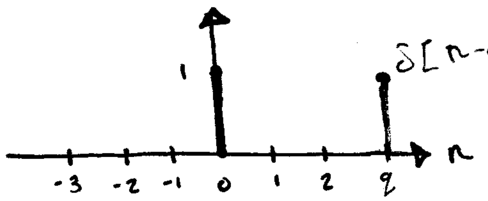
- Conv. with the unit pulse

$\delta[n]$



$$x[n] \otimes \delta[n] = \sum_{i=-\infty}^{\infty} x[i]\underbrace{\delta[n-i]}_{\substack{=0 \\ \text{if } n-i \neq 0}}$$

If $n-i = 0$, $\delta[n-i] = 1$
$i = n$

$$x[n] \otimes \delta[n] = x[n]\delta[n-n]$$
$$= x[n]$$

- Conv. with a shifted unit pulse

$$x[n] \otimes \delta[n-q]$$
$$= \sum_{i=-\infty}^{\infty} x[i]\,\delta[n-q-i]$$

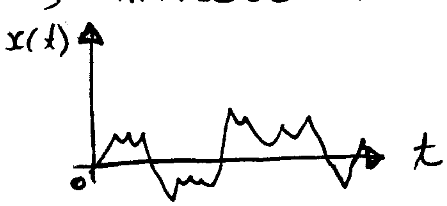
$$\delta[n-q-i] = 1$$
$$n-q-i = 0$$
$$i = n-q$$

$$= x[n-q]\,\delta[\underbrace{n-q-(n-q)}_{0}]$$
$$= x[n-q]$$

Chapter 3 : Tools for Signal Processing

3.1 Properties of Continuous FT

1) Introduction

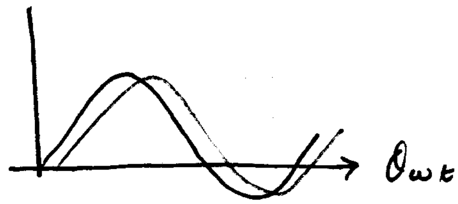


A signal consists of sinusoids with different frequencies, magnitudes, and phase functions.

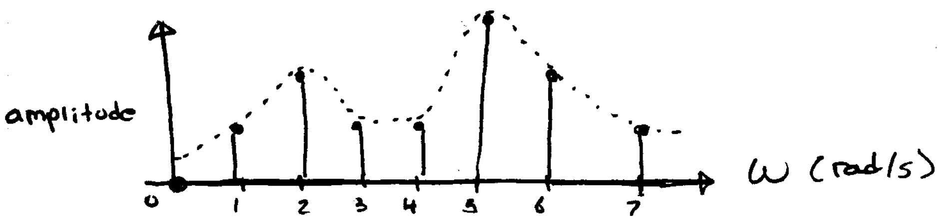
$$x(t) = A_0\cos(\omega_0 t + \theta_0) + A_1\cos(\omega_1 t + \theta_1) + \dots + A_n\cos(\omega_n t + \theta_n)$$
$$= \sum_{n=0}^{\infty} A_n\cos(\omega_n t + \theta_n)$$

where A_n = magnitudes
ω_n = Freq. (rad/s)
$\omega = 2\pi f$
f = Hz
θ_n = Phase angles

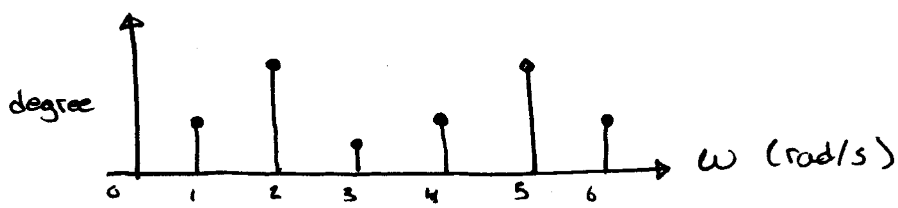
time delay, 0.001 sec



amplitude spectrum,



Phase spectrum,

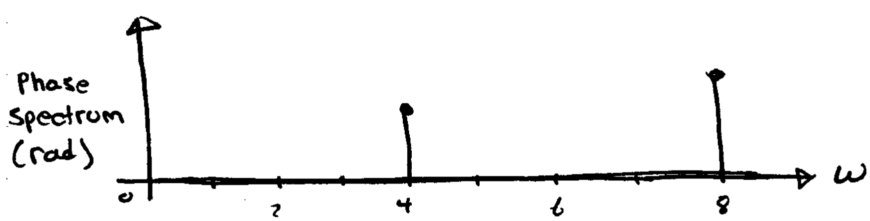


$\boxed{\text{Example}}$

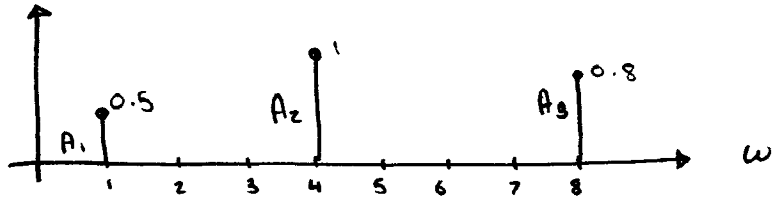
$x(t) = A_1 \cos t + A_2 \cos(4t + \pi/3) + A_3 \cos(8t + \pi/2)$

$\omega_1 = 1$ rad/s ; $\omega_2 = 4$; $\omega_3 = 8$

$0, \pi/3, \pi/2$



line spectrum



amp. spectrum

$$\left(\begin{array}{l} t = 0.200 \\ A1 = 0.5, \quad A2 = 2, \quad A_3 = 1 \end{array} \right)$$

$X_t = A_1 \cos(1 \times t) + A_2 \cos(4t + Pi/3)$ ... etc., plot(x)

2 Continuous FT (CFT)

Given $x(t)$

→ lower case letter - time domain signal

capital letter - Freq. Fn

$$X(\omega) = \int x(t)\, e^{-j\omega t}\, dt$$

Freq. : ω = Frequency variable $-\infty < \omega < \infty$

$j = \sqrt{-1}$

$e^{-j\omega t} \longrightarrow$ Complex valued

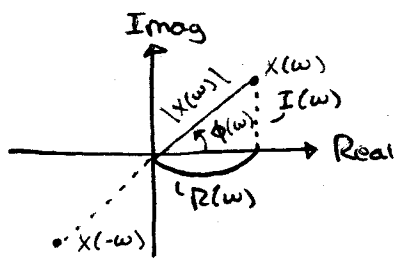
$X(\omega) \longrightarrow$ complex valued Fxn

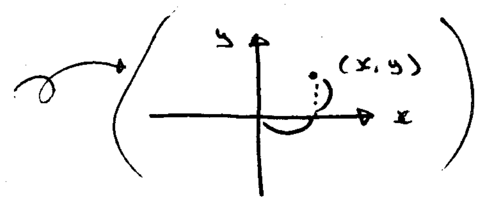
— Rectangular representation

$X(\omega) = R(\omega) + jI(\omega)$

$R(\omega)$ = Real part

$I(\omega)$ = Imaginary part



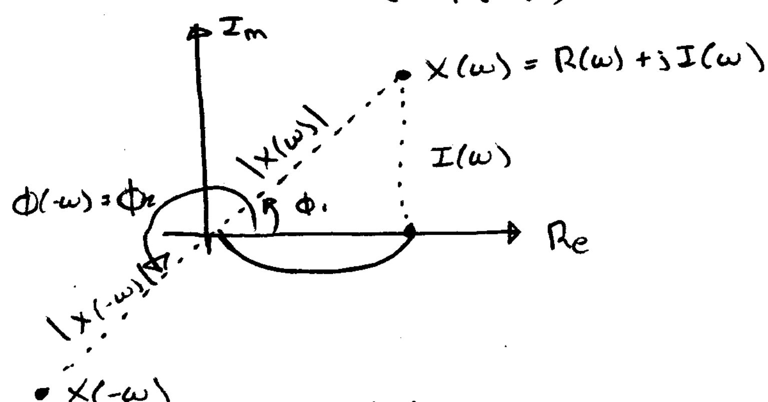


Polar representation

$X(\omega) = |X(\omega)| e^{j\phi(\omega)}$

$|X(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$

$\phi(\omega) = \tan^{-1}\left(\dfrac{I(\omega)}{R(\omega)} \right)$



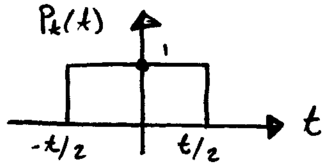
$X(-\omega) \sim$ complex conjugate $X(\omega)$

$|X(-\omega)| = |X(\omega)|$

$$\begin{cases} \phi(-\omega) = \phi(\omega) + \pi \\ \phi(-\omega) = -\phi(\omega) \end{cases}$$

EXAMPLE 3.2

$$P_t(t) = \begin{cases} 1 & -t/2 < t < t/2 \\ 0 & \text{otherwise} \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{\infty} | x(t)\, e^{-j\omega t}\, dt | < \infty$$

Solution :

$$X(\omega) = \int_{-T/2}^{T/2} 1 * e^{-j\omega t}\, dt$$

Euler's Formula :

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \theta = -\omega t$$

$$X(\omega) = \int_{-T/2}^{T/2} [\cos(\omega t) - j\sin(\omega t)]\, dt$$

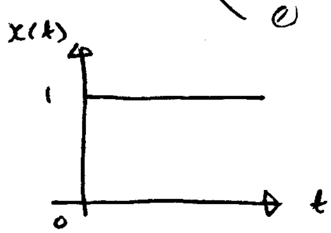
$$= [\, 1/\omega \sin(\omega t) + j\, 1/\omega \cos(\omega t)\,]_{-T/2}^{T/2}$$

$$= 1/\omega [\sin(\omega t/2) - \sin(-\omega t/2)] + j/\omega [\cos[\omega t/2 - \cos(-\omega t/2)]$$

$$= 2/\omega \sin(\omega t/2) + \cancel{j\, 1/\omega \cos(\omega t/2)}$$

Example 3.3

$$x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Solution

$$X(\omega) = \int_0^{\infty} 1 e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega} \int_0^{\infty} e^{-j\omega t}\, d(-j\omega t)$$

$$X(\omega) = -\left(\frac{1}{j\omega}\right) e^{-j\omega t} \Big|_0^{\infty}$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega\infty} - 1\right] = -\frac{1}{j\omega}\left[\cos(-\omega t) + j\sin(-\omega t)\right]_0^{\infty}$$

$$= -\frac{1}{j\omega}\left[\ \ldots\ \text{DNE}\right.$$

A Signal

a series of sinusoids

ω_i, θ_i, A_i

$A_i \sim \omega_i$: amplitude spectrum

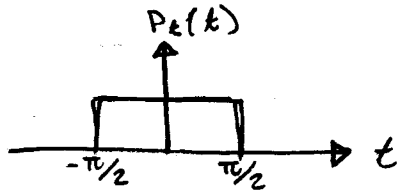
$\theta_i \sim \omega_i$: Phase spectrum

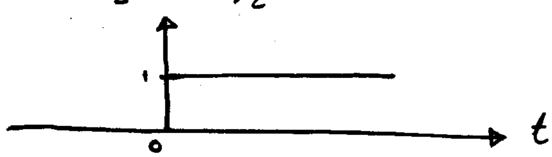
CFT (continuous fourier transform)

$x(t)$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \quad -\infty < \omega < \infty$$

$$\int_{-\infty}^{\infty} |x(t) e^{-j\omega t}| dt < \infty$$





$$x(\omega) = \int_0^{\infty} 1 * e^{-j\omega t} dt$$

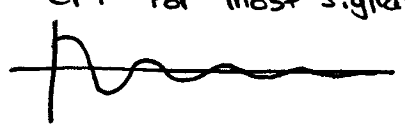
$$= \int_0^{\infty} [\cos(-\omega t) + j\sin(-\omega t)] dt$$

$$= \int_0^{\infty} [\cos(\omega t) - j\sin(\omega t)] dt$$

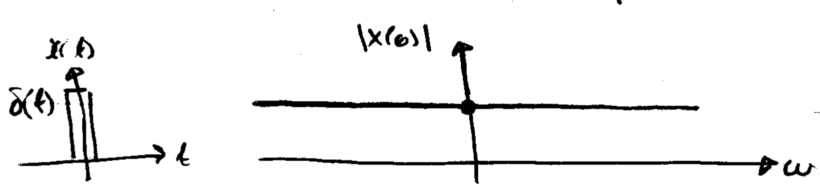
where $\rightarrow$ $= \int_0^{\infty} |\cos(\omega t)| dt = \frac{1}{\omega} |\sin(\omega t)| \Big|_0^{\infty}$

$$= \frac{1}{\omega} |\sin(\omega * \infty)|$$

- most signals with the product of $e^{-j\omega t}$ don't satisfy the sufficient integral conditions
- CFT for most signals don't exist in the original sense



- CFT is undertaken in a generalized sense
- use FT pairs + FT properties to do FT



Common Fourier Transform Pairs (handout)

- Inverse FT

Given $X(\omega)$ Filter, controls

IFT :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad (-\infty < t < \infty)$$

CFT :

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad (-\infty < \omega < \infty)$$

FT Pairs,

$x(t) \longleftrightarrow X(\omega)$

$\boxed{\text{Example 3}}$

$X(\omega) = \cos(\omega t)$, $x(t) = ?$

Based on Euler's formula :

$e^{j\theta} = \cos\theta + j\sin\theta \longleftarrow (1)$

$e^{-j\theta} = \cos\theta - j\sin\theta \longleftarrow (2)$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$

$$\rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Solution

$$\begin{aligned} X(\omega) &= \cos(2\omega) \\ &= (1/2)\left[e^{2j\omega} + e^{-2j\omega}\right] \\ &\longleftrightarrow (1/2)\left[\delta(t+2) + \delta(t-2)\right] \end{aligned}$$

$-j\omega c = j2\omega$

4) Some properties of the CFT

- Linearity

$x(t) \longleftrightarrow X(\omega)$, $v(t) \longleftrightarrow V(\omega)$

$ax(t) + bv(t) \longleftrightarrow aX(\omega) + bV(\omega)$

Proof

$$\int_{-\infty}^{\infty} [ax(t) + bv(t)] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [a x(t) e^{-j\omega t} + b v(t) e^{-j\omega t}] \, dt$$
$$= \underbrace{a \int_{-\infty}^{\infty} x(t) e^{-j\omega t}}_{X(\omega)} dt + \underbrace{b \int_{-\infty}^{\infty} v(t) e^{-j\omega t}}_{V(\omega)} dt$$

- Shifts in time

If $x(t) \longleftrightarrow X(\omega)$

$x(t-c) \longleftrightarrow X(\omega) e^{-j\omega c}$

Proof :

$\int x(t-c) e^{-j\omega t} dt$; $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Let :

$\lambda = t - c$

$t = \lambda + c$

$dt = d\lambda$

$$\int_{-\infty}^{\infty} x(t-c) e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda + c)} d\lambda$$
$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} \cdot e^{-\lambda\omega c} d\lambda$$
$$= e^{-j\omega c} \underbrace{\int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda}_{X(\omega)}$$

- Time scaling

$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt}$

$x(t) \longleftrightarrow X(\omega)$

$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$; $a > 0$

$\int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$

Let $\lambda = at$, $t = \frac{1}{a}\lambda$, $dt = \frac{1}{a} d\lambda$

$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega \frac{1}{a}\lambda} (1/a) \, d\lambda$$
$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j(\omega/a)\lambda} d\lambda$$
$$= \frac{1}{a} X(\omega/a)$$

- Time reversal

$x(t) \longleftrightarrow X(\omega)$

$x(-t) \longleftrightarrow X(-\omega)$

Proof :

$\int_{-\infty}^{\infty} x(-t) e^{-j\omega t} dt$

Let $\lambda = -t$, $t = -\lambda$, $dt = -d\lambda$

$$= \int_{+\infty}^{-\infty} x(\lambda) e^{-j\omega(-\lambda)} (-d\lambda)$$
$$= + \int_{+\infty}^{-\infty} x(\lambda) e^{-j(-\omega)\lambda} \, d\lambda$$
$$= X(-\omega)$$

Convolution in the time domain

- If $x(t) \longleftrightarrow X(\omega)$, $v(t) \longleftrightarrow V(\omega)$

A2Q1 $\rightarrow$ $x(t) \otimes v(t) \longleftrightarrow X(\omega) * V(\omega)$

$$\xrightarrow[x(t)]{\text{Input}} \boxed{\text{System}}^{\,v(t)} \xrightarrow[y(t)]{\text{output}}$$

$y(t) = x(t) \otimes v(t)$ ✗ slow

Input $x(t) \longleftrightarrow X(\omega)$

$v(t) \longleftrightarrow V(\omega)$

$\downarrow$

$X(\omega) * V(\omega)$ ✓ Faster

$\downarrow$ IFT

$y(t)$

$\boxed{\text{Example}}$

$X(\omega) = \dfrac{1}{j\omega + 1}$ $\qquad x(t) \longleftrightarrow X(\omega)$

① $v(t) = x(t) e^{j3t}$; $\omega_0 = 3$

$\longleftrightarrow X(\omega - 3)$

$= \dfrac{1}{j(\omega-3)+1}$ $\qquad$ multiply by $-j(\omega-3)+1$

$= \dfrac{1}{1 + j(\omega-3)}$ $= \dfrac{1 - j(\omega-3)}{1^2 - [j(\omega-3)]^2}$

$= \dfrac{1 - j(\omega-3)}{1 - j^2(\omega-3)^2}$ $\rightarrow j^2 = (\sqrt{-1})^2 = -1$

$= \dfrac{1 - j(\omega-3)}{1 + (\omega-3)^2}$

- $x(2t-1)$ $\qquad$ Superposition

$= x(2(t - 0.5))$

$\longleftrightarrow \left(\frac{1}{2}\right) X\left(\frac{\omega}{2}\right) e^{-j\omega 0.5}$

$= \left(\frac{1}{2}\right) e^{-j\frac{\omega}{2}} \dfrac{1}{j\frac{\omega}{2} + 1}$

Q1 ~ Q4