Mechatronics Lab: (CB-1034)
No tutorial this week.

→ Matlab used for assignments (15%)
↳ 5 assignments total

→ 3 lab sessions
↳ 1 lab report

↳ Wang using course website on Flash, not D2L

Chapter 1 - Introduction

(1.1) - overview

→ Signal process: to extract representative features for advanced analysis

→ pattern classification (diagnostics)

→ modeling (forecasting)

→ control

(1.2) - maintenance strategies

→ run-to-break

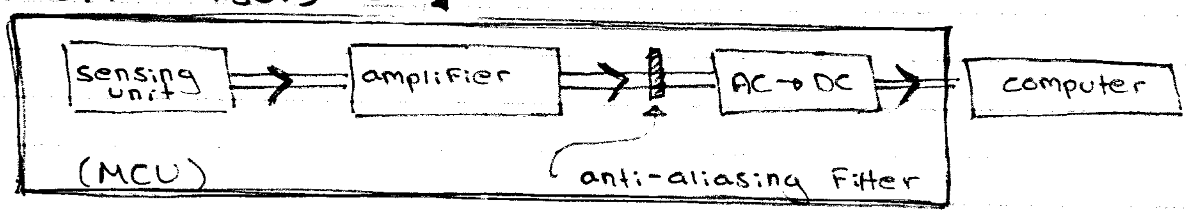
→ preventative maintenance ($\gtrsim 25\%$)
↳ periodically shut down machine for maintenance
↳ unnecessary downtimes

→ predictive maintenance (research state)
↳ condition monitoring → recognize defects
↳ predict the remaining useful life of the faulty component
↳ schedule maintenance operations

- literature review, discuss the maintenance strategies

Condition monitoring
- recognize the defects at its earliest stage
- Prevent machinery performance degredation, malfunction, failure.

Smart sensors



micro controller unit (?)

(1.3) - approaches to Fault detection

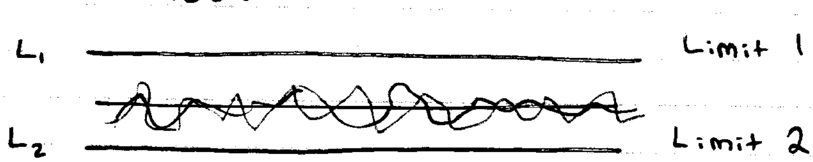
1. classical approaches ~ biological sense
   - looking
   - listening
   - touching
   - Smelling

2. Automatic diagnosis
   - Analytical model

diff. equations

   - Signal processing - based

3. Monitoring
   - limit checking
   - index



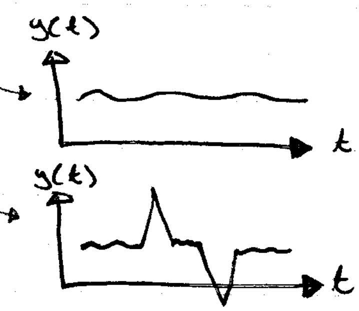
Chapter 2 - Introduction to Signals & Systems

(2.1) - Signal classification

- deterministic (inputs, outputs)
- random (statistical quantities, mean, std.)

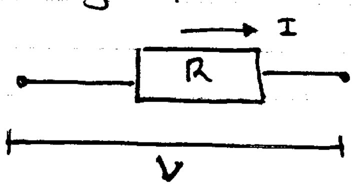
e.g. $y(t) = 28^{32(t)}$

signals
- continuous
- transient



Signals
- stationary (statistical quantities don't change w/ time)
- non-stationary (change w.r.t. time)

Enough power



Energy $= I^2 R = \frac{V^2}{R=1}$

(if $R=1$, then) $= I^2 = V^2$

Energy $= \int x^2(t)\,dt$

I^2

V^2

$\boxed{\text{Sensor}} \rightarrow x(t)$

Sep. 5/19

(2.2) - System Properties

1) Causality

@ any time t_1, $y(t_1) \sim x(t_1)$

$x(t) \longleftrightarrow y(t)$ $\qquad t \leq t_1$

$x(t) \longrightarrow$ System $\longrightarrow y(t)$

Output $y(t_1)$ depends on $x(t)\big|_{t \leq t_1}$

Not depending on its future input $x(t)\big|_{t > t_1}$

System $\longrightarrow$ Causal

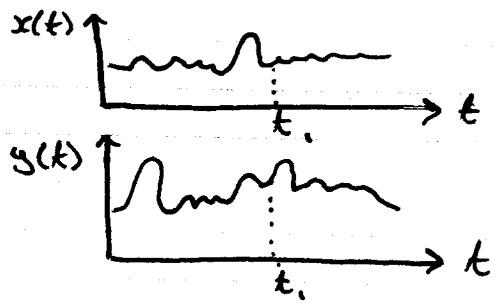
Real systems are causal

- Assume no initial energy

$y(t) = x^2(t) + \cancel{1}$ initial energy

when $x(t) = 0$, $y(t) = 0$

$y(t) = x^2(t)$



EX1 $y(t) = 3x(t+1)$

$t = 1$

$y(t)\big|_{t=1} = 3x(t+1)$

$\hookrightarrow$ non-causal

offline processing

diagnostics

2) Linearity:

- Additive

$$\xrightarrow{x(t)} \boxed{\text{System}} \xrightarrow{y(t)}$$

Input	Output
$x_1(t)$	$y_1(t)$
$x_2(t)$	$y_2(t)$

Additive: $x_1 + x_2$ → $y_1 + y_2$

If the output is a sum, the system can be considered additive

- Homogeneous

Input	output
$x_1(t)$	$y_1(t)$

homogeneous: $a\,x_1(t)$ → $a\,y_1(t)$

output is ... scaled by same amount as input

- linear ~ additive + homogeneous

Input	Output
linear: $ax_1(t) + bx_2(t)$	$ay_1 + by_2$

EX2 $\quad y(t) = t\,x(t)$

Input	Output
$x_1(t) = u$	$y_1 = tu$
$x_2 = 3u$	$3y_2 = 3tu$
$x_3 = x_1 + x_2 = 4u$	$y_3 = 4tu$

$y_3 = 4tu = y_1 + y_2$ = linear $\quad (y_1 + y_2 = y_3)$

EX3 $\quad y(t) = x^2(t)$

Input	Output
$x_1 = u$	$y_1 = u^2$
$x_2 = 3u$	$y_2 = 9u^2$
$x_3 = 4u$	$y_3 = 16u^2$

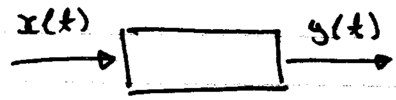
$y_1 + y_2 = 10u^2 \neq y_3$ non-linear

Linear System

↳ Superposition depends on this

3) Time Invariance

Input	Output
$x(t)$	$y(t)$



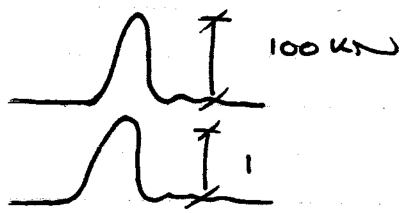
Shifted input $x(t-t_1)$, $y(t-t_1)$

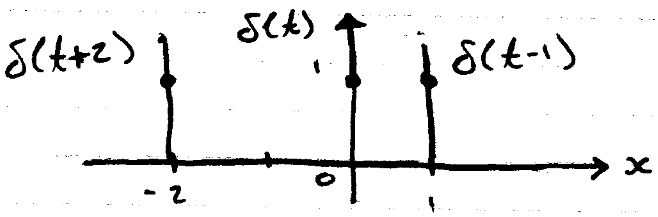
t_1 = number   time invariant

- System properties don't change with time

impulse fn

$$\delta(t) \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$





basically, input has shift, output has corresponding shift.

$t_1 = 1 > 0$

then $\delta(t-t_1) = \delta(t-1)$

$t_1 = -2 < 0$

then $\delta(t-t_1) = \delta(t+2)$

EX4 $y(t) = x^2(t) \cancel{+ 3}$

Input	output
$x_1(t) = x(t)$	$y_1 = x^2(t)$
$x_2(t) = x(t-t_1)$	$y_2 = x^2(t-t_1)$

Time Invariant

EX5 $y(t) = t\,x(t)$

Input	Output
$x_1(t) = x(t)$	$y_1 = t\,x(t)$
$x_2 = x(t - t_1)$	$y_2 = t\,x(t - t_1)$

$\#$

$y_3(t)|_{t = t - t_1}$
$= y(t - t_1) = (t - t_1)\,x(t - t_1)$

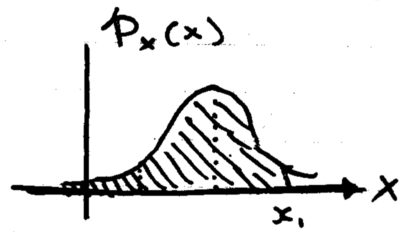
In signal processing,
Assume : casual, linear, time invariant

(2.3) - Review of Statistical Quantities

① Probability fn

Random variable X

Probability Distribution fn
$p_x(x)$



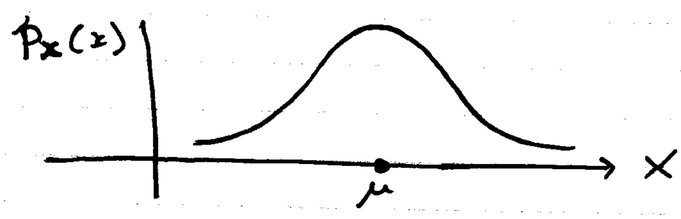
Probability of P_x = Prob $(x \le x_1)$

$$P_x = \int_{-\infty}^{x_1} p_x(x)\,dx$$

$p_x(x) \geq 0$

$$\int_{-\infty}^{\infty} p_x(x)\,dx = 1$$

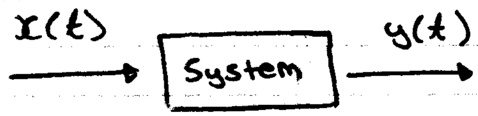
Temp +



Gaussian probability density function (pdf)

$$p_x = \frac{1}{2\pi\sigma} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

μ = mean
σ = st. d
σ = variance



$y(t) = 3x(t)e^{t} + \cancel{I.E.}$ I.E. (initial energy)

- Causal
  non-causal : offline processing
- linear
  Superposition

Input	Output
$x_1 = u$	y_1
$x_2 = 3u$	y_2
$x_3 = x_1 + x_2 = 4u$	$y_3 = y_1 + y_2$

- Time-invariance

Input	Output
$x_1 = x(t)$	$y_1 = 3x(t)e^{x}$
$x_2 = x(t-2)$	$y_2 = 3x(t-2)e^{x}$

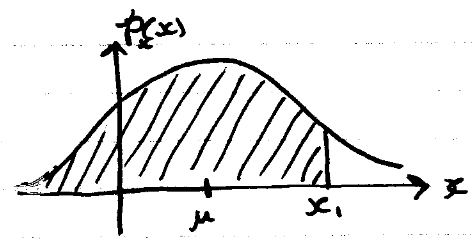
$\neq$

$$y_3(t-2) = 3x(t-2)e^{t-2}$$

(2.3) Review of Probability Concepts

pdf - prob. distribution function

$P_x(x)$



prob.

$P_x(x \leq x_1) = \int_{-\infty}^{x_1} P_x(x)\,dx$, Gaussian pdf

2) Statistical moments

- 1st order moment

$$\mu = \int_{-\infty}^{\infty} x P_x(x)\,dx$$
$$= E\{x\}$$

expectation

- 2nd-order moment

$Var(x) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \, p_x(x) dx$

Variance

Standard dev. σ $= E\{(x-\mu)^2\}$

$\sigma^2 = E\{(x^2 - 2x\mu + \mu^2)\}$

$= E\{x^2\} - 2\mu E\{x\} + \mu^2$

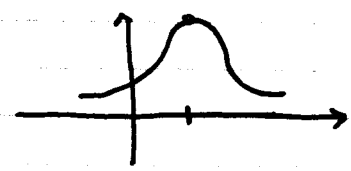
$= E\{x^2\} - 2\mu^2 + \mu^2$

- 3^{rd} moment

$\mu_3 = E\{(x-\mu)^3\}$

$SK = \frac{\mu^3}{\sigma^3}$; skewness

(unitless)

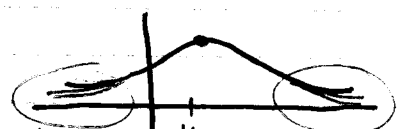


- 4^{th} moment

$\mu_4 = E\{(x-\mu)^4\}$

Kurtosis

$KU = \frac{\mu^4}{\sigma^4}$



Two Variables

$cov(x_1, x_2)$

Coefficient

$\rho_{12}^2 = \frac{cov^2(x_1, x_2)}{Var(x_1) Var(x_2)}$

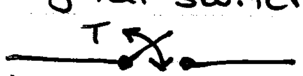
(2.4) Sampling + Aliasing

Collect data, analog signal

Computer ⟶ digital signal

- Sampling

digital switch

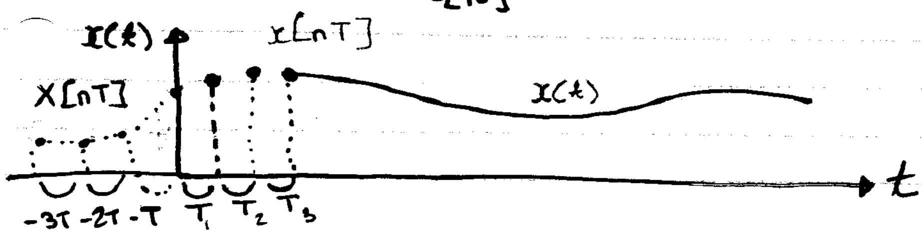


analog signal $x(t)$

digital signal $x[n]$

round brackets for analog signal

square brackets for digital signal



T = time interval (sec)

Analog to Digital converter (ADC)

↳ (DAC) inverse

- control motor, etc.

$x[nT] = x(t)\big|_{t=nT}$

For $n = 0, 1, 2 \ldots$

(in theory, $n = -2, -1, 0, 1 \ldots$)

Sampling Frequency,

$f_s = \frac{1}{T}$ (Hz)

$f_s = 20$ Hz, $T = 1/f_s = 1/20 = 0.05$ sec

$f_s = 15000$ Hz, $T = 1/f_s \ldots$

$x[n]$

n = discrete time value

$f_s \uparrow\uparrow$, $T = 1/f_s \downarrow\downarrow$

data size $\uparrow\uparrow$, Processing speed $\downarrow\downarrow$

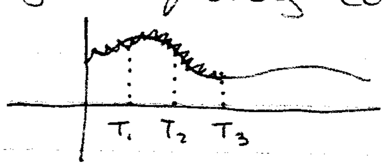
$T \geq 0.003$ sec

online control, real time (diagnosis prognosis)

machine learning

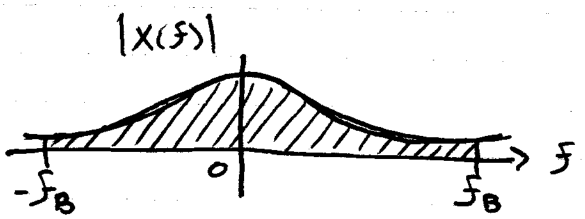
If $f_s \downarrow\downarrow$, $T = 1/f_s \uparrow\uparrow$

high frequency components would be lost

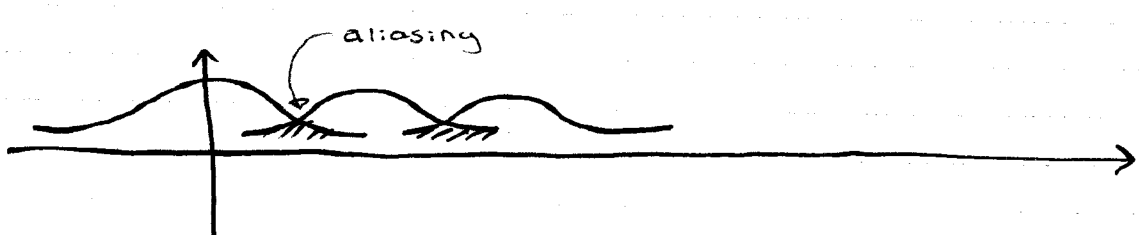


- Actual signals are time-limited

  $-f_B \leq f \leq f_B$

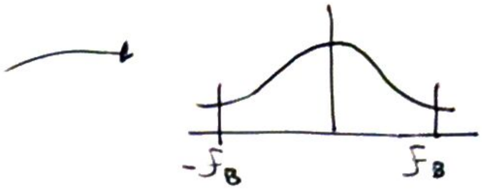


Sampling ($2f_B$)



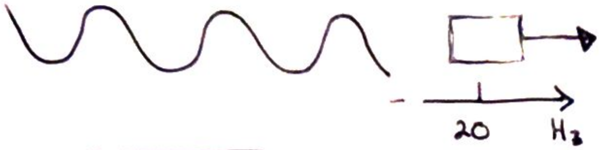
2) Aliasing

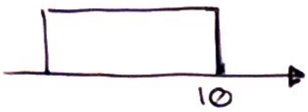
All the signals are time limited



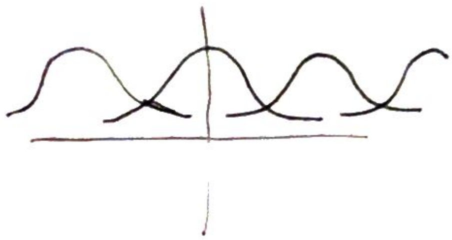
$f_s = 15{,}000$ Hz

Frequency components would be non-limited





Sampled continuously:

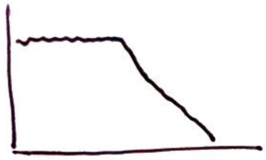


$x[n] = x[nT] = x(t)\big|_{t=nT}$

$n = \ldots -2, -1, 0, 1, 2 \ldots$

$f_{max} = f_B$ Hz,

FFT



Most important Freq. components $(-f_B, f_B)$

$f_s = 2f_B$

$[0, f_B]$ or $[-f_B, f_B]$

Contains extra frequency components overlapped from high freq. region to the low freq. region.

$(f_B, +\infty)$

~ aliasing

$f_s = 40$ shots/sec

$= 20 \sim 25$ pictures/sec

Before doing ADC

anti-aliasing filter to remove freq. (Low pass filter) analog filter

components higher than f_B

Nyquist Freq

$f_n = 2f_B$

Sampling Freq

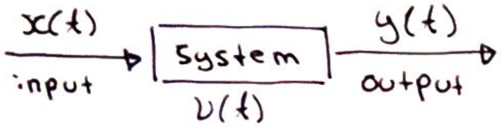
$f_s \geq f_n = 2f_B$

CD

20 KHZ

$f_s = 44.1$ KHZ

(2.4) - Convolution representation



$y(t) = x(t) \otimes v(t)$

1) Convolution for continuous signals

$$x(t) \otimes v(t) = \int_{-\infty}^{\infty} x(\lambda)\, v(t-\lambda)\, d\lambda$$

$$\begin{cases} x(t) = 0, \text{ if } t < 0 \\ v(t) = 0, \text{ if } t < 0 \end{cases}$$

, if $\lambda < 0$
$x(\lambda) = 0$

$v(t-\lambda) = 0$, if $t - \lambda < 0$
$t < \lambda$

$$x(t) \otimes v(t) = \int_{0}^{t} x(\lambda)\, v(t-\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

$$\int_{-\infty}^{\infty} |v(t)|\, dt < \infty$$

2) Conv. for discrete signals

$x[n]$, $v[n]$

$\ldots, -2, -1, 0 < n < 1, 2, \ldots$

$$x[n] \otimes v[n] = \sum_{i=-\infty}^{+\infty} x[i]\, v[n-i]$$

if $x[n] = 0$; $n < 0$

if $v[n] = 0$; $n < 0$

$x[i] = 0$ if $i < 0$

$v[n-i] = 0$ if $n - i < 0$, $i > n$

$$x[n] \otimes v[n] = \begin{cases} \sum_{i=0}^{n} x[i]\, v[n-i] & ; \; i = 0, 1, 2, \ldots, n \\ 0 & ; \text{ if } i < 0, i > n \end{cases}$$

Convolution computation procedures:

1) Changing the discrete time index n to i in signals $x[n]$ and $v[n]$. The resulting signals $x[i]$ and $v[i]$ are then functions of the discrete-time index i.

2) Determining $v[n-i]$. The signal $v[n-i]$ is a folded and shifted version of the signal $v[i]$. More precisely, $v[-i]$ is $v[i]$ folded about the vertical axis, and $v[n-i]$ is $v[-i]$ shifted by n steps. If $n > 0$, $v[n-i]$ is an n-step right shift of $v[-i]$. In contrast, if $n < 0$, $v[n-i]$ is an n-step left shift of $v[-i]$

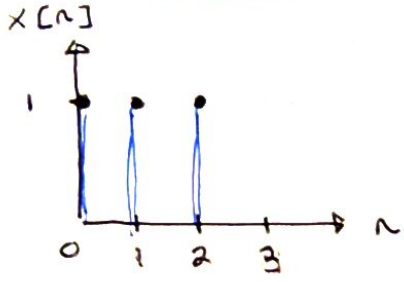
3) Computing the convolution

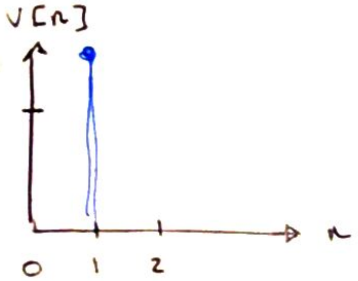
$v[n] \longrightarrow v[i]$, changed index

$\longrightarrow v[-i]$ Folding

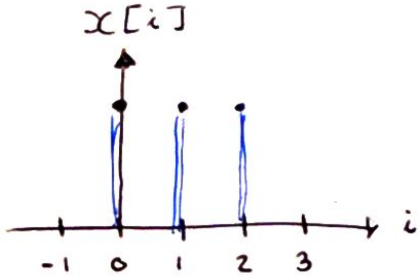
$\longrightarrow v[n-i]$ Shifting

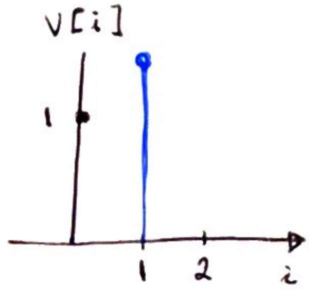
Example 1

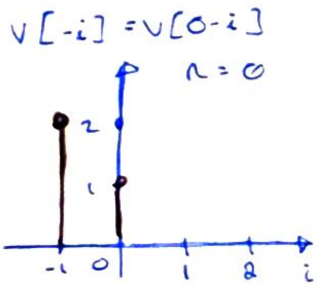


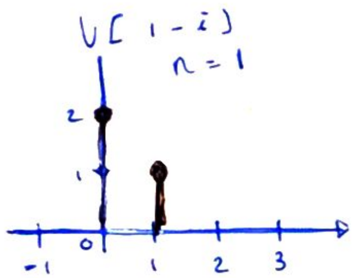


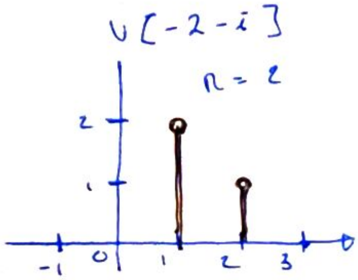
Solution











. . . etc.

3) Conv. Properties

- Associativity,

  $x[n]$, $v[n]$, $w[n]$

  $x[n] \otimes (v[n] \otimes w[n])$

  $= (x[n] \otimes v[n]) \otimes w[n]$

- Communitivity,

  $x[n] \otimes v[n] = v[n] \otimes x[n]$

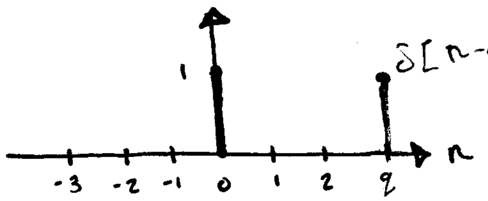
  $\sum_{i=-\infty}^{+\infty} x[i]v[n-i] = \sum_{i=-\infty}^{\infty} v[i]x[n-i]$

- Distributivity with Addition

$x[n] \otimes v[n] + w[n]) = x[n] \otimes v[n] + x[n] \otimes w[n]$

- Conv. with the unit pulse

$\delta[n]$



$$x[n] \otimes \delta[n] = \sum_{i=-\infty}^{\infty} x[i]\underbrace{\delta[n-i]}_{\substack{=\emptyset \\ \text{if } n-i \neq \emptyset}}$$

If $n - i = \emptyset$, $\delta[n-i] = 1$
$i = n$

$$x[n] \otimes \delta[n] = x[n]\delta[n-n]$$
$$= x[n]$$

- Conv. with a shifted unit pulse

$$x[n] \otimes \delta[n-q]$$
$$= \sum_{i=-\infty}^{\infty} x[i]\delta[n-q-i]$$

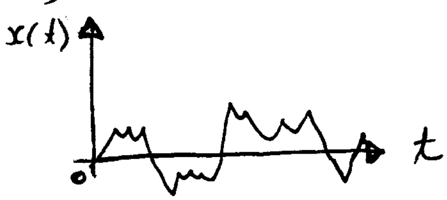
$$\delta[n-q-i] = 1$$
$$n - q - i = \emptyset$$
$$i = n - q$$

$$= x[n-q]\,\underset{1}{\delta}\underbrace{[n-q-(n-q)]}_{\emptyset}$$
$$= x[n-q]$$

Chapter 3 : Tools for Signal Processing

3.1 Properties of Continuous FT

1) Introduction

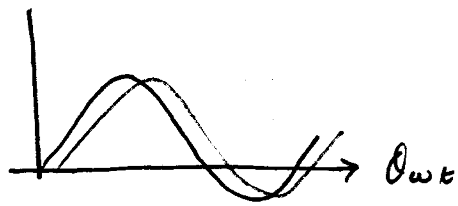


A signal consists of sinusoids with different frequencies, magnitudes, and phase functions.

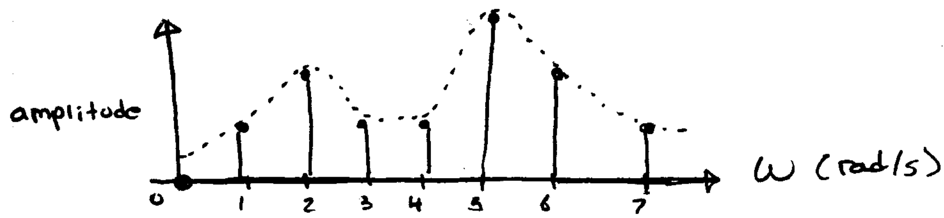
$$x(t) = A_0\cos(\omega_0 t + \theta_0) + A_1\cos(\omega_1 t + \theta_1) + \dots + A_n\cos(\omega_n t + \theta_n)$$
$$= \sum_{n=0} A_n \cos(\omega_n t + \theta_n)$$

where A_n = magnitudes
ω_n = Freq. (rad/s)
$\omega = 2\pi f$
f = Hz
θ_n = Phase angles

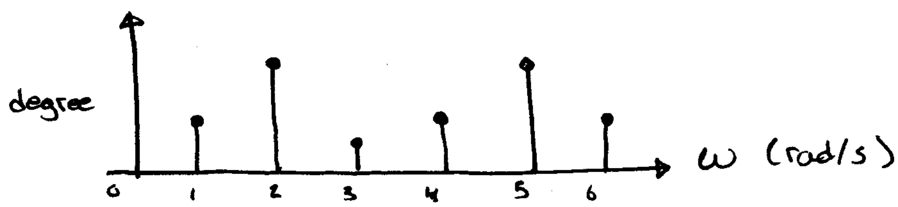
time delay, 0.001 sec



amplitude spectrum,



Phase spectrum,

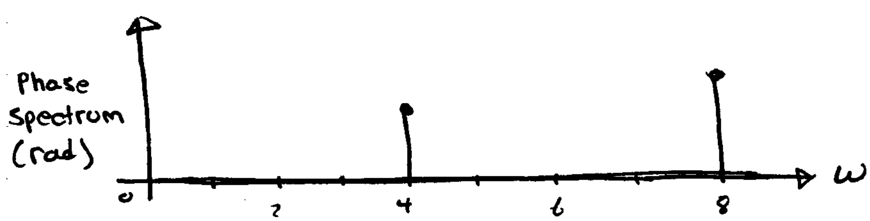


$\boxed{\text{Example}}$

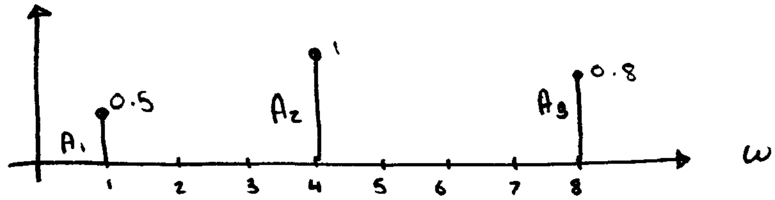
$x(t) = A_1\cos t + A_2\cos(4t + \pi/3) + A_3\cos(8t + \pi/2)$

$\omega_1 = 1$ rad/s ; $\omega_2 = 4$; $\omega_3 = 8$

$\varnothing,\ \pi/3,\ \pi/2$



line spectrum



amp. spectrum

$\left(\begin{array}{l} t = 0.200 \\ A1 = 0.5,\quad A2 = 2,\quad A_3 = 1 \end{array} \right)$

X = $A_1\cos(1*t) + A_2\cos(4t + \text{Pi}/3)$ ... etc., Plot(x)

2 Continuous FT (CFT)

Given $x(t)$ → lower case letter - time domain signal

capital letter - freq. fn

$$X(\omega) = \int x(t) e^{-j\omega t} dt$$

freq. : ω = frequency variable $-\infty < \omega < \infty$

$j = \sqrt{-1}$

$e^{-j\omega t} \longrightarrow$ Complex valued

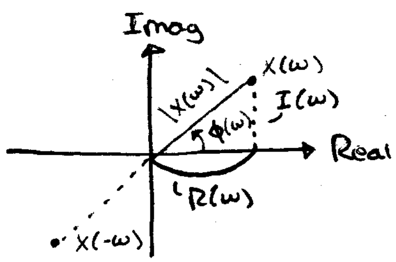
$X(\omega) \longrightarrow$ complex valued fxn

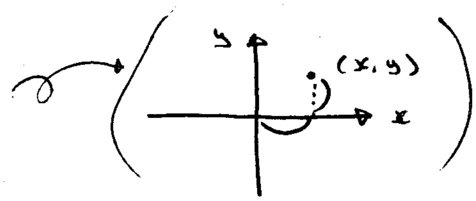
— Rectangular representation

$X(\omega) = R(\omega) + jI(\omega)$

$R(\omega)$ = Real part

$I(\omega)$ = Imaginary part



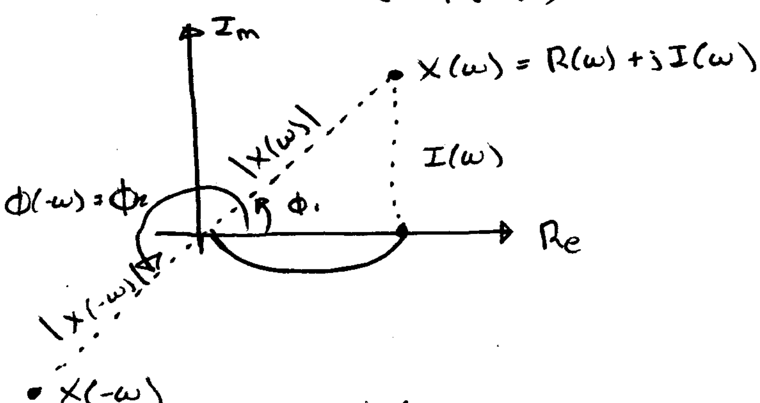


Polar representation

$X(\omega) = |X(\omega)| e^{j\phi(\omega)}$

$|X(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$

$\phi(\omega) = \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right)$



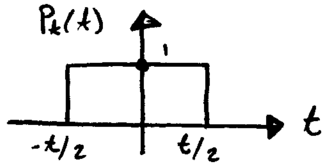
$X(-\omega) \sim$ complex conjugate $X(\omega)$

$|X(-\omega)| = |X(\omega)|$

$$\begin{cases} \phi(-\omega) = \phi(\omega) + \pi \\ \phi(-\omega) = -\phi(\omega) \end{cases}$$

$\boxed{\text{EXAMPLE 3.2}}$

$$P_t(t) = \begin{cases} 1 & -t/2 < t < t/2 \\ 0 & \text{otherwise} \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$
$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-j\omega t}\, dt \right| < \infty$$

Solution :

$$X(\omega) = \int_{-\tau/2}^{\tau/2} 1 * e^{-j\omega t}\, dt$$

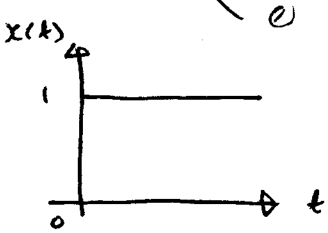
Euler's Formula :

$e^{j\theta} = \cos\theta + j\sin\theta \qquad \theta = -\omega t$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} \left[\cos(\omega t) - j\sin(\omega t) \right] dt$$
$$= \left[{}^{1}/\omega \sin(\omega t) + j\, {}^{1}/\omega \cos(\omega t) \right]_{-\tau/2}^{\tau/2}$$
$$= {}^{1}/\omega \left[\sin(\omega t/2) - \sin(-\omega t/2) \right] + j/\omega \left[\cos[\omega t/2 - \cos(-\omega t/2) \right]$$
$$= 2/\omega \sin(\omega t/2) + \cancel{j\, {}^{1}/\omega \cos(\omega t/2)}$$

Example 3.3

$$x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Solution

$$X(\omega) = \int_0^{\infty} 1 e^{-j\omega t}\, dt$$
$$= \frac{1}{-j\omega} \int_0^{\infty} e^{-j\omega t}\, d(-j\omega t)$$
$$X(\omega) = -\left(\frac{1}{j\omega}\right) e^{-j\omega t} \Big|_0^{\infty}$$
$$= -\frac{1}{j\omega}\left[e^{-j\omega\infty} - 1 \right] = -\frac{1}{j\omega}\left[\cos(-\omega t) + j\sin(-\omega t) \right]_0^{\infty}$$
$$= -\frac{1}{j\omega}\left[\;\ldots \quad \text{DNE} \right.$$

A Signal

a series of sinusoids

ω_i, θ_i, A_i

$A_i \sim \omega_i$: amplitude spectrum

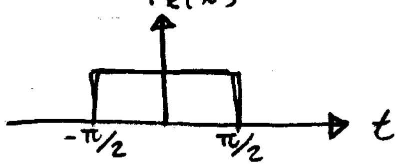
$\theta_i \sim \omega_i$: Phase spectrum

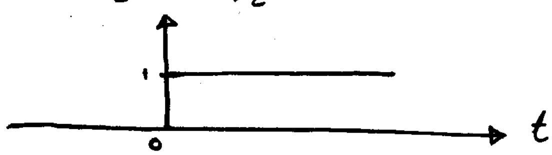
CFT (continuous fourier transform)

$x(t)$

$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \quad -\infty < \omega < \infty$

$\int_{-\infty}^{\infty} | x(t) e^{-j\omega t} | dt < \infty$,



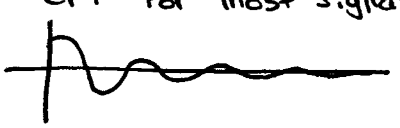


$$X(\omega) = \int_0^{\infty} 1 * e^{-j\omega t} dt$$
$$= \int_0^{\infty} [\cos(-\omega t) + j\sin(-\omega t)] dt$$
$$= \int_0^{\infty} [\cos(\omega t) - j\sin(\omega t)] dt$$

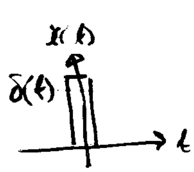
where $\rightarrow$ $= \int_0^{\infty} | \cos(\omega t) | dt = \frac{1}{\omega} | \sin(\omega t) | _0^{\infty}$

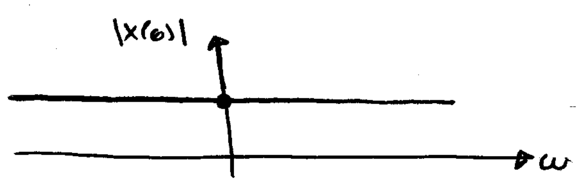
$$= \frac{1}{\omega} | \sin(\omega \infty) |$$

- most signals with the product of $e^{-j\omega t}$ don't satisfy the sufficient integral conditions
- CFT for most signals don't exist in the original sense



- CFT is undertaken in a generalized sense
- use FT pairs + FT properties to do FT





Common Fourier Transform Pairs (handout)

- Inverse FT

Given $X(\omega)$ Filter, controls

IFT :

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad (-\infty < t < \infty)$$

CFT :

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (-\infty < \omega < \infty)$$

FT Pairs,

$x(t) \longleftrightarrow X(\omega)$

$\boxed{\text{Example 3}}$

$X(\omega) = \cos(\omega t)$, $x(t) = ?$

Based on Euler's formula :

$e^{j\theta} = \cos\theta + j\sin\theta \longleftarrow (1)$

$e^{-j\theta} = \cos\theta - j\sin\theta \longleftarrow (2)$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$

$$\rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Solution

$$X(\omega) = \cos(2\omega)$$
$$= (1/2)[e^{2j\omega} + e^{-2j\omega}]$$
$$\longleftrightarrow (1/2)[\delta(t+2) + \delta(t-2)]$$

$-j\omega c = j2\omega$

4) Some properties of the CFT

- Linearity

$x(t) \longleftrightarrow X(\omega)$, $v(t) \longleftrightarrow V(\omega)$

$ax(t) + bv(t) \longleftrightarrow aX(\omega) + bV(\omega)$

Proof

$$\int_{-\infty}^{\infty} [ax(t) + bv(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [a x(t) e^{-j\omega t} + b v(t) e^{-j\omega t}]\, dt$$
$$= \underbrace{a \int_{-\infty}^{\infty} x(t) e^{-j\omega t}}_{X(\omega)} dt + \underbrace{b \int_{-\infty}^{\infty} v(t) e^{-j\omega t}}_{V(\omega)} dt$$

- Shifts in time

IF $x(t) \longleftrightarrow X(\omega)$

$x(t-c) \longleftrightarrow X(\omega) e^{-j\omega c}$

Proof :

$\int x(t-c) e^{-j\omega t}\, dt \qquad ; \qquad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

Let :

$\lambda = t - c$

$t = \lambda + c$

$dt = d\lambda$

$$\int_{-\infty}^{\infty} x(t-c) e^{-j\omega t}\, dt$$
$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda + c)}\, d\lambda$$
$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} \cdot e^{-\lambda\omega c}\, d\lambda$$
$$= e^{-j\omega c} \underbrace{\int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda}\, d\lambda}_{X(\omega)}$$

- Time scaling

$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt}$

$x(t) \longleftrightarrow X(\omega)$

$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right) \qquad ; \qquad a > 0$

$\int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt$

Let $\lambda = at$, $t = \frac{1}{a}\lambda$, $dt = \frac{1}{a} d\lambda$

$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega \frac{1}{a}\lambda} (1/a)\, d\lambda$$
$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j(\omega/a)\lambda}\, d\lambda$$
$$= \frac{1}{a} X(\omega/a)$$

- Time reversal

$x(t) \longleftrightarrow X(\omega)$

$x(-t) \longleftrightarrow X(-\omega)$

Proof :

$\int_{-\infty}^{\infty} x(-t) e^{-j\omega t}\, dt$

Let $\lambda = -t$, $t = -\lambda$, $dt = -d\lambda$

$= \int_{+\infty}^{-\infty} x(\lambda) e^{-j\omega(-\lambda)} (-d\lambda)$

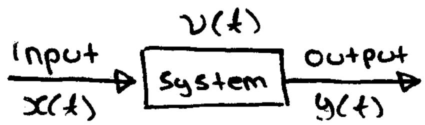
$= + \int_{+\infty}^{-\infty} x(\lambda) e^{-j(-\omega)\lambda} \, d\lambda$

$= X(-\omega)$

Convolution in the time domain

- If $x(t) \longleftrightarrow X(\omega)$, $v(t) \longleftrightarrow V(\omega)$

A2Q1 $\rightarrow$ $x(t) \otimes v(t) \longleftrightarrow X(\omega) * V(\omega)$



$y(t) = x(t) \otimes v(t)$ $\times$ slow

Input $x(t) \longleftrightarrow X(\omega)$

$v(t) \longleftrightarrow V(\omega)$

$\downarrow$

$X(\omega) * V(\omega)$ ✓ Faster

$\downarrow$ IFT

$y(t)$

$\boxed{\text{Example}}$

$X(\omega) = \frac{1}{j\omega + 1}$ $x(t) \longleftrightarrow X(\omega)$

① $v(t) = x(t) e^{j3t}$; $\omega_0 = 3$

$\longleftrightarrow X(\omega - 3)$

$= \frac{1}{j(\omega-3)+1}$ multiply by $-j(\omega-3)+1$

$= \frac{1}{1 + j(\omega-3)}$ $= \frac{1 - j(\omega-3)}{1^2 - [j(\omega-3)]^2}$

$= \frac{1 - j(\omega-3)}{1 - j^2(\omega-3)^2}$ $\rightarrow j^2 = (\sqrt{-1})^2 = -1$

$= \frac{1 - j(\omega-3)}{1 + (\omega-3)^2}$

- $x(2t-1)$ Superposition

$= x(2(t - 0.5))$

$\longleftrightarrow \left(\frac{1}{2}\right) X\left(\frac{\omega}{2}\right) e^{-j\omega 0.5}$

$= \left(\frac{1}{2}\right) e^{-j\frac{\omega}{2}} \frac{1}{j\frac{\omega}{2} + 1}$

Q1 ~ Q4

$$\int |\underbrace{x(t)e^{-j\omega t}}_{\text{not integrable}}|dt < \infty$$
$$= \infty \quad \text{DNE}$$

Example $X(\omega) = \dfrac{1}{j\omega + 1}$

(3) $v(t) = t^2 x(t)$

Solution: $\dfrac{d}{d\omega} X(\omega) = -\dfrac{j}{(j\omega+1)^2}$

$$\frac{d^2}{d\omega^2} X(\omega) = -\frac{j}{(j\omega+1)^3}(-2)j$$
$$= \frac{2(-1)}{(j\omega+1)^3} = -\frac{2}{(1+j\omega)^3}$$

$$V(\omega) = j^2 \frac{-2}{(1+j\omega)^3} = \frac{2}{(1+j\omega)^3}$$

Note: rectangular, polar

$$V(\omega) = \frac{2(1-j\omega)^3}{[(1+j\omega)^3(1-j\omega)^3]} = \frac{2(1-j)(1-j\omega)^2}{(1+\omega^2)^3}$$
$$= Re + jIm$$

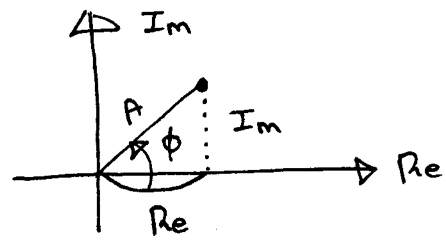
(4) $v(t) = x(t)\cos(t)$

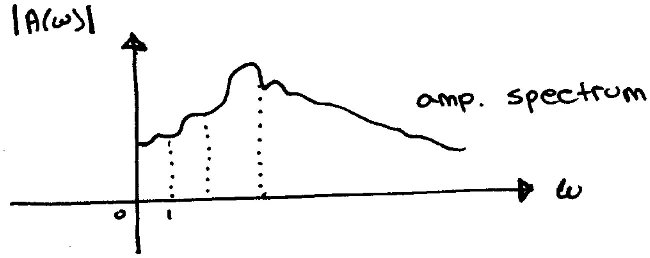
$$v(t) \longleftrightarrow V(\omega) = \left(\tfrac{1}{2}\right)[X(\omega+4) + X(\omega-4)]$$

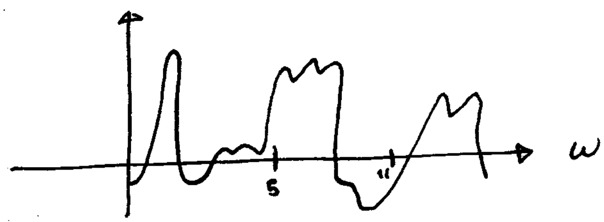
$$V(\omega) = \left(\frac{1}{2}\right)\left[\frac{1}{j(\omega+4)+1} + \frac{1}{j(\omega-4)+1}\right]$$
$$= \left(\frac{1}{2}\right)\left[\frac{1-j(\omega+4)}{[1+j(\omega+4)][1-j(\omega-4)]} + \frac{1-j(\omega-4)}{[1+(j(\omega-4)][1-j(\omega+4)]}\right]$$
$$= \left(\frac{1}{2}\right)\left[\frac{1-j(\omega+4)}{1+(\omega+4)^2} + \frac{1-j(\omega-4)}{1+(\omega-4)^2}\right]$$
$$= \left(\frac{1}{2}\right)\underbrace{\left[\frac{1}{1+(\omega+4)^2} + \frac{1}{1+(\omega-4)^2}\right]}_{Re} + \left(\frac{j}{2}\right)\underbrace{\left[\frac{-(\omega+4)}{1+(\omega+4)^2} + \frac{-(\omega-4)}{1(\omega-4)^2}\right]}_{Im}$$

$$= \sqrt{Re^2 + Im^2} = e^{j\phi}$$
$$\phi = \arctan\left[\frac{Im}{Re}\right]$$

$$V(\omega) = A(\omega)e^{j\phi(\omega)}$$
$$A(\omega) = \sqrt{Re^2(\omega) + Im^2(\omega)}$$
$$\phi(\omega) = \arctan\left(\frac{Im(\omega)}{Re(\omega)}\right)$$







$\boxed{\text{Example}}$ ICFT

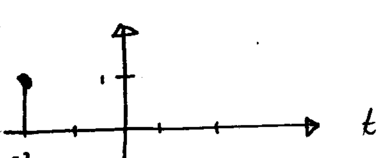
(1) $X(\omega) = \sin(2\omega)$
$x(t) = ?$

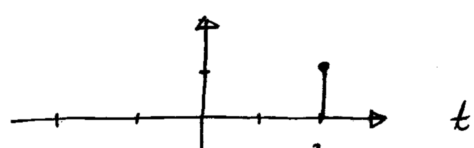
$$\begin{cases} \cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2} \\[2ex] \sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} \end{cases}$$

(modified Euler Formula)

$$X(\omega) = (-j/2)[\underbrace{e^{j2\omega}}_{c=-2} - \underbrace{e^{-j2\omega}}_{c=2}]$$

$$\longleftrightarrow (-j/2)[\delta(t+2) - \delta(t-2)]$$



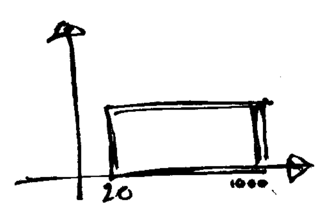


(2) $X(\omega) = \cos^2(2\omega)$

$$= \left[\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right]^2$$

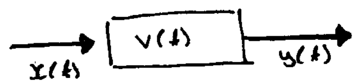
$$= \left(\frac{1}{4}\right)\left[e^{j4\omega} + e^{-j4\omega} + 2e^{j2\omega - j2\omega}\right]$$

$$x(t) = \left(\frac{1}{4}\right)\left[\delta(t+4) + \delta(t-4) + 2\delta(t)\right]$$



* Passive Filter : 1, 2, 3 order (heat)

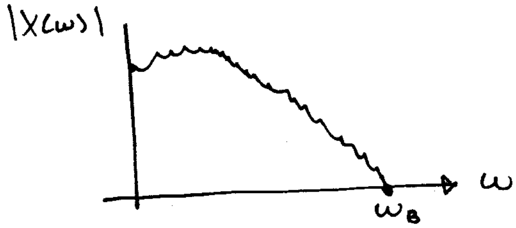
* active filter : op - amp



$x(t) \downarrow$

$X(\omega) * V(\omega) = Y(\omega) \xrightarrow{ICFT} y(t)$

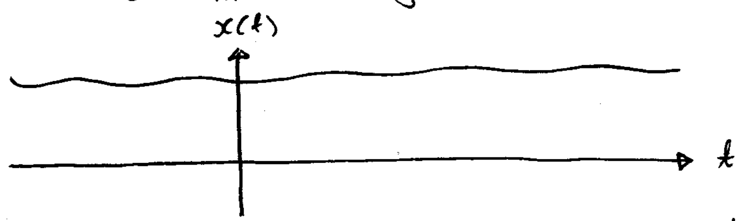
(3.3) Band Limited Signals



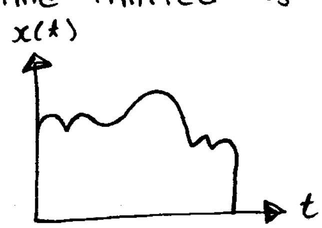
$|X_\omega| \approx 0 , \ \omega \geq \omega_B$

$\longrightarrow$ Band limited signal

* Band limited signal cannot be time limited signal

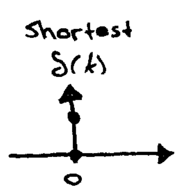


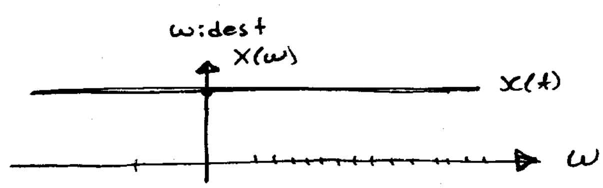
* Time limited signal $x(t)$ cannot be band limited.



* If $x(t)$ is longer $\longrightarrow$ band $\downarrow$
  .... shorter $\longrightarrow$ band $\uparrow$

$\delta(t)$ Band





$x(at) \longleftrightarrow \frac{1}{a} X(\omega/a)$

(3.4) Continuous Time FT (CTFT)

Given $x[n]$, $n = 0, 1, 2, \cdots ; N-1$

DTFT : $X(\Omega) = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$

$-\infty < \Omega < \infty$

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \longrightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$\sum_{n=0}^{N-1} |x[n]| < \infty$

- $X(\Omega)$ is a periodic Fxn with 2π

$X(\Omega + 2\pi) = \sum_{n=0}^{N-1} x[n] e^{-j(\Omega + 2\pi)n}$

$= \sum_{n=0}^{N-1} x[n] e^{-j\Omega n} * e^{-j2\pi n}$

$e^{-j2\pi n} = \cos(-2\pi n) + j\sin(-2\pi n)$

$= \underbrace{\cos(2\pi n)}_{\text{always } = 1} - j\underbrace{\sin(2\pi n)}_{\text{always } = 0} = 1$

Then $X(\Omega + 2\pi) = X(\Omega)$

$0 \le \Omega \le 2\pi$, $-\pi < \Omega < \pi$

IDTFT :

$x[n] = \int_{0}^{2\pi} X(\omega) e^{j\Omega n} d\Omega$

$n = 0, 1, 2, \cdots ; N-1$

$e^{j\Omega n} = e^{j(\Omega + 2\pi)n}$

$= e^{j\Omega n} * \underbrace{e^{j2\pi n}}_{1}$

3.4 DTFT

$x[n]$: $n = 0, 1, 2, 3, \ldots, N-1$

$x[n]$: if $n < 0$, $n \geq N$

$$X(\Omega) = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

$X(\Omega) \sim$ periodic fxn. $\quad -\infty < \Omega < \infty$ (continuous) $\quad 2\pi$ (discrete)

IDTFT

$$x[n] = \frac{1}{2\pi}\int_{-\infty \,(0/-\pi)}^{\infty \,(2\pi/\pi)} \underbrace{X(\Omega)e^{j\Omega n}}_{\text{Periodic fxn w/ } 2\pi} d\Omega$$

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega, \quad n = 0, 1, \ldots, N-1$$

Example 3.4 : Compute the DTFT of a discrete-time signal defined by:

$$x[n] = \begin{cases} 0 & ; \quad n < 0 \\ a^n & ; \quad 0 \leq n \leq q \\ 0 & ; \quad n > q \end{cases}$$

Where a is a nonzero real constant and q is a positive integer.

Solution :

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{q} a^n e^{-j\Omega n}$$
$$= \sum_{n=0}^{q} (ae^{-j\Omega})^n$$

$$\sum_{n=P_1}^{P_2} r^n = \frac{r^{P_1} - r^{P_2+1}}{1-r}$$

$P_1 = 0, \quad P_2 = q, \quad r = ae^{-j\Omega}$

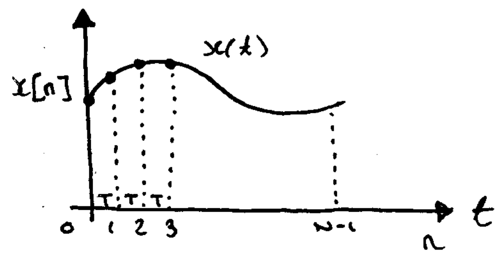
$$X(\Omega) = \frac{(ae^{-j\Omega})^0 - (ae^{-j\Omega})^{q+1}}{1 - ae^{-j\Omega}}$$

$$= \frac{1 - (ae^{-j\Omega})^{q+1}}{(1 - ae^{-j\Omega})}$$

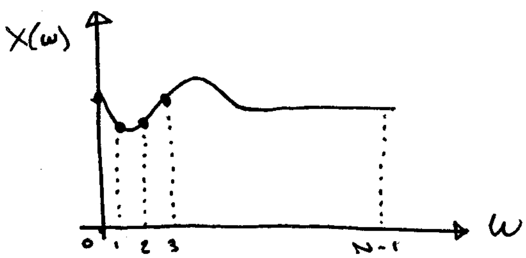
3.5 DFT

$x(t)$, ADC, f_s Hz

$T = 1/f_s$ (sec)



$$x[n] = x(t)\big|_{t=nT} \quad ; \quad n = 0, 1, 2, 3, \ldots, N-1$$



$f_s = 1/T$

$\Delta f = f_s/N$

$\omega = 2\pi f$

$\Delta\omega = 2\pi\Delta f = 2\pi f_s/N$

$\Delta f = f_s/N$ (Hz)

$\Delta\omega = 2\pi\Delta f = 2\pi(f_s/N)$ rad/s

DTF

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

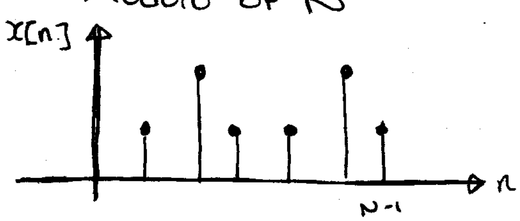
$k\Delta\omega$ $\quad\quad$ $k = 0, 1, \ldots, N-1$

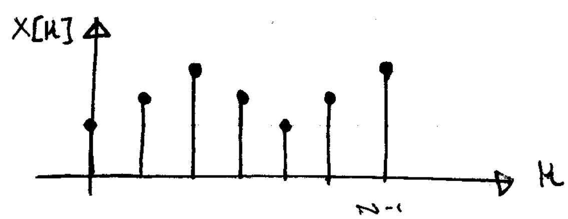
$k\Delta\Omega$

$\underset{n=0,1,2,\ldots,N-1}{x[n]} = 0$, if $n < 0$, $n \geq N$

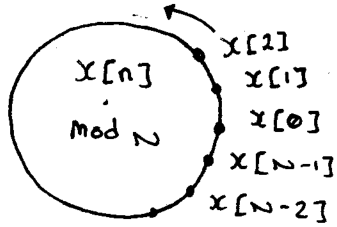
$\underset{k=0,1,2,\ldots,N-1}{X[k]} = 0$, if $k < 0$, $k \geq N$

Modulo of N





Circular representation



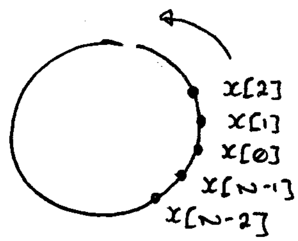
$N = 1024$

$1024 - 1$

$x[-2] = x[N-2]$

$x[-n] = x[N-n]$

$x[n+N] \quad x[n]$



$N = 1024$

$x[-k] = X[N-k]$

$x[k+N] = X[k]$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$X[k] = Re[k] + jIm[k]$$

$$x[n] = 0, \quad \text{if } n < 0, \quad n \geq N \qquad ; \; n = 0, 1, 2, \ldots, N-1$$

$$X[k] = 0, \quad \text{if } k < 0, \quad k \geq N \qquad ; \; k = 0, 1, 2, \ldots, N-1$$

$$X[k] = \sqrt{Re^2 + Im^2}\, e^{j\phi[k]}$$

$$\phi[k] = \arctan\left[\frac{Im(k)}{Re(k)}\right]$$

- IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \qquad ; \; n = 0, 1, 2, \ldots, N-1$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Example 3.5 : Suppose that $x[0]=1$, $x[1]=2$, $x[2]=2$, $x[3]=1$, and $x[n]=0$ for all other integers n. Compute DFT.

Solution : $x[n] = [1, 2, 2, 1]$, $N = 4$

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j2\pi kn/4}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x[n] \left[\cos\left(\frac{-2\pi kn}{N}\right) + j\sin\left(\frac{-2\pi kn}{N}\right)\right]$$

$$= \underbrace{\sum_{n=0}^{N-1} x[n] \cos(2\pi kn/N)}_{Re[k]} - \underbrace{j\sum_{n=0}^{N-1} x[n] \sin(2\pi kn/N)}_{Im}$$

$$Re[k] = \sum_{n=0}^{3} x[n] \cos(\cancel{2}\pi kn/\cancel{4}\,2)$$

$k=1$

$$Re[1] = x[0]\cos(\pi \times 1 \times 0/2) + x[1]\cos(\pi \times 1 \times 1/2) + x[2]\cos(\pi \times 1 \times 2/2) + \dots$$

$$\dots + x[3]\cos(\pi \times 1 \times 3/2)$$

$$= (1 \times 1) + (2 \times 0) + (2 \times (-1)) + (1 \times 0)$$

If $k = 0$:

$$Re[0] = x[0]\cos\left(\frac{\pi \times 0 \times 0}{2}\right) + x[1]\cos\left(\frac{\pi \times 0 \times 1}{2}\right) + x[2]\cos\left(\frac{\pi \times 0 \times 2}{2}\right) + x[3]\cos\left(\frac{\pi \times 0 \times 3}{2}\right)$$

$$= (1 \times 1) + (2 \times 1) + (2 \times 1) + (1 \times 1) = 6$$

$$Re[k] = \begin{cases} 6 & ; k = 0 \\ -1 & ; k = 1 \\ 0 & ; k = 2 \\ -1 & ; k = 3 \end{cases}$$

If $k = 0$:

$$Im[k] = \sum_{n=0}^{3} x[n] \sin\left(\frac{\pi \times k \times n}{2}\right)$$

$$= 1 \times \sin\left(\frac{\pi \times 0 \times 0}{2}\right) + 2 \times \sin\left(\frac{\pi \times 0 \times 1}{2}\right) + 2 \times \sin\left(\frac{\pi \times 0 \times 2}{3}\right) + \dots$$

$$\dots\ 1 \times \sin\left(\frac{\pi \times 0 \times 3}{2}\right) = 0$$

If $k = 1$:

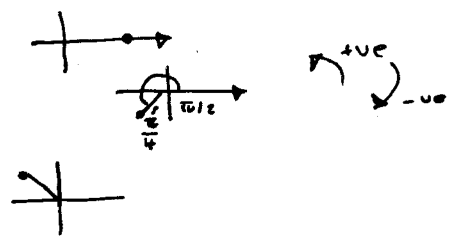
$$Im[1] = \sum_{n=0}^{3} x[n] \sin\left(\frac{\pi kn}{2}\right) = 1$$

$$Im[k] \begin{cases} 0 & ; k = 0 \\ -1 & ; k = 1 \\ 0 & ; k = 2 \\ 1 & ; k = 3 \end{cases}$$

$$X[k] = Re[k] + Im[k] \begin{cases} 6 & ; \quad k = 0 \\ -1 - j & ; \quad k = 1 \\ 0 & ; \quad k = 2 \\ -1 + j & ; \quad k = 3 \end{cases}$$

Polar representation,

$$X[k] = \begin{cases} 6e^{j0} & ; \quad k = 0 \\ \sqrt{2}e^{-j(3\pi/4)} & ; \quad k = 1 \\ 0 & ; \quad k = 2 \\ \sqrt{2}e^{j(3\pi/4)} & ; \end{cases}$$



No class on Thursday.
↳ Midterm covers material until end of today.

Example 3.6:

Consider the signal with the rectangular form of the DFT given by:

$$X[k] = \begin{cases} 6 & ; k = 0 \\ -1-j & ; k = 1 \\ 0 & ; k = 2 \\ -1+j & ; k = 3 \end{cases}$$

Compute the inverse DFT

DFT:

$x[n] = \quad n = 0, 1, 2 \ldots N-1$

mod N

$$X[k] = \sum_{0}^{N-1} x[n]\, e^{-j(2\pi kn/N)}$$

$k = 0, 1, 2 \ldots N-1$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$
$$e^{j\theta} = \cos\theta + j\sin\theta$$

IDFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}$$

$n = 0, 1, 2, \ldots, N-1$

Solution:

$N = 4$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\cos\left(\frac{2\pi kn}{4}\right) + j\sin\left(\frac{2\pi kn}{4}\right)\right)$$

$$= \frac{1}{4} \sum_{0}^{3} X[k] \left(\cos\left(\frac{\pi kn}{2}\right) + j\sin\left(\frac{\pi kn}{2}\right)\right)$$

when $n = 0$:

$$x[0] = \frac{1}{4}\Big[\big(6 \times \cancel{\cos(0)}^{1} + j\cancel{\sin(0)}^{0}\big) + (-1-j) \times \big(\cancel{\cos(0)}^{1} + j\cancel{\sin(0)}^{0}\big) \ldots$$

$$\ldots + (0) + (-1+j) \times \big(\cancel{\cos(0)}^{1} + j\cancel{\sin(0)}^{0}\big) = 1$$

$$x[1] = \left(\frac{1}{4}\right)\Big[6 \times \big(\cancel{\cos(0)}^{1} + j\cancel{\sin(0)}^{0}\big) + (-1-j) \times \big(\cancel{\cos(\pi/2)}^{0} + j\cancel{\sin(\pi/2)}^{1}\big) \ldots$$

$$\ldots + 0 + (-1+j) \times \Big(\cancel{\cos\left(\tfrac{3\pi}{2}\right)}^{0} + j\cancel{\sin\left(\tfrac{3\pi}{2}\right)}^{-1}\Big)\Big] =$$

$$= \left(\frac{1}{4}\right)\Big[6 + (-1-j)j + 0 + [-1+j](-j)\Big]$$

$$= 2$$

$x[2] = 2$

$x[3] = 1$

4) Properties of DFT

$x[n] \longleftrightarrow X[k]\ ,\ v[n] \longleftrightarrow V[k]$

- Linearity

$$a\,x[n] + b\,v[n] \longleftrightarrow a\,X[k] + b\,V[k]$$

- Circular time shift

$$x[n-q,\ \text{mod } N] \longleftrightarrow X[k]e^{-j2\pi kq/N}$$

Proof :

$$X[k] = \sum_{n=0}^{N-1} x[n-q,\ \text{mod } N]e^{-j2\pi kq/N}$$

$$u = n-q \quad ; \quad n = u+q$$

limits : $-q\ ,\ N-1-q$

$$X[k] = \sum_{-q}^{N-1-q} x[u,\ \text{mod } N]e^{-j2\pi k(u+q)/N}$$

$$= \sum_{-q}^{N-1-q} x[u,\ \text{mod } N]e^{-j2\pi ku/N} \cdot e^{-j2\pi kq/N}$$

$$= X[k]e^{-j2\pi kq/N}$$

- Time Reversal

$$x[-n,\ \text{mod } N] \longleftrightarrow X[-k,\ \text{mod } N]$$

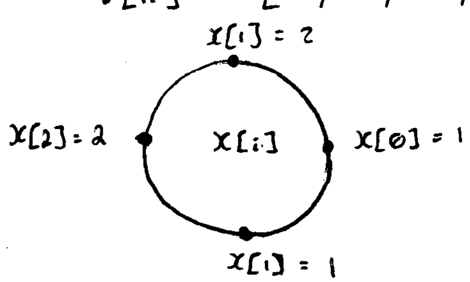
- Circular convolution

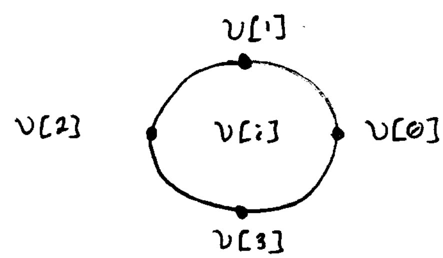
$$y[n] = \underset{(i)}{x[n] \circledast v[n]} = \sum_{i=-\infty}^{\infty} x[i]v[n-i]$$

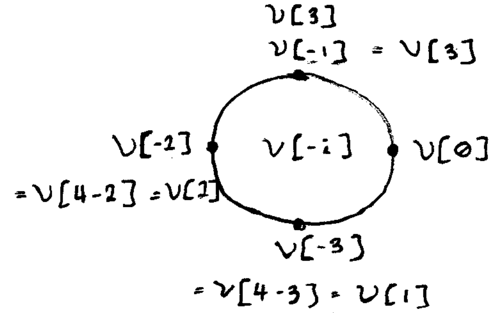
$$x[n] \circledast v[n] = \sum_{i=0}^{N-1} x[i]v[n-i,\ \text{mod } N]$$

$$x[n] = [1,\ 2,\ 2,\ 1]$$

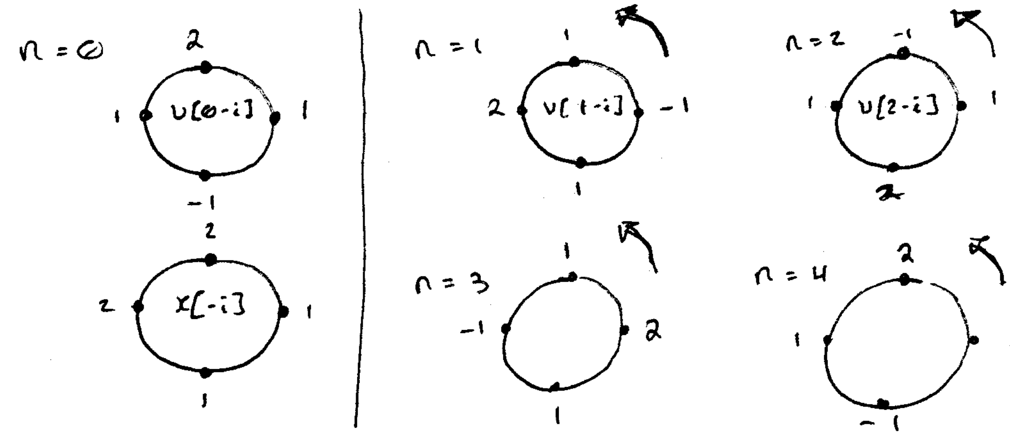
$$v[n] = [1, -1,\ 1,\ 2]$$







$v[-n] = v[N-n]$



$n = 0$

$1 + 4 + 2 - 1 = 6$

$n = 1$

$-1 + 2 + 4 + 1 = 6$

$n = 2$

$1 + (-2) + 2 + 2 = 3$

$n = 3$

$2 + 2 + (-2) + 1 = 3$

$n = 4$

$1 + 4 + 2 - 1 = 6$

5) Relationship between DTFT & DFT

DTFT:

$$X(\Omega) = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$$

DFT:

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi k n}{N}}$$

$$\Omega = \frac{2\pi k}{N} \qquad f_s \qquad T = \frac{1}{f_s}$$

$$\Delta f = \frac{f_s}{N} = \frac{1}{NT}$$

$$\Delta\omega = 2\pi\,\Delta f = 2\pi\,k/N$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Given k, $k = 0, 1, 2 \ldots N-1$

# of multiplications

$\geq N^2$

If $N = 1024$, $N^2 = 1024^2 = 1{,}048{,}576$

FFT (decimation-in-time)

$$\frac{N \log_2(N)}{2}$$

$N = 1024$; $\# = 5129$

If N is an even integer, $N/2$ is an integer

$$x[n] \begin{cases} a[n] = x[2n], & n = 0, 1, 2 \ldots (N/2 - 1) \\ b[n] = x[2n+1], & n = 0, 1, 2 \ldots (N/2 - 1) \end{cases}$$

If $N/2$ is an even integer,

$$a[n] \begin{cases} a_1[n] \begin{cases} \text{etc...} \\ \ldots \end{cases} \\ a_2[n] \begin{cases} \ldots \\ \ldots \end{cases} \end{cases}$$

$$b[n] \begin{cases} b_1[n] \\ b_2[n] \end{cases}$$

- N should be an integer

  1025

  $N = 2^q$

(3.4) FFT

$x[n]$, $n = 0, 1, 2, \dots, N-1$

$N = 1024$

IF N is an even number

$$x[n] \rightarrow a[n] = x[2n], \quad n = 0, 1, \dots, \frac{N}{2} - 1$$
$$ \rightarrow b[n] = x[2n+1], \quad n = 0, 1, \dots, \frac{N}{2} - 1$$

$N/2 \sim$ even #

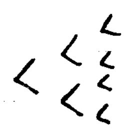
$$a[n] \rightarrow a_1[n] \rightarrow a_3[n], \; a_4[n]$$
$$a[n] \rightarrow a_2[n]$$

$N = 1026,\ 1025$

$N = 1024 = 2^{10}$

- Bit reversing

$8 = 2^3$



Real-time online

Time point n	Binary	Reverse Bit word	Order
0	000	000	$x[0]$
1	001	100	$x[1]$
2	010	010	$x[2]$
3	011	110	$x[3]$
4	100	001	$x[4]$
5	101	101	$x[5]$
6	110	011	$x[6]$
7	111	111	$x[7]$

(3.5) The Laplace Transform & Transfer Function Representation

1) LT computation

Given $x(t)$, two-sided LT

$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

$s =$ Freq variable (complex)

CFT : $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

$s = j\omega$

For one sided LT

$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

ILT : Given $X(s)$

$x(t) = \int_{-\infty}^{\infty} X(s)e^{st}\,dt$

$x(t) \longleftrightarrow X(s)$

$v(t) \longleftrightarrow V(s)$

Example 3.7

(a) $20t \longleftrightarrow 20\ 1/s^2$

(b) $2e^{3t} \longleftrightarrow 2\left(\frac{1}{s-3}\right)$

(c) $5\cos(2t) \longleftrightarrow 5\left(\frac{s^2}{s^2+4}\right)$

(d) $2e^{-t}\sin(3t) \longleftrightarrow 2\left(\frac{3}{(s+1)^2+3^2}\right)$

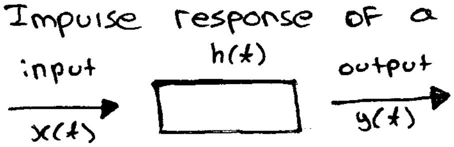
(e) $1 + 2t + e^{-t} \longleftrightarrow \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s+1}$

(f) $2t + 3\frac{dx(t)}{dt} \longleftrightarrow \frac{2}{s^2} + 3\left[sX(s) - x(0)\right]$

(g) $2\int x(t)dt \longleftrightarrow 2\,(1/s)\,X(s)$

2) Transfer Function Representation

- Impulse response of a system is $h(t)$



$y(t) = x(t) \otimes h(t)$

$Y(s) = X(s)H(s)$

TF : $H(s) = \frac{Y(s)}{X(s)}$

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s + b_0}{1 * s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0}$$

$N \geq M$

Factorial Form:

$$H(s) = \frac{b_M (s-z_1)(s-z_2) * \dots * (s-z_m)}{(s-p_1)(s-p_2) * \dots * (s-p_n)}$$

- $p_1, p_2, \dots, p_n$ ~ roots of denominator polynomial (poles)
- $z_1, z_2, \dots, z_m$ ~ roots of the numerator (zeros)

→ MATLAB : roots m

→ H(s) properties ~ poles, zeros

- N = order of the H(s)

$\boxed{\text{Example 3.8}}$ Given the following frequency function, determine the roots + order of the system

$-3 \pm j$

$$H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2(s^2 + 6s + 10)}{s^3 + 6s^2 + 10s + 8}$$

Solution:

$$H(s) = \frac{2(s+3-j)(s+3+j)}{(s+4)(s+1-j)(s+1+j)}$$

- 3rd order system
  - roots: $-4, \ -1 \pm j$
- zeros : $-3 \pm j$
- poles : $-4 \ ; -1 \pm j$

$$\text{input} \xrightarrow[x(t)]{X(s)} \boxed{\text{system}}_{h(t)}^{H(s)} \xrightarrow[y(t)]{Y(s)} \text{output}$$

Solution to get $y(t)$

- $h(t) \longrightarrow H(s)$
- $x(t) \longrightarrow X(s)$
- $Y(s) = H(s) * X(s)$
- $y(t) \xleftrightarrow{ILT} Y(s)$

Example 3.10

Given the following frequency function, determine it's corresponding time signal.

$$H(s) = \frac{s+2}{s^3 + 4s^2 + 3s}$$

Poles :

$$H(s) = \frac{s+2}{s(s^2+4s+3)}$$

$$= \frac{s+2}{s(s+3)(s+1)}$$

zeros : -2

poles : $0, \ -3, \ -1$

$$H(s) = \frac{a}{s} + \frac{b}{s+3} + \frac{c}{s+1}$$

(3.5) LT & TF

$x(t)$

LT: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

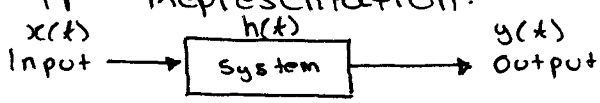
$-\infty < s < \infty$

$s = j\omega$

ILT: $x(t) = \int_{-\infty}^{\infty} X(s)e^{st}\,ds$

$-\infty < s < \infty$

TF Representation:



$y(t) = h(t) \otimes x(t)$

$Y(s) = H(s) * X(s)$

$\longrightarrow$ TF : $H(s) = \dfrac{Y(s)}{X(s)}$

$$H(s) = \frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Roots:

$z_1, z_2, \ldots, z_m$ ~ zeros

$p_1, p_2, \ldots, p_N$ ~ poles

order $= N$

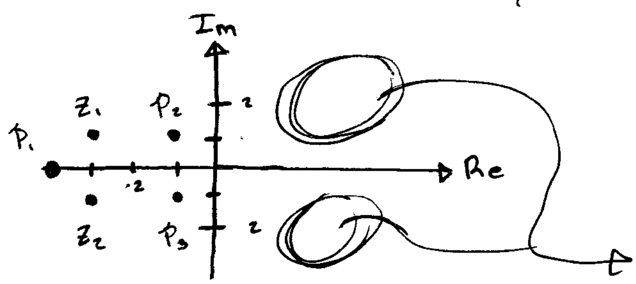
$N \uparrow\uparrow$ – complexity increases
– costs increase
– heat generated increase

↳ as temperature ↑, μ ↓ (viscosity decreases)

$\boxed{\text{Example 3.8}}$ Determine roots + order of system:

$$H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8}$$

$\rightarrow$ $z_1 = -3 + j$, $z_2 = -3 - j$; $\rightarrow$ $p_1 = -4$, $p_2 = -1 + j$, $p_3 = -1 - j$



If poles were here, system would be unstable.

$\boxed{\text{Example 3.10}}$ Given frequency fxn, determine time signal

$$H(s) = \frac{s+2}{s^3 + 4s^2 + 3s}$$

$\rightarrow$ $z_1 = -2$; $p_1 = 0$, $p_2 = -1$, $p_3 = -3$

$$H(s) = \frac{a}{s-0} + \frac{b}{s+1} + \frac{c}{s+3}$$

Method I: direct comparison (3 unknowns)

$$\frac{s+2}{s(s+1)(s+3)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+3}$$

$$= \frac{a(s+1)(s+3) + b(s)(s+3) + c(s)(s+1)}{s(s+1)(s+3)}$$

$$a(s^2 + 4s + 3) + b(s^2 + 3s) + c(s^2 + s) = s + 2$$

$$\rightarrow \underline{as^2} + \underline{a4s} + \underline{a3} + \underline{bs^2} + \underline{b3s} + \underline{cs^2} + \underline{cs} = \underline{s} + \underline{2}$$

$$\begin{cases} a + b + c = 0 \\ 4a + 3b + c = 1 \\ 3a = 2 \end{cases}$$

Method II

$$\frac{s+2}{s(s+1)(s+3)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+3}$$

Multiple s. Let $s = 0$

$$\left.\frac{s+2}{\cancel{s}(s+1)(s+3)} * \cancel{s}\right|_{s=0} = \left.\frac{a}{\cancel{s}} * \cancel{s} + \cancel{\frac{b}{s+1} * s} + \cancel{\frac{c}{s+3} * s}\right|_{s=0}$$

$$(2/3) = a$$

$* s+1$, let $s = -1$

$$\rightarrow \left.\frac{s+2}{s(\cancel{s+1})(s+3)} \times (\cancel{s+1})\right|_{s=-1} = \left.\cancel{\frac{a}{s}(s+1)} + \frac{b}{\cancel{s+1}} * (\cancel{s+1}) + \cancel{\frac{c}{s+3}(s+1)}\right|_{s=-1}$$

$$\rightarrow \frac{-1+2}{(-1)(2)} \Rightarrow \frac{-1}{2} = \emptyset + b + \emptyset$$

then $b = (^{-1}/2)$

$* s+3$, let $s = -3$

$$\rightarrow \left.\frac{s+2}{s(s+1)(\cancel{s+3})} (\cancel{s+3})\right|_{s=-3} = \left.\cancel{\frac{a}{s}(s+3)} + \cancel{\frac{b}{s+1}(s+3)} + \frac{c}{\cancel{s+3}}(\cancel{s+3})\right|_{s=-3}$$

$$\rightarrow \frac{(-3)+2}{(-3)(-2)} \Rightarrow \frac{-1}{6} = \emptyset + \emptyset + c$$

then $c = (^{-1}/6)$

$$H(s) = \frac{(2/3)}{s} + \frac{(^{-1}/2)}{s+1} + \frac{(^{-1}/6)}{s+3}$$

$$h(t) = (2/3)e^{-0t} - (1/2)e^{-t} - (1/6)e^{-3t}$$

3.6 Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$x[n]$, $-\infty < n < \infty$

DTFT:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

The Z-transform :

$x[n]$

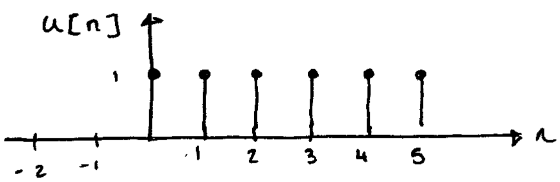
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = e^{j\Omega}$$

$$\Omega = \omega T \quad \& \quad s = j\omega$$

$\boxed{\text{Example 4.8}}$

$$u[n] = \begin{cases} 1 & ; \; n = 0, 1, 2, \ldots, \infty \\ 0 & ; \; n = -\infty, \ldots, -2, -1, 0 \end{cases}$$



$$U(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \times z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \ldots$$

$$\sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2}}{1 - r} = \frac{(z^{-1})^0 - (z^{-1})^\infty}{(z^{-1})}$$

$$= \frac{1 - 0}{1 - 1/z} = \frac{z}{z - 1}$$

Analog Filter < passive Filter / active Filter

R.C. op. amp

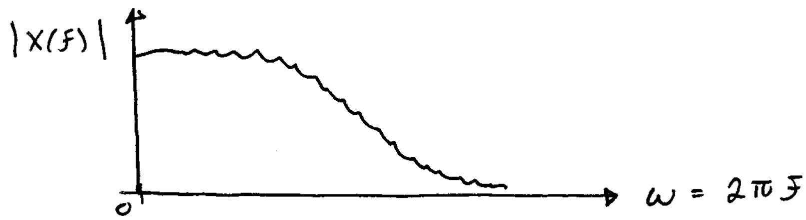
Inductor

3.7 Window Fxn

Physical quantity → $\boxed{\text{sensor}}$ → output $x(t)$

$\downarrow$ ADC

$x[n]$



$\frac{30000}{60}$ rev/min

500 Hz

$\frac{1800}{60}$ rpm

= 30 Hz

$x[n]$ length(x) = ∞

Infinite length, $n = 0, 1, 2, 3, \dots$

$x[n] = \cos(40\pi n)$, $n = 0, 1, \dots, \infty$

$\omega = 40\pi$; $F = 20$ Hz

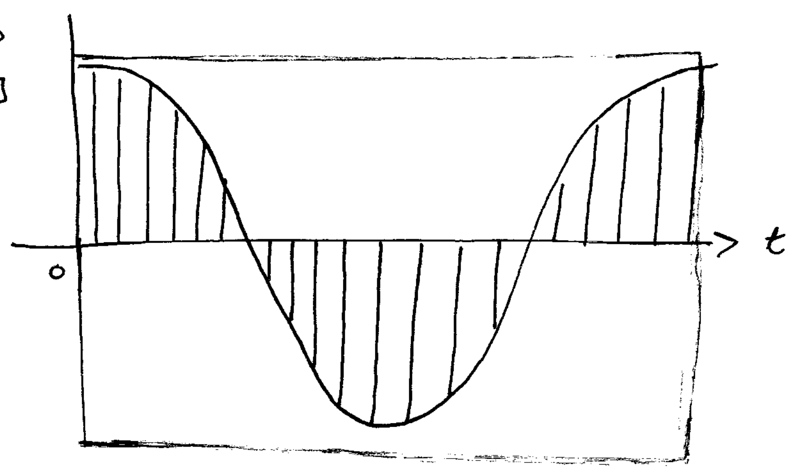
$N = 10\,000$

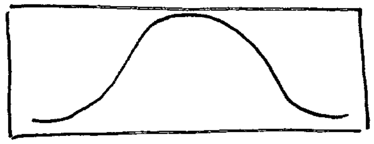
$N = 1000$

$N = 100 \longrightarrow$ leakage

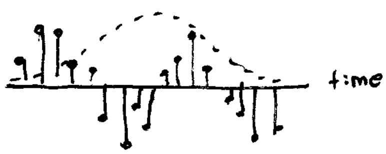
- Signal length
- $x(t)$

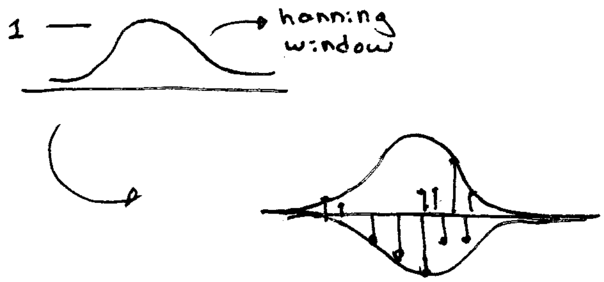
  $x[n]$



- rectangular 
- Hanning window $W[n] = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)$; $n = 0, 1, \cdots N-1$
- Hamming window $W[n] = 0.5 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$; $n = 0, 1, \cdots, N-1$

$x[n] * w[n]$

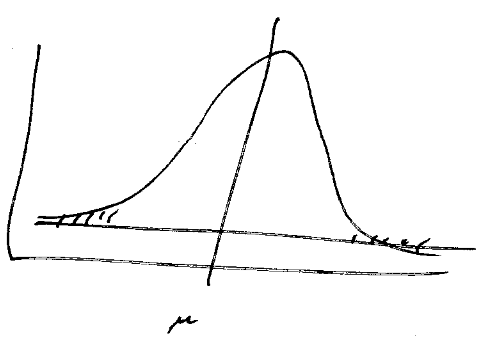




3.8 Kurtosis Analysis

$$KU = \frac{\mu^4}{\sigma^4} = \frac{E\{(x-\mu)^4\}}{\sigma^4}$$

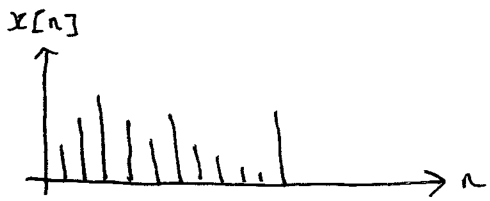
Pulses will change pdf properties @ tail . . .



$x[n]$, $n = 0, 1, 2 \ldots, (N-1)$

N

- infinitely long $>$ leakage
- rectangular window



$x'[n] = x[n] .* w[n]$

$x'[n] \neq x[n]$

on the amplitude spectrum, leakage $\downarrow$

$\rightarrow$ $x(t) = \cos(40\pi t)$

$\omega = 40\pi$ rad/s

$f = \omega / 2\pi = 20$ Hz

MATLAB code

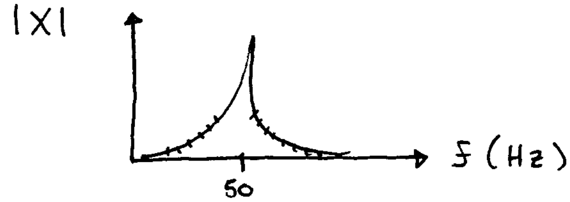
```
• fs = 100 Hz ;
• T = 1/fs ; (sec)- used for demonstration
• t = 0 : T : 1 ;
• x = x' ;
• Xr = abs(fft(x)) ;
• w = hanning (length(x)) ;
• xw = x.*w ;
• Xamp = abs ( fft(xw)) ;
• L2