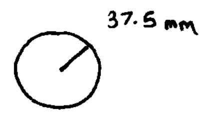


$$\Sigma F_z = 0 : \quad -\iint_{A'} \sigma_{zz} \, dx\,dy + \iint_{A'} (\sigma_{zz} + d\sigma_{zz})\, dx\,dy$$

$$- \tau\, dz \cdot t = 0$$

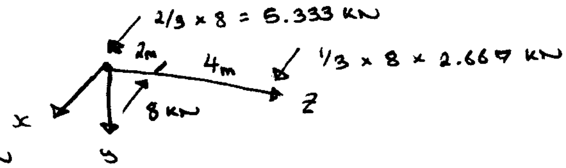
$$\therefore \quad \tau \cdot t = q = \iint_{A'} \frac{d\sigma_{zz}}{dz} dx\,dy$$

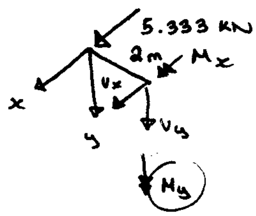




Find the location and magnitude of the max normal stress in the beam.

Solution :

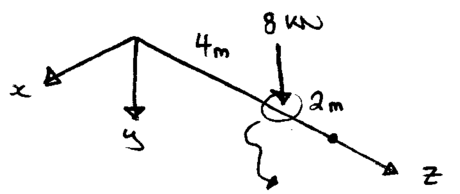




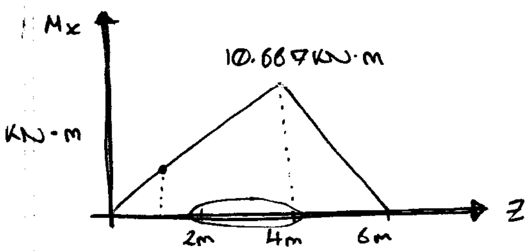
$\Sigma M_y = 0$

$M_y - 5.333 \times 2 = 0$

$M_y = 10.667$ KN·m



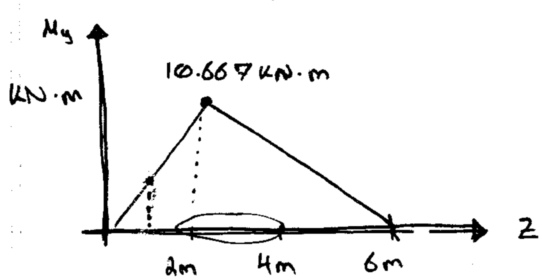
$M_x = 10.667$ KN·m

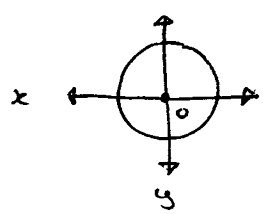


Observation, max normal occurs between $z = 2$, $z = 4$

when $2 \le z \le 4$

$$\begin{cases} M_x(z) = \frac{10.667}{4} z \\ M_y(z) = (6 - z)\left(\frac{10.667}{4}\right) \end{cases}$$





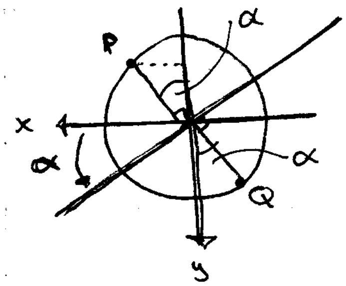
Since $I_x = I_y = (1/4)\pi r^4$

$= (1/4)\pi(37.5)^4$

$= 1.553 \times 10^6 \text{ mm}^4$

$2 \le z \le 4$

$$\therefore \tan\alpha = \frac{M_y I_x + M_x I_{xy}}{M_x I_y + M_y I_{xy}} = \frac{M_y}{M_x} \Rightarrow \boxed{\tan\alpha = \frac{6 - z}{z}}$$



$P(r\sin\alpha,\ -r\cos\alpha)$

$Q(-r\sin\alpha,\ r\cos\alpha)$

Normal Stress

$$\sigma_{zz} = \frac{M_x I_y + M_y \cancel{I_{xy}}^{0}}{\Delta} y - \frac{M_y I_x + \cancel{M_x I_{xy}}^{0}}{\Delta} x$$

(where $\Delta = I_x I_y$)

(where $I_{xy} = 0$ for circular cross-section)

$$= \frac{M_x}{I_x} y - \frac{M_y}{I_x} x$$

can use either way

OR

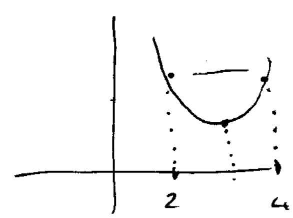
$$\sigma_{zz} = \boxed{\frac{M_x(y - x\tan\alpha)}{I_x - \cancel{I_{xy}}^{0}\tan\alpha}}$$

$$\sigma_{zz} = \frac{M_x}{I_x}\,(y - x\tan\alpha)$$

$M_x \to z$, $\tan\alpha \leftarrow z$, I_x const.

$$\sigma_{zz,P} = \frac{M_x}{I_x}(-r\cos\alpha - r\sin\alpha \cdot \tan\alpha)$$
$$= -\frac{M_x r}{I_x} \cdot \frac{1}{\cos\alpha}$$

$$\sigma_{zz,Q} = \frac{M_y}{I_x}(r\cos\alpha - (-r\sin\alpha)\cdot\tan\alpha)$$
$$= \frac{M_x \cdot r}{I_x} \cdot \frac{1}{\cos\alpha}$$



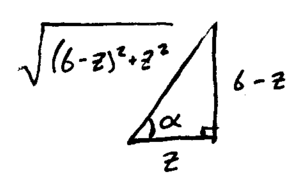
$$\therefore\ \sigma_{zz,max}^{(z)} = \frac{M_x \cdot r}{I_x} \cdot \frac{1}{\cos\alpha}$$

Since $M_x = \frac{10.667}{4} z$ kN·m (N·m) $= \frac{10.667}{4} z \times 10^3$ (N·mm)

$I_x = 1.553 \times 10^6\ mm^4$

$r = 37.5\ mm$

$\tan\alpha = \frac{6-z}{z}$



min occurs @ 3000 mm

max occurs @ 2000 mm, 4000 mm

$\sigma_{zz}(2000) = \sigma_{zz}(4000)$

$\Rightarrow$ 288.0 MPa

AT $z = 2000$ $\tan\alpha = \frac{6000-z}{z} = 2$

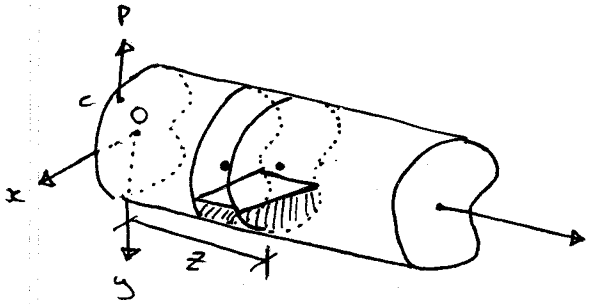
$z = 4000$ $\tan\alpha = \frac{6000-z}{z} = 1/2$

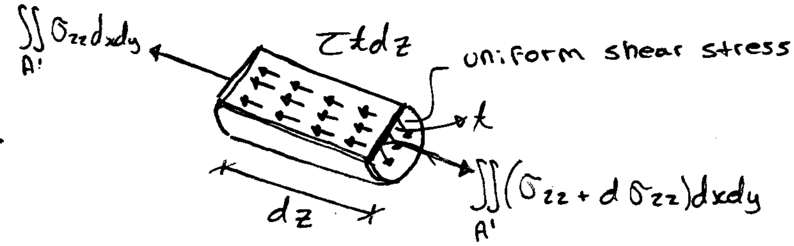
$$\cos\alpha = \frac{z}{\sqrt{(6-z)^2 + z^2}} \longrightarrow \frac{z}{\sqrt{(6000-z)^2+z^2}}$$

$$\therefore\ \sigma_{zz\,max}(z) = \frac{(10.667)z \times 10^3}{4} \times \frac{(37.5)}{(1.553\times10^6)} \Big/ \frac{z}{\sqrt{(6000-z)^2+z^2}}$$

$$\therefore\ \sigma_{zz\,max}(z) = 64.39\times10^{-3} \times \sqrt{(6000-z)^2 + z^2}$$

when $2000\,mm \le z \le 4000\,mm$

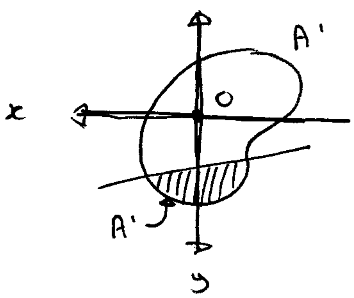


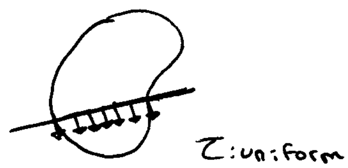


$\Sigma F_z = 0$:

$$\iint_{A'} (\sigma_{zz}+d\sigma_{zz})\,dx\,dy - \iint_{A'} \cancel{\sigma_{zz}}\,dx\,dy - \tau t\,dz = 0$$

$$= q = \tau t = \iint_{A'} \frac{d\sigma_{zz}}{dz}\,dx\,dy$$





Since $\sigma_{zz} = \dfrac{M_x I_y + M_y I_{xy}}{\Delta}\, y - \dfrac{M_y I_x + M_x I_{xy}}{\Delta}\, x$

$$\Rightarrow \frac{d\sigma_{zz}}{dz} = \frac{\frac{dM_x}{dz} I_y + \frac{dM_y}{dz} I_{xy}}{\Delta}\, y - \frac{\frac{dM_y}{dz} I_x + \frac{dM_x}{dz} I_{xy}}{\Delta}\, x$$

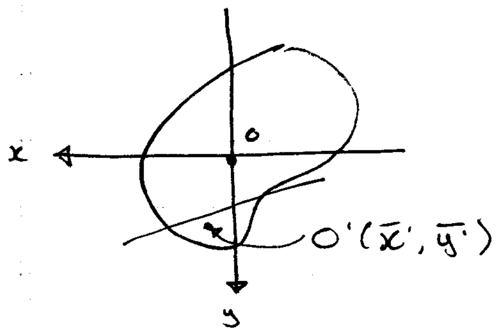
because of: $\dfrac{dM_x}{dz} = V_y \qquad \dfrac{dM_y}{dz} = -V_x$

$$\Rightarrow \frac{d\sigma_{zz}}{dz} = \frac{V_y I_y - V_x I_{xy}}{\Delta}\, y - \frac{-V_x I_x + V_y I_{xy}}{\Delta}\, x$$

$$= \frac{V_y I_y - V_x I_{xy}}{\Delta}\, y + \frac{V_x I_x + V_y I_{xy}}{\Delta}\, x$$

$$\Rightarrow q = \tau t = \iint_{A'} \frac{d\sigma_{zz}}{dz}\,dx\,dy$$

$$= \frac{V_y I_y - V_x I_{xy}}{\Delta} \iint_{A'} y\,dx\,dy + \frac{V_x I_x - V_y I_{xy}}{\Delta} \iint_{A'} x\,dx\,dy$$



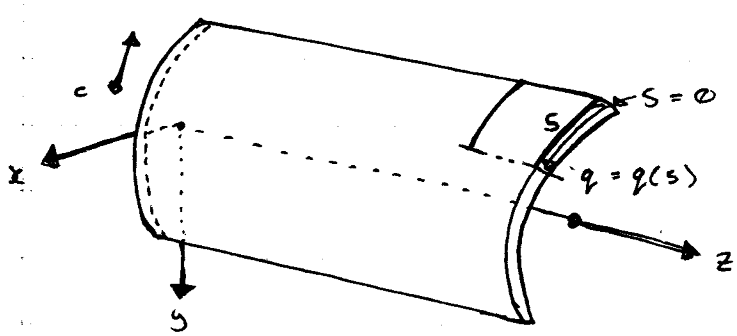
$$\iint_{A'} y\,dx\,dy = A'\bar{y}'$$
$$\iint_{A'} x\,dx\,dy = A'\bar{x}'$$

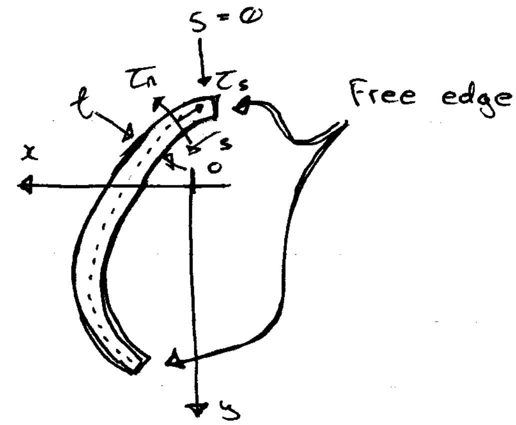
$$\boxed{q = \tau t = \frac{V_y I_y + V_x I_{xy}}{\Delta} A'\bar{y}' + \frac{V_x I_x + V_y I_{xy}}{\Delta} A'\bar{x}'}$$

If $I_{xy} = 0$, $q = (V_y / I_x) A'\bar{y}' + (V_x / I_y) A'\bar{x}'$

$\boxed{\text{If } V_x = 0, \quad q = (V_y / I_x) A'\bar{y}'}$

2. Thin-wall open section



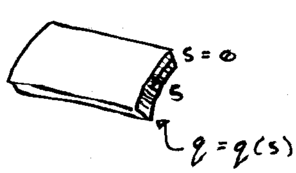


1° $\tau_n = 0$ (thickness very small)

2° τ_s is uniform through thickness of the wall

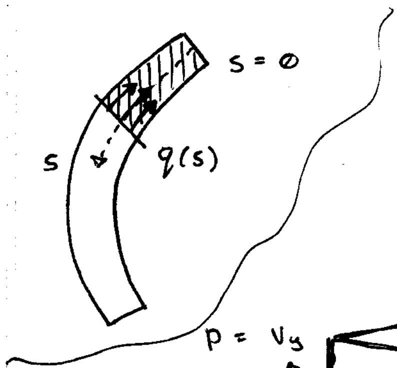
3° $q = \tau t$: shear flow

4° Shear flow is zero at the free edge

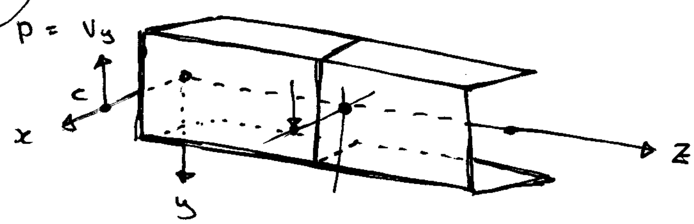


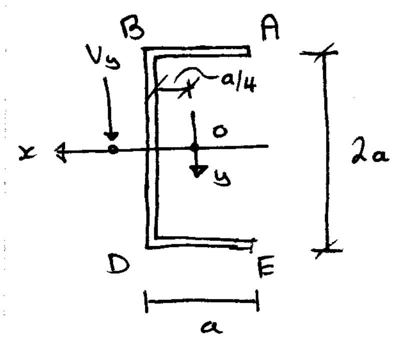
Positive shear flow:

the shear flow points into the area

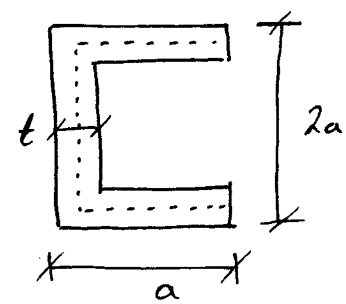


Example: Determine the shear flow in a C channel section due to a shear force V_y through it's shear center.





$(t \ll a)$

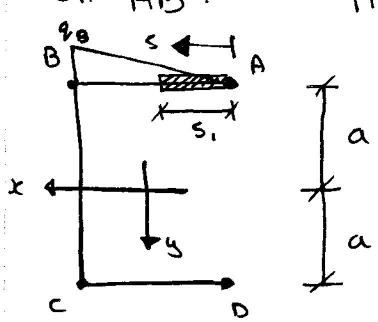


$$q = \tau t = \frac{V_y}{I_x} A'\bar{y}'$$

$$I_x = \left(\tfrac{1}{12}\right) t (2a)^3 + \left[\left(\tfrac{1}{12}\right)(a)(t)^3 + at\cdot a^2\right] \times 2$$

$$= (8/3)\, a^3 t \qquad (at^3 \ll a^3 t)$$

On AB: $\quad A' = s_1 t \qquad \bar{y}' = -a$



$$q(s_1) = \frac{V_y}{I_x} A'\bar{y}'$$

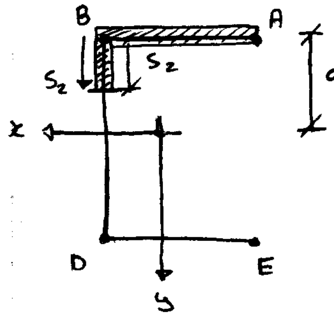
$$= \frac{V_y}{8/3\, a^3 t}(s_1 t)(-a) = -\frac{3V_y (s_1)}{8a^2}$$

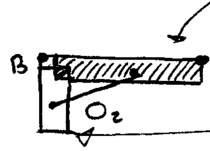
(when $0 \le s_1 \le a$)

at B, $s_1 = a$

$$q_B = -\frac{3V_y}{8a^2}(a) = \frac{-3V_y}{8a}$$

On BD:





$$A'\bar{y}' = A_1\bar{y}_1' + A_2\bar{y}_2'$$

$$q(s_2) = q_B + \frac{V_y}{I_x} A_2 \bar{y}_2'$$

$$\Rightarrow \frac{-3V_y}{8a} - \frac{V_y}{8/3\, a^3 t}\cdot(s_2 t)(a - s_2/2)$$

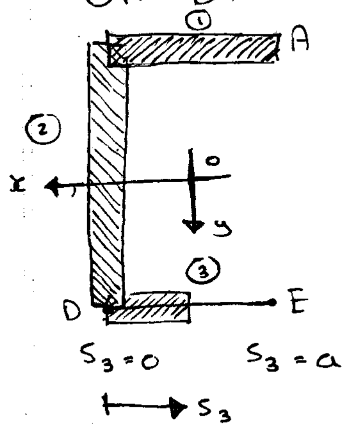
$$= \frac{3V_y}{8a}\left(\frac{1}{2}\left(\frac{s_2}{a}\right)^2 - \frac{s_2}{a} - 1\right)$$

$$\boxed{\begin{aligned} A_2 &= s_2 t \\ \bar{y}_2' &= -(a - s/2) \end{aligned}}$$

At D_1, $s_2 = 2a$

$$q_D = q(2a) = -\frac{3V_y}{8a} = q_B$$

On DE



$$q(s_3) = (V_y / I_x) A' \bar{y}'$$

$$A'\bar{y}' = A_1'\bar{y}_1' + A_2'\bar{y}_2' + A_3'\bar{y}_3'$$

$$q(s_3) = \frac{V_y}{I_x}\left[A_1'\bar{y}_1' + A_2'\bar{y}_2' + A_3'\bar{y}_3'\right]$$

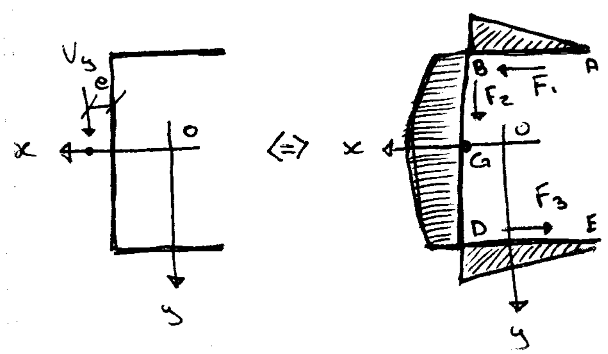
$$= q_D + \frac{V_y}{I_x} A_3'\bar{y}_3'$$

$$q(s_3) = -\frac{3V_y}{8a} + \frac{V_y}{8a^3 t} s_3 t$$

$$= \frac{-3V_y}{8a} + \frac{3V_y}{8a} \cdot \frac{s_3}{a}$$

@ E, $s_3 = a$

$$\therefore q_E = \frac{-3V_y}{8a} + \frac{3V_y}{8a} = \emptyset$$



Since $q_B = \frac{3V_y}{8a}$

$$\therefore F_1 = \frac{1}{2} q_B \cdot AB$$

$$= \frac{1}{2}\left(\frac{3V_y}{8a}\right) a = \frac{3}{16} V_y$$

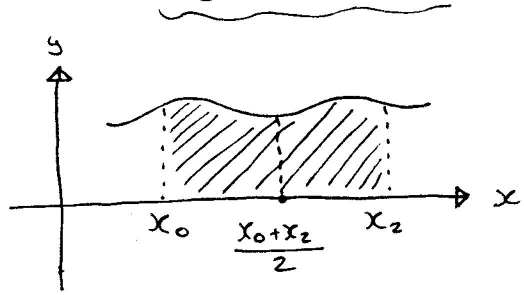
D is the moment center

$$F_1 \cdot 2a = V_y e$$

$$\Rightarrow e = \frac{F_1 \cdot 2a}{V_y} = \frac{3/16 V_y \, 2a}{V_y} = \frac{3a}{8}$$

negative

$$F_2 = -\int_0^{2a} q(s_2)\, ds_2$$



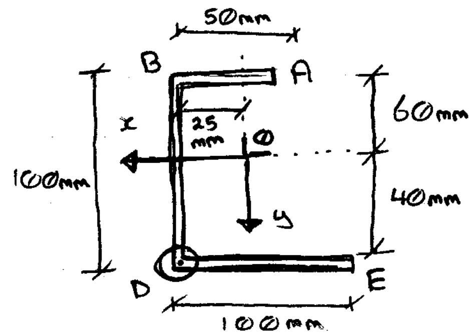
$$\int_{x_0}^{x_2} f(x)dx = \frac{x_2 - x_0}{6}(f_0 + 4f_1 + f_2)$$

$$f_0 = f(x_0), \quad f = f\left(\frac{x_0 + x_2}{2}\right)$$

$$f_2 = f(x_2)$$

$$F_2 = \frac{2a}{6}\left(q_B + 4q_G + q_D\right)$$
$$= \frac{a}{3}\left(\frac{-3V_y}{8a} - 4\left(\frac{9V_y}{16a}\right) - \frac{3V_y}{8a}\right)$$
$$F_2 = -V_y$$

Example: Find the shear center of a C-section.



$t = 4\,mm$

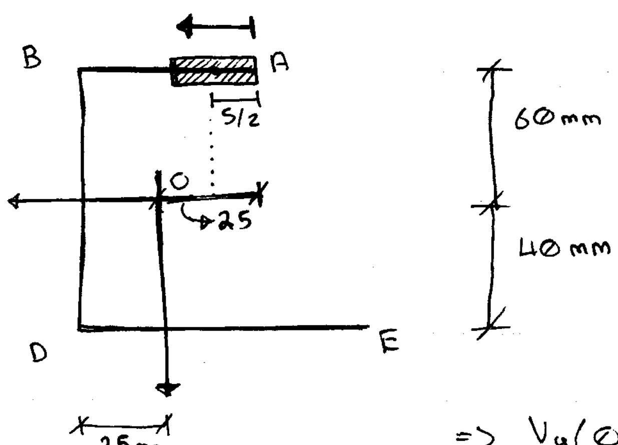
Solution: 1° Find the centroid and moment of area about centroid axes

$I_x = 1.733 \times 10^6\ mm^4$

$I_y = 0.876 \times 10^6\ mm^4$

$I_{xy} = -0.500 \times 10^6\ mm^4$

2° Find internal shear flow due to the internal shear forces V_x and V_y



At $0 \leq s \leq 50$

$A' = st$

Centroid $O_i\,(s/2 - 25,\ -60)$

$$\therefore\ q(s) = \frac{V_y I_y - V_x I_{xy}}{\Delta} A'\bar{y}' + \ldots$$
$$\ldots + \frac{V_x I_x - V_y I_{xy}}{\Delta} A'\bar{x}'$$

$$\Rightarrow \frac{V_y(0.876)(10^6) - V_x(-0.500)(10^6)}{(1.268 \times 10^{12})} \cdot s\,(4)(-60) \ldots$$

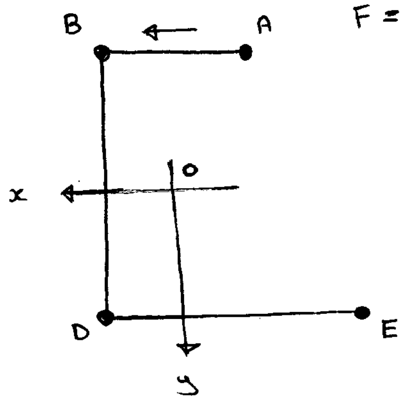
$$\ldots \frac{V_x(1.733)(10^6) - V_y(-0.500)(10^6)}{(1.268 \times 10^{12})} s\,(4)(s/2 - 25)$$

$$\Rightarrow V_x\left[2.7335(s-50) - 94.635\right](10^6) + V_y\left[0.7885(s-50) - 165.795\right](10^6)$$

Resultant on AB :

$$F = \int_0^{50} q(s)\,ds$$

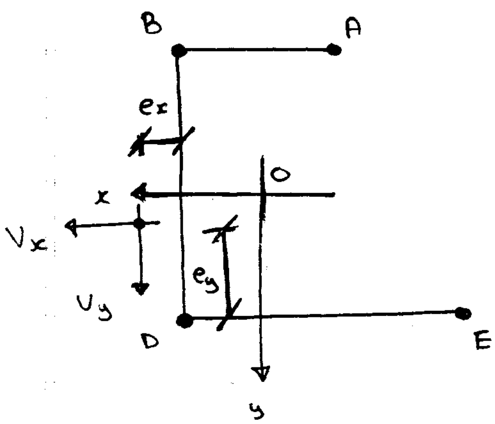
$$= -(0.1752\,V_x + 0.2237\,V_y)$$



$F = 0.1752\,V_x + 0.2237\,V_y$

$$\Sigma M_D = (0.1752V_x + 0.2237V_y)(100)$$

$$= 17.52V_x + 22.37V_y$$

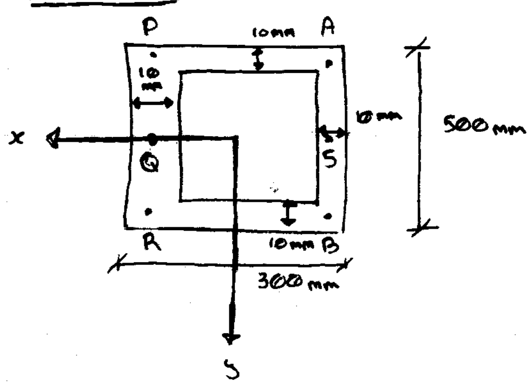


Shear Force V_x and V_y

$$\Sigma M_D = V_x e_y + V_y e_x$$

$$\Rightarrow \begin{cases} e_y = 17.52 \text{ mm} \\ e_x = 22.37 \text{ mm} \end{cases}$$

Example Box Beam



1° thin wall

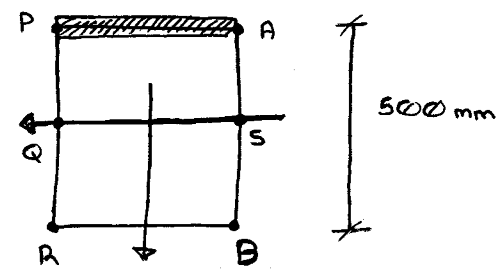
2° x-axis is symmetric axis

3° only internal shear force

V_y is needed

Take $V_y = I_x$

4° cut at point A



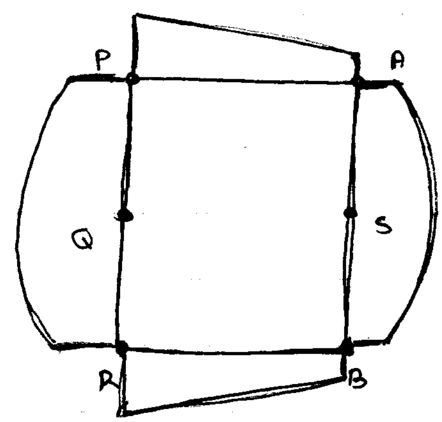
→ 4° cont'd

$$q_P = q_A + \frac{V_y}{I_x} A'\bar{y}'$$
$$= q_A + (300)(10)(-250)$$
$$= q_A - 750000$$

$$q_Q = q_P + \frac{V_y}{I_x} A'\bar{y}'$$
$$= (q_A - 750000) + (250)(20)(-125)$$
$$= q_A - 1375000$$

$q_R = q_P$; $q_B = q_A$

→ $$q_S = q_B + \frac{V_y}{I_x} A'\bar{y}'$$
$$= q_A + (250)(10)(125)$$
$$= q_A + 313500$$



Angle of twist (per unit length):

$$\theta = \frac{1}{2GA} \oint q/t \, dl = 0$$

$$\Rightarrow \int_{AP} q/t \, dl + \int_{PR} q/t \, dl + \int_{RB} q/t \, dl + \int_{BA} q/t \, dl$$

$$\Rightarrow \frac{1}{10}\int_{AP} q \, dl + \frac{1}{20}\int_{PR} q \, dl + \frac{1}{10} \int_{RB} q \, dl + \frac{1}{10}\int_{BA} q \, dl$$

$$\int_{AP} q \, dl = \frac{1}{2}(q_A + q_P)(AP) = \frac{1}{2}(q_A + q_A - 750000)(300)$$
$$= 300 q_A - 112500000$$

$$\int_{PR} q\,dl = \frac{PR}{6}(q_P + 4q_Q + q_R)$$

$$= \frac{500}{6}(q_A - 750000 + 4(q_A - 1375000) + q_A - 750000)$$

$$= 500\,q_A - 20(2916667)$$

$$\int_{RB} q\,dl = \frac{1}{2}(q_R + q_B)(RB)$$

$$= 300(q_A) - 112500000$$

$$\int_{BA} q\,dl = \frac{AB}{6}(q_B + 4q_S + q_A)$$

$$= 500(q_A) + 1041667 \times 10$$

$$\Rightarrow \left(\frac{1}{10}\right)(300q_A - 112500000)$$

$$+ \left(\frac{1}{20}\right)(500q_A - (20)(2916667))$$

$$+ \left(\frac{1}{10}\right)(300\,q_A - 112500000)$$

$$+ \left(\frac{1}{10}\right)(500q_A + 10416670) = 0$$

$$\Rightarrow q_A = 305556 \text{ N/mm}$$

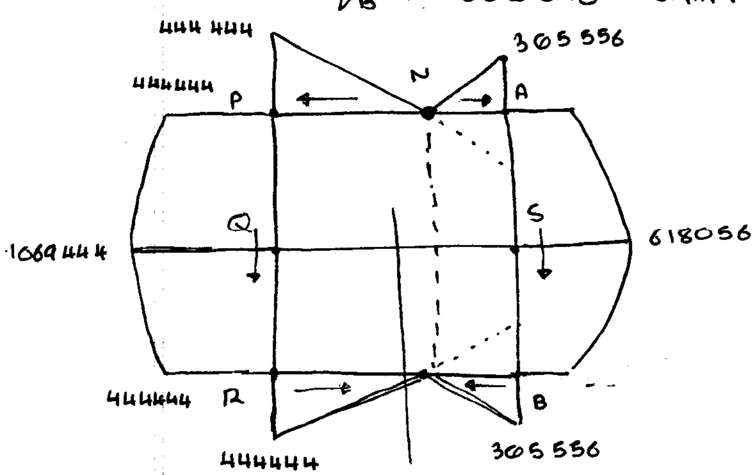
$\longrightarrow$ $q_P = -444\,444$ N/mm

$q_R = -444444$ N/mm

$q_B = 305556$ N/mm

$q_Q = -1069444$ N/mm

$q_S = 618056$ N/mm



1° Symmetrical ?

2° Edge (Parallel) Shear Force

$\Rightarrow$ quadratic

3° Edge (perpendicular) Shear Force

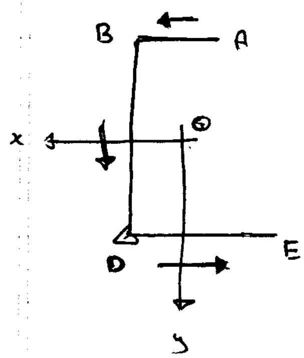
$\Rightarrow$ linear

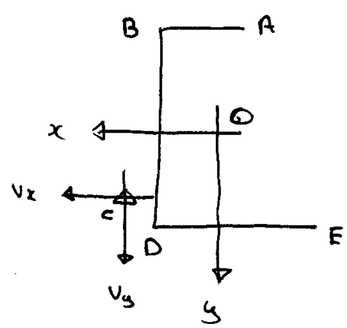
Finding shear center (using Point A)

A : moment center

$$\boxed{e = 203.0 \text{ mm}}$$

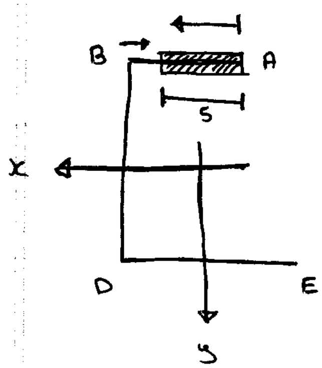
Here $I_x = 687500000 \text{ mm}^4$





Find shear flow on $\underline{\underline{AB}}$

D is the moment center



Find shear flow on $\underline{\underline{AB}}$

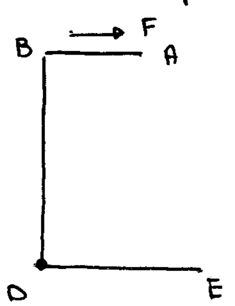
The resultant on AB

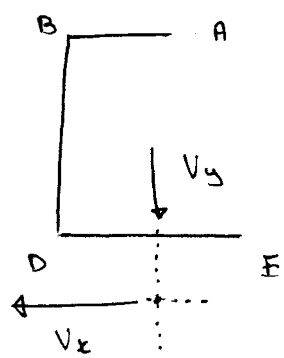
$$F = \int_A^B q(s)\,ds$$

$$= C_1 V_x + C_2 V_y$$

C_1 and C_2 are constants

If F is positive,





Method 2 :

Step 1 : Consider shear force V_x only

↳ identify the line of action of $\underline{\underline{V_x}}$

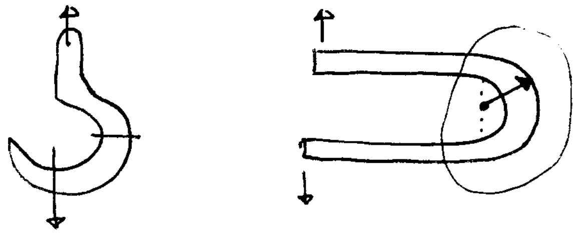
Step 2 : Consider V_y only

↳ identify the line of action of $\underline{\underline{V_y}}$

Chapter 9 - Curved Beams

9.1 - Introduction

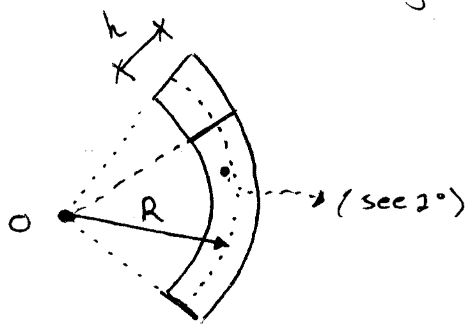
Crane hook:



9.2 Circumfrential stresses in a curved beam

Geometry: ① The cross-section has a symmetric axis, and the beam has a symmetric plane

② The area of cross-section is constant through the axis of the beam.

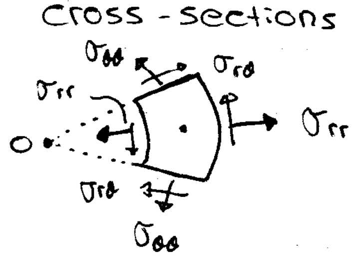


$\frac{R}{h} > 5 \Rightarrow$ straight beam theory

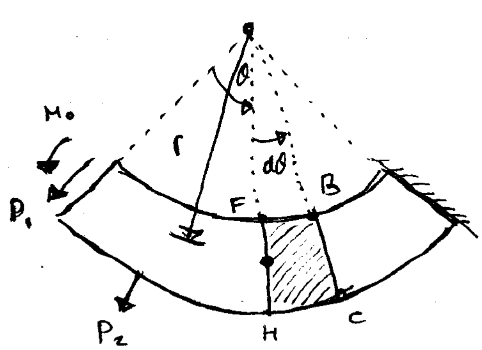
Deformation:

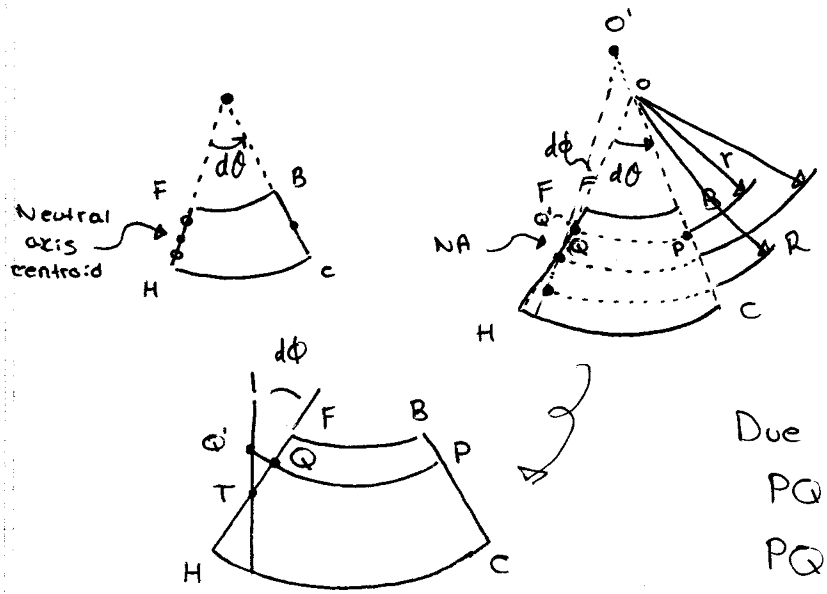
1° Plane cross-sections remain plane after loading

2°



σ_{yy} and $\sigma_{r\theta}$ are sufficiently small





Due to the rotation

$$PQ \to PQ'$$

$$PQ = r\,d\theta$$

$$QQ' = TQ \cdot d\phi$$

$$TQ = R_n - r$$

$$QQ' = (R_n - r)\,d\phi$$

The normal strain of the line segment PQ

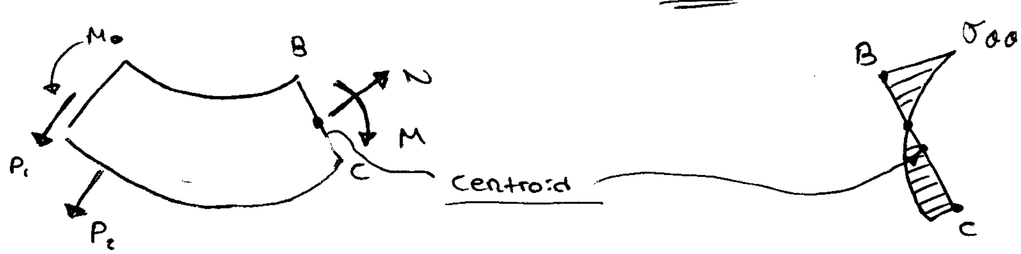
$$\epsilon_{\theta\theta} = \frac{QQ'}{PQ} = \frac{(R_n - r)\,d\phi}{r\,d\theta}$$

Define

$$\omega = d\phi / d\theta \quad \Rightarrow \quad \epsilon_{\theta\theta} = \frac{R_n - r}{r}\,\omega$$

Stress : $\sigma_{\theta\theta} = E\epsilon_{\theta\theta} = E\omega\,\frac{R_n - r}{r}$

Statics : Cross-section <u>BC</u>



$$\begin{cases} N = \iint_A \sigma_{\theta\theta}\,dA \\ M = \iint_A \sigma_{\theta\theta}(R - r)\,dA \end{cases}$$

Sub $\sigma_{\theta\theta}$ into the above eqns

$$\begin{cases} \iint_A E\omega\,\frac{R_n - r}{r}\,dA = N \\ \iint_A E\omega\,\frac{R_n - r}{r}(R - r)\,dA = M \end{cases} \Rightarrow E\omega = \frac{A_m}{A(RA_m - A)} \cdot M - \frac{N}{A}$$

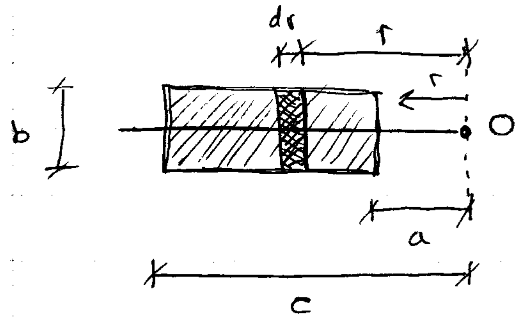
$$\left\{ \begin{array}{l} \boxed{E\omega = \dfrac{A_m}{A(RA_m - A)} \cdot M - \dfrac{N}{A}} \\[2ex] \boxed{R_n = \dfrac{MA}{MA_m + N(A - RA_m)}} \end{array} \right.$$

Here, $A_m = \iint_A \frac{1}{r}\, dA$

$A = \iint_A dA$ area of cross section

$$\Rightarrow \quad \boxed{\sigma_{\theta\theta} = \frac{N}{A} + \frac{M}{A(RA_m - A)} \left(\frac{A}{r} - A_m \right)}$$

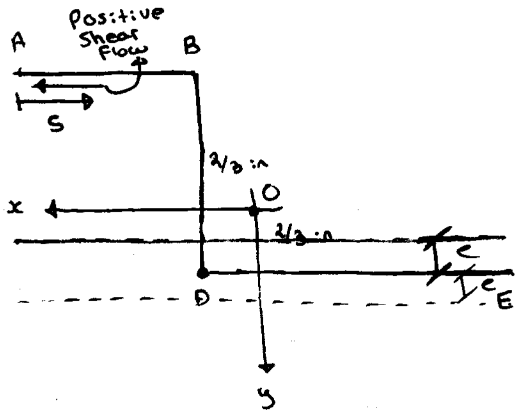
Calculate A_m :



$$A_m = \iint_A \frac{1}{r}\, dA$$

$$= \int_a^b \frac{1}{r}\, b \cdot dr = b \ln(c/a)$$

Nov. 22/18



D : moment center

Internal shear forces are positive

Step 1: positive shear force V_x

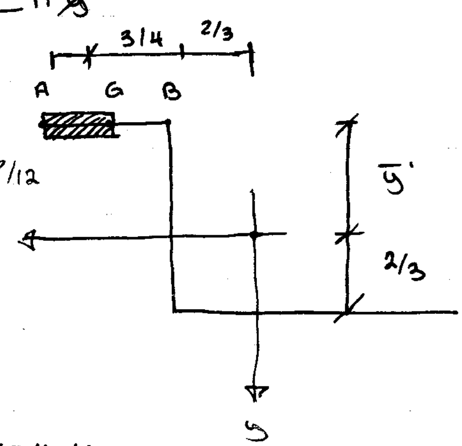
if sheer flow is positive, the line of action of V_x is above D

$$e = \frac{\frac{AB}{6}(q_A + 4q_G + q_B) \times BD}{V_x}$$

$$q = \frac{V_yI_y - V_xI_{xy}}{\Delta}A'\bar{y}' + \frac{V_xI_x - V_yI_{xy}}{\Delta}A'\cancel{\bar{y}}'$$ should be $\bar{x}'$

$(V_y = 0)$

$$q = -\frac{V_xI_{xy}}{\Delta}A'\bar{y}' + \frac{V_xI_x}{\Delta}A'\cancel{\bar{y}}'$$ $\bar{x}'$



At G, $A' = 0.5t$

$\bar{x}' = 3/4 + 2/3 = 17/12$

$\bar{y}' = -4/3$

$q_G = 0.02944\,V_x$

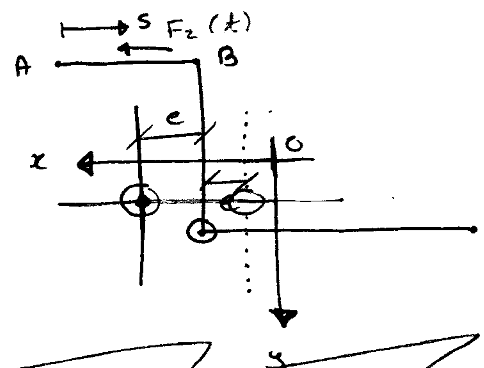
At B, $q_B = -0.01674\,V_x$

the resultant $F = \frac{AB}{6}(q_A + 4q_G + q_B) = 0.01684\,V_x$

red line, distance

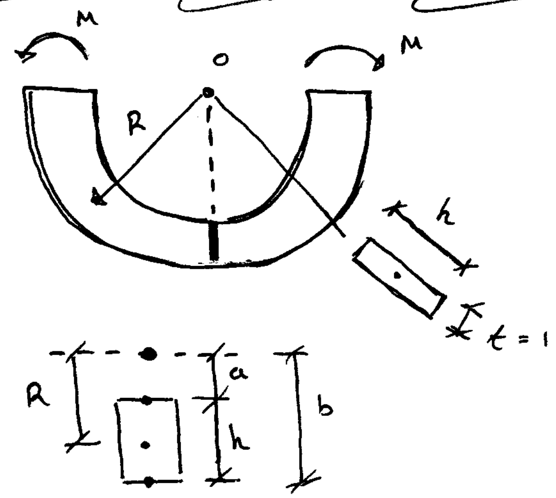
$$e = \frac{F \times BD}{V_x} = 0.03368 \text{ in}$$

Step 2 : Positive internal shear force V_y



The resultant shear flow on AB F_z,

if F_z is positive,



1° curved beam theory

$$\sigma_{\theta\theta} = \frac{M}{A(RA_m - A)}\left(\frac{A}{r} - A_m\right)$$

M : internal bending moment

A : area of cross-section $[A = th]$

R : centroidal radius

$$A_m = \iint_A \frac{1}{r}\, dA$$

$$\rightarrow = t \ln\left(\frac{R+h/2}{R-h/2}\right)$$

Case : $\dfrac{R}{h} = 1$

Inner radius : $a = R - \frac{h}{2} = \frac{h}{2}$

Outer radius : $b = R + \frac{h}{2} = \frac{3h}{2}$

$\Rightarrow b = 3a$

$$\sigma_{\theta\theta} = \frac{M}{th(h\,t\ln(3) - th)}\left(\frac{th}{r} - t\ln(3)\right)$$

$$\sigma_{\theta\theta} = \frac{M}{th^2(\ln 3 - 1)}\left(\frac{h}{r} - \ln(3)\right)$$

At inner surface, $r = a = h/2$

$$\sigma_{\theta\theta,max} = \frac{M}{th^2} \cdot \frac{1}{\ln(3)-1}(2-\ln 3)$$

$$= 2.28518 \frac{M}{th^2} \quad \Leftarrow \text{ approximate solution}$$

Elasticity :

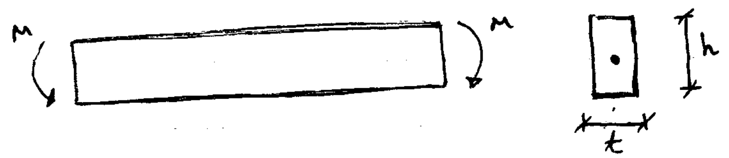
$$\sigma_{\theta\theta} = \frac{4M}{Q}\left[\frac{-a^2b^2}{r^2}\ln\left(\frac{b}{a}\right) + b^2\ln\left(\frac{r}{b}\right) + a^2\ln\left(\frac{a}{r}\right) + b^2 - a^2\right]$$

$$Q = 4a^2b^2\left(\ln\left(\frac{b}{a}\right)\right)^2 - (b^2-a^2)$$

at $r = a$

$$\sigma_{\theta\theta} = 2.29199 \frac{M}{th^2} \quad \Leftarrow \text{ exact solution}$$

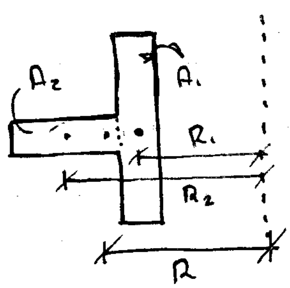
Straight Beam Theory :



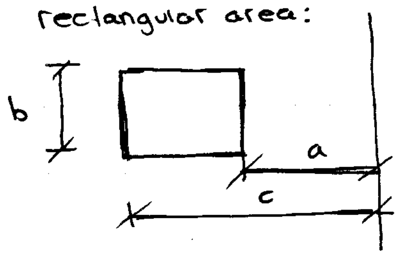
$$\sigma_{max} = \frac{M}{I}\cdot\frac{h}{2} = \frac{M}{\frac{1}{12}th^3}\cdot\frac{h}{2}$$

$$= 6\frac{M}{th^2}$$

composite area :



rectangular area:



$$A_M = \iint \frac{1}{r}\,dA$$

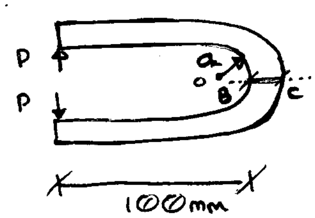
$$= A_m = \iint_{A_1} \frac{1}{r}\,dA + \iint_{A_2} \frac{1}{r}\,dA$$

$$\rightarrow A_m = A_{m_1} + A_{m_2}$$

$$R = \frac{A_1R_1 + A_2R_2}{A_1 + A_2}$$

$$A_m = b\ln\left(\frac{c}{a}\right)$$

Example :

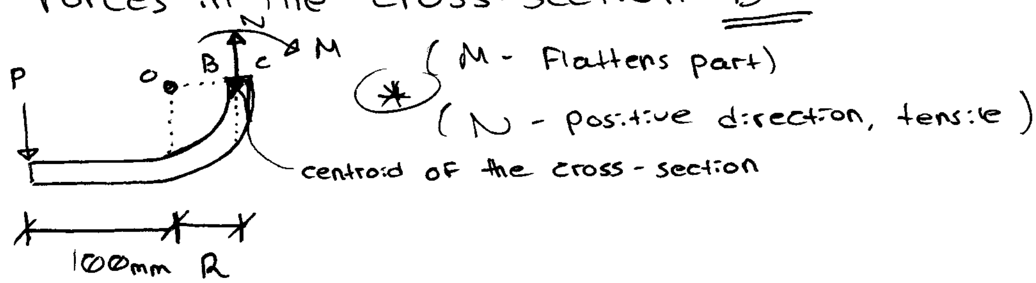


$a = 30\,mm$

Determine the value of the max tensile and max compressive stresses in the frame.

Solution : BC

Internal Forces in the cross-section BC



$$\hookrightarrow R = 30 + (50/2) = 55\,mm$$

$$\Rightarrow N = P = 9.50\ kN = 9500\,N$$

$$M = P(100 + 55) = 1472500\ N\cdot mm$$

Geometry

$$R = 55\,mm$$

$$A = 50 \times 50 = 2500\,mm^2$$

$$A_m = b\ln(c/a) = 50\ln\left(\frac{80}{30}\right)$$

$$\Rightarrow A_m = 49.0415\,mm \text{ (use all decimals)}$$

Normal stress

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M}{A(RA_m - A)}\left(\frac{A}{r} - A_m\right)$$

$$= \frac{9500}{2500} + \frac{1472500}{(2500)((55 \times 49.0415) - 2500))} \times \left(\frac{2500}{r} - 49.0415\right)$$

r : mm

$\sigma_{\theta\theta}$: MPa

At $r = a = 30$ mm

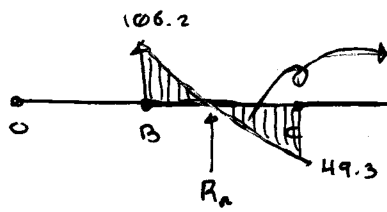
$$\sigma_{\theta\theta} = 3.8 + 2.9856\left(\frac{2500}{30} - 49.0415\right)$$
$$= 106.2 \text{ MPa}$$

At $r = 30 + 50 = 80$ mm

$$\sigma_{\theta\theta} = 3.8 + 2.9856\left(\frac{2500}{80} - 49.0415\right)$$
$$= -49.3 \text{ MPa}$$

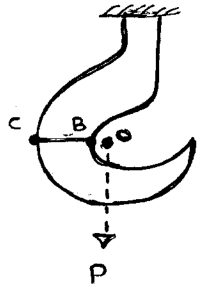
1° neutral axis ? $R_n = 52.3355$ mm

2° distribution of $\sigma_{\theta\theta}$ in the radial direction



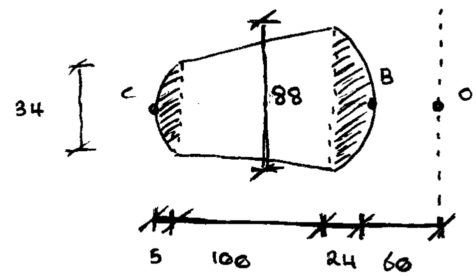
not a straight line !!

Example :



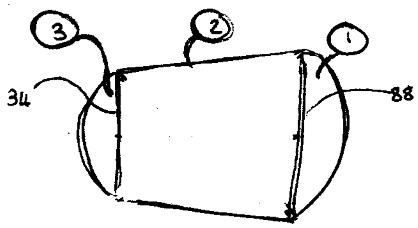
$Y = 500$ MPa

$SF = 2.00$



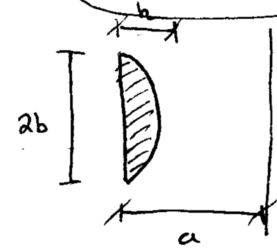
Find the max load the crane hook can support

Solution : A, R, A_m (Pg 324 - 325)



elliptical

For area 1 :



$a = 60 + 24 = 84$

$2b = 88 \rightarrow b = 44$

$h = 24$

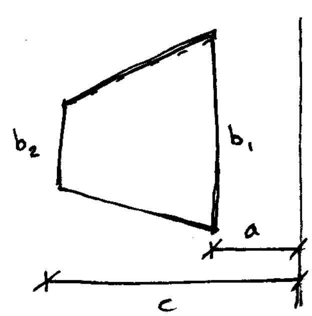
$$A_1 = \frac{\pi b h}{2} = \frac{\pi(44)(24)}{2}$$

$$A_1 = 1658.76$$

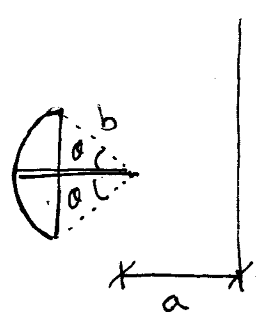
$$R_1 = a - \frac{4h}{3\pi} = 84 - \frac{(4)(24)}{(3\pi)} = 73.81$$

$$A_m' = 2b + \frac{\pi b}{h}\left(a - \sqrt{a^2-h^2}\right) - \frac{2b}{h}\sqrt{a^2-h^2}\arcsin\left(\frac{h}{a}\right) = 22.64$$

For area 2 :

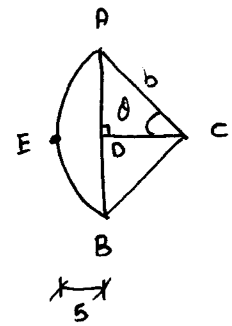


$$\left.\begin{array}{l} a = 24 + 60 = 84 \\ c = 100 + a = 184 \\ b_1 = 88 \ , \ b_2 = 34 \end{array}\right\} \begin{array}{l} A_2 = 6100 \\ R_2 = 136.62 \\ A_{m_2} = 50.57 \end{array}$$



Area 3

→ circular arc



$AC = b$

$CD = b - 5$

$AD = \frac{1}{2}(34) = 17$

$\triangle ACD \; : \; AC^2 = AD^2 + CD^2$

$b^2 = 17^2 + (b-5)^2$

$\rightarrow b = 31.4$

$\sin\theta = \frac{AD}{AC} = \frac{17}{31.4} \Rightarrow \theta = 32.78°$

and $a = 100 + 24 + 60 - (31.4 - 5)$

$= 157.6 > b = 31.4$

$A_3 = 115.27$

$R_3 = 186.01$

$Am_3 = 0.62$

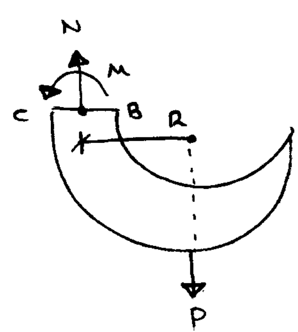
For the cross section:

$A = A_1 + A_2 + A_3 = 7874.03 \text{ mm}^2$

$Am = Am_1 + Am_2 + Am_3 = 73.83 \text{ mm}$

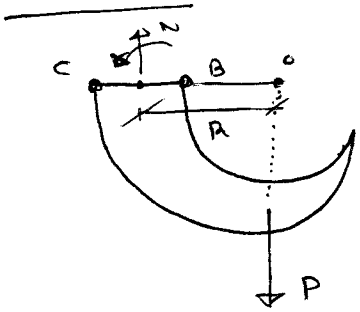
$$R_1 = \frac{A_1R_1 + A_2R_2 + A_3R_3}{A} = 116.37 \text{ mm}$$

Statics:



. . . (*)

Statics



$N = P$

$M = PR$

$$\sigma_B = \frac{N}{A} + \frac{M}{A(RA_m - A)} \cdot \left(\frac{A}{r_B} - A_m\right) = 0.001309\,P$$

$$\sigma_c = \frac{N}{A} + \frac{M}{A(RA_m - A)} \cdot \left(\frac{A}{r_c} - A_m\right) = -0.000535\,P$$

$$\therefore \sigma_{max} = 0.001309\,P$$

$$\sigma_y / F_s = 0.001309\,P$$

$$500/2 = 0.001309\,P$$

$$P = 190900\ N$$

Chapter 11 : The thick-wall cylinder

Geometry : A thick wall cylinder

wall thickness is constant

* closed cylinder : with end caps
* open cylinder : w/o end caps

Loading : Internal pressure p_1

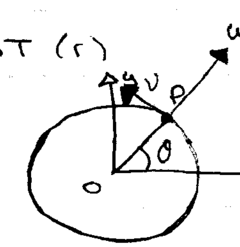
External pressure p_2

Axial load, P

Temperature Change $\Delta T(r)$

Deformation : axisymmetric :

$P(x,y,z) \Rightarrow P(r,\theta,z)$



Displacement:

$$\begin{cases} u(r,\theta,z) \\ v(r,\theta,z) \\ w(r,\theta,z) \end{cases}$$

$$\begin{cases} u = u(r, z) \\ v = \emptyset \\ w = w(r, z) \end{cases}$$

Consider the cross-section far away from the end caps :

$$\begin{cases} u = u(r) \\ v = \emptyset \end{cases}$$

$$\epsilon_{zz} = \partial w / \partial z = \text{const.}$$

Disp - strain :

$$\epsilon_{rr} = \partial u / \partial r = du/dr$$

$$\epsilon_{\theta\theta} = \frac{1}{r} \partial u / \partial \theta + \frac{u}{r} = \frac{u}{r}$$

$$\epsilon_{zz} = \text{const.}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{\partial v}{\partial r}\right) = \emptyset$$

$$\epsilon_{\theta z} = \frac{1}{2}\left(\partial u / \partial z + \frac{1}{r}\ \partial w / \partial z\right) = \emptyset$$

$$\epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) = \emptyset$$

Compatability equation

$$d/dr\,(r\,\epsilon_{\theta\theta}) = \epsilon_{rr}$$

Hooke's Law

$$\begin{cases} \epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right) + \alpha \Delta T \\ \epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{zz} + \sigma_{rr})\right) + \alpha \Delta T \\ \epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right) + \alpha \Delta T = \text{const.} \end{cases}$$

Equilibrium eq-n-s :

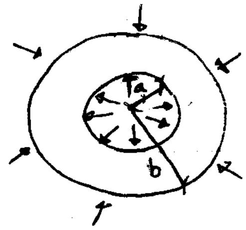
$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\cancel{\frac{\partial \sigma_{r\theta}}{\partial \theta}}^{\,0} + \cancel{\frac{\partial \sigma_{rz}}{\partial z}}^{\,0} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \emptyset$$

$$\rightarrow \frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \emptyset$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = \emptyset$$

$$\cancel{\frac{\partial \sigma_{rz}}{\partial r}} + \cancel{\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \cancel{\frac{\sigma_{rz}}{r}} = 0$$

Boundary Conditions :



At $r = a$, $\sigma_{rr} = -P_1$

At $r = b$, $\sigma_{rr} = -P_2$

$\Rightarrow$ Find σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, u and w

Solution for constant temperature :

$$\sigma_{rr} = \frac{P_1 a^2 - P_2 b^2}{b^2 - a^2} - \frac{a^2 b^2}{r^2(b^2-a^2)}(P_1 - P_2)$$

$$\sigma_{\theta\theta} = \frac{P_1 a^2 - P_2 b^2}{b^2 - a^2} - \frac{a^2 b^2}{r^2(b^2-a^2)}(P_1 - P_2)$$

$$\sigma_{zz} = \frac{P_1 a^2 - P_2 b^2}{b^2 - a^2} + \frac{P}{\pi(b^2-a^2)}$$

$$u = \frac{r}{E(b^2-a^2)}\left[(1-2\nu)(P_1 a^2 - P_2 b^2) + (1+\nu)\left(\frac{a^2 b^2}{r}\right)(P_1 - P_2) - \frac{\nu P}{\pi}\right]$$

Example : Cylinder with internal pressure P_1 only.

Find the max shear stress.

Solution: Since $P_2 = 0$,

$$\sigma_{rr} = \frac{P_1 a^2}{b^2-a^2} - \frac{a^2 b^2}{r^2(b^2-a^2)} P_1$$

$$= \frac{P_1 a^2}{b^2-a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{P_1 a^2}{b^2-a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_{zz} = \frac{P_1 a^2}{b^2-a^2}$$

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$: the principal stresses $(\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr})$

$\therefore \tau_{max}(r) = \dfrac{\sigma_{\theta\theta} - \sigma_{rr}}{2} = \dfrac{P_1 a^2 b^2}{(b^2-a^2) r^2}$

$\therefore$ max shear occurs at the inner surface where $r = a$

$\therefore \tau_{max} = \dfrac{P_1 b^2}{b^2 - a^2}$

$\tau_{max} = \dfrac{P_1 \, (b/a)^2}{(b/a)^2 - 1}$

Case study: $b/a = 3$

P_1 produces the allowable τ_{max}

$\Rightarrow \tau_{max} = (P_1) \cdot (9/8)$

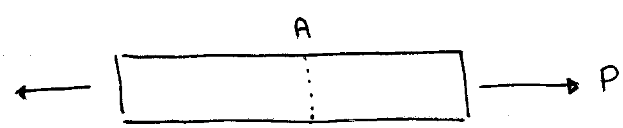
To maintain the max shear τ_{max} under the new internal pressure $1.1 P_1$, the new cylinder should have the ratio b/a:

$$\tau_{max} = \frac{1.1 P_1 \, (b/a)^2}{(b/a)^2 - 1} = (P_1) . (9/8)$$

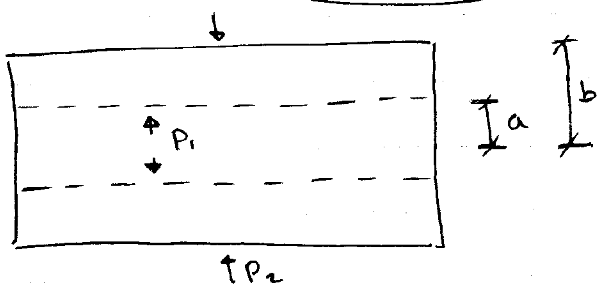
$\rightarrow \dfrac{1.1 \, (b/a)^2}{(b/a)^2 - 1} = 9/8$

$\hookrightarrow b/a = 6.7$

Case:



$\sigma = P/A$



$\sigma_{\theta\theta} = \dfrac{P_1 a^2 - P_2 b^2}{b^2 - a^2} \ldots$

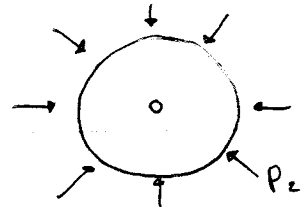
$\ldots + \dfrac{a^2 b^2}{r^2 (b^2 - a^2)} (P_1 - P_2)$

$b \to \infty \qquad (b/a \to \infty)$

$\sigma_{\theta\theta} = -P_2$ At the inner surface, $r = a$

$\sigma_{\theta\theta} = -P_2 + P_1 - P_2$

$$\boxed{\sigma_{\theta\theta} = P_1 - 2P_2}$$



IF $P_1 = 0$, $\sigma_{\theta\theta} = -2P_2$

11.7 Rotating Disc of Constant Thickness

Geometry : t = const., $t << b$ (outer radius)

State of Stress : plane stress & axisymmetry

$$\sigma_{zr} = \sigma_{z\theta} = \sigma_{zz} = \sigma_{r\theta} = 0$$

and σ_{rr} and $\sigma_{\theta\theta}$ are functions of r

Equation of motion :

$$\begin{cases} \partial\sigma_{rr}/\partial r + \frac{1}{r}\cancel{\frac{\partial\sigma_{r\theta}}{\partial\theta}}^{0} + \cancel{\frac{\partial\sigma_{rz}}{\partial z}}^{0} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho r\omega^2 = 0 \\ \cancel{\frac{\partial\sigma_{r\theta}}{\partial r}}^{0} + \frac{1}{r}\cancel{\frac{\partial\sigma_{\theta\theta}}{r\,\partial\theta}}^{0} + \cancel{\frac{2\sigma_{r\theta}}{r}}^{0} = 0 \end{cases}$$

$$\Rightarrow \quad d\sigma_{rr}/dr + (\sigma_{rr} - \sigma_{\theta\theta})/r + \rho r\omega^2 = 0$$

Stress-strain - temperature :

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}(\epsilon_{rr} - \nu\epsilon_{\theta\theta}) - \frac{E\alpha T}{1-\nu} \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\epsilon_{\theta\theta} - \nu\epsilon_{rr}) - \frac{E\alpha T}{1-\nu} \end{cases}$$

Strain - disp :

$$\epsilon_{rr} = \partial u/\partial r = du/dr$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\partial u/\partial\theta + u/r = u/r$$

The solution of the displacement : $u = u(r)$

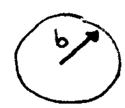
$$u(r) = \frac{1-\nu^2}{8E}\rho\omega^2 r^3 + \frac{\alpha(1+\nu)}{r}\int_a^r rT\,dr + C_1 r + C_2/r$$

C_1 and C_2 are unknown constants.

The Stress ↙ temp. change

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\frac{u}{r}\right] - \frac{E\alpha T}{1-\nu} \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[\nu\frac{du}{dr} + \frac{u}{r}\right] - \frac{E\alpha T}{1-\nu} \end{cases}$$

Case 1 : Solid Disk



(w/ const. temperature : $T = \Delta T = 0$)

Traction free at $r = b$

Boundary condition :

At $r = b$, $\sigma_{rr} = 0$

$$\sigma_{px} = \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n$$
$$\sigma_{py} = \sigma_{xy} + \sigma_{yy} m + \sigma_{yz} n$$
$$\sigma_{pz} = \sigma_{xz} + \sigma_{yz} m + \sigma_{zz} n$$

$$\sigma_{pr} = \sigma_{rr} l + \cancel{\sigma_{r\theta} m}^{\,0} + \cancel{\sigma_{rz} n}^{\,0}$$
$$\sigma_{p\theta} = \cancel{\sigma_{r\theta} l}^{\,0} + \cancel{\sigma_{\theta\theta} m}^{\,0} + \cancel{\sigma_{\theta z} n}^{\,0}$$
$$\sigma_{pz} =$$

At $r = b$, $\sigma_{rr} = 0$

At $r = 0$, $|u| < \infty$

$C_2 = 0$

After solving for C_1, we have:

$$\sigma_{rr} = [(3+\nu)/8]\, \rho\omega^2 (b^2 - r^2)$$
$$\sigma_{\theta\theta} = [(3+\nu)/8]\, \rho b^2 \omega^2 - \frac{1+3\nu}{8} \rho\omega^2 r^2$$

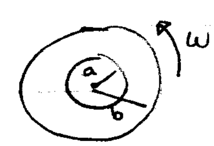
The displacement:

$$u(r) = \frac{1}{8E} \rho\omega^2 [(1-\nu)(3+\nu) b^2 r - (1-\nu^2) r^3]$$

The max normal stress occurs at the center of the solid disk:

$$\sigma_{rr,\,max} = \sigma_{\theta\theta,\,max} = \frac{3+\nu}{8} \rho b^2 \omega^2$$

Case 2: A disk with a hole and $T = 0$



At $r = a$, $\sigma_{rr} = 0$

At $r = b$, $\sigma_{rr} = 0$

$$\Rightarrow \sigma_{rr} = \frac{3+\nu}{8} \rho\omega^2 \left[b^2 + a^2 - \frac{a^2 b^2}{r^2} - r^2 \right]$$
$$\sigma_{\theta\theta} = \frac{3+\nu}{8} \rho\omega^2 \left[b^2 + a^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right]$$

$(\sigma_{\theta\theta} > \sigma_{rr})$

The maximum normal stress

For σ_{rr}: $\dfrac{d\sigma_{rr}}{dr} = 0$

$r = \sqrt{ab}$ and $\sigma_{rr,max} = \dfrac{3+\nu}{8} \rho\omega^2 (b-a)^2$

For $\sigma_{\theta\theta}$:

Max $\sigma_{\theta\theta}$ occurs at $r = a$

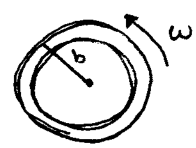
and $\sigma_{\theta\theta,max} = \left(\frac{3+\nu}{4}\right)\rho\omega^2\left(b^2 + \frac{r\nu}{3+\nu}a^2\right)$

Consider when $a \to 0$

$\sigma_{\theta\theta,max} \to \left(\frac{3+\nu}{4}\right)\rho\omega^2 b^2$

Special case : $a \xrightarrow[\text{approaches}]{} b$

$\sigma_{\theta\theta,max} \longrightarrow \rho(b\omega)^2$



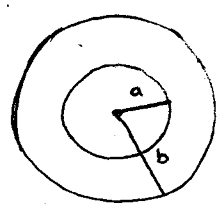
Example :

$a = 0.1\,m$

$b = 0.3\,m$

$E = 200\,GPa$

$\nu = 0.29$



$\rho = 7850\ kg/m^3$

$Y = 620\ MPa$

* max shear stress criterion

The disc is traction free at $r = a$, and $r = b$

$T = 0$

Find a) the max angular velocity ω

b) at the yield velocity, what is the change in thickness in the radial direction.

Solution : $\tau_{max} = \frac{\sigma_{\theta\theta,max}}{2} = \frac{Y}{2}$

$\Rightarrow \sigma_{\theta\theta,max} = Y$

$\Rightarrow \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right) Y$

$\Rightarrow \omega = 1020.77$ rad/s

b) Since $u(r) = \frac{\rho\omega^2}{E}\left[\frac{(1-\nu)(3+\nu)}{8}(b^2+a^2)r + \frac{(1+\nu)(3+\nu)}{8}\frac{a^2b^2}{r} - \frac{r\nu^2}{8}r^3\right]$

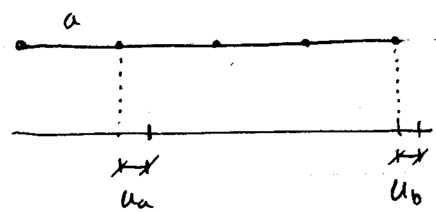
At $r = a$

$u_a = u(a) = \frac{\rho\omega^2}{E}\cdot a\cdot\left[(1-\nu)a^2 + (3+\nu)b^2\right]$

$= 0.0003100$

At $r=b$:

$$u_b = u(b)$$
$$= \frac{\rho\omega^2}{4E} \cdot b \cdot [(3+\nu)a^2 + (1-\nu)b^2)$$
$$= 0.0002969 \text{ m}$$



New inner radius $a' = a + u_a$

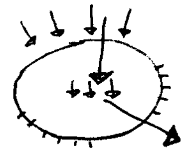
New outer radius $b' = b + u_b$

$\therefore$ New thickness in the radial direction

$$h' = b' - a' = (b + u_b) - (a + u_a)$$
$$= 0.2 + 0.0002969 - 0.0003100$$
$$= 0.1999999869 \text{ m}$$

Theory of Elasticity

Stress, strain



Disp: $u(x,y,z), v(x,y,z), w(x,y,z)$

Stress: $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz}$ } 15 unknowns

Strain: $\epsilon_{xx}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yy}, \epsilon_{yz}, \epsilon_{zz}$

Strain Disp:

$$\begin{cases} \epsilon_{xx} = \partial u/\partial x \ \cdots \\ \epsilon_{xy} = \tfrac{1}{2}(\partial u/\partial y + \partial v/\partial x) \ \cdots \end{cases}$$

Hooke's Law:

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \ \cdots$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \ \cdots$$

Equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$

$$\vdots$$

Boundary conditions:

∂uP : $u = \bar{u}, v = \bar{v}, w = \bar{w}$

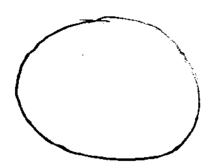
$\partial \sigma P$: $\sigma_{px} = \bar{f_x}, \sigma_{py} = \bar{f_y}, \sigma_{yz} = \bar{f_z}$

Compatability Conditions (6)

* Torsion of a general cross-section:

— Semi-inverse method

$u = -\theta yz, \quad v = \theta xz$



CROSS-SECTION

Stress Function $\phi(x,y)$:

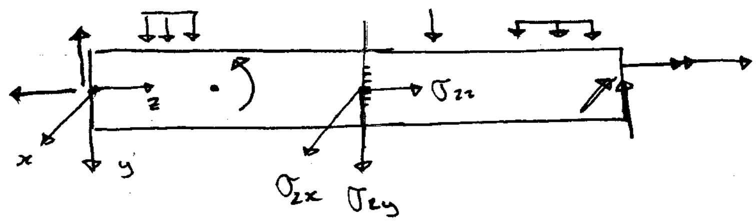
$$\left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \right. \qquad \left\{ \begin{array}{l} \phi = \text{const along boundary} \\ *T = 2\iint_A \phi \, dA \end{array} \right.$$

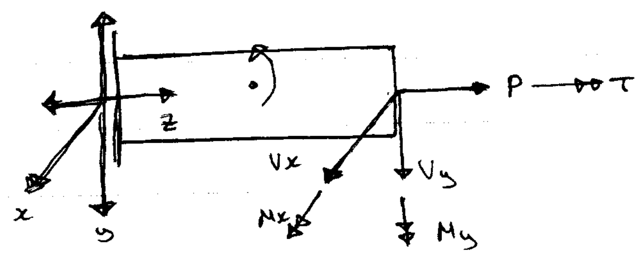
$\sigma_{xz} = \partial\phi / \partial y$
$\sigma_{yz} = -\partial\phi / \partial x$

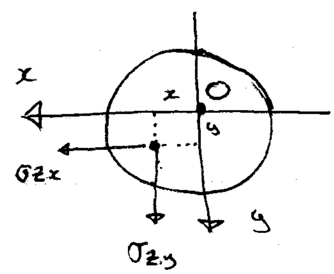
* Thick cylinder and rotating disk

* * σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz}
* * $u(r)$

Mechanics of Materials Method :







$$\iint_A \sigma_{zz}\, dA = P$$
$$\iint_A \sigma_{zx}\, dA = V_x$$
$$\iint_A \sigma_{zy}\, dA = V_y$$
$$\iint_A y\,\sigma_{zz}\, dA = M_x$$
$$\iint_A x\,\sigma_{zz}\, dA = M_y$$
$$\iint_A (x\sigma_{zy} - y\sigma_{zx})\, dA = T$$

* Thin-wall member

1° torsion → open
→ closed

2° bending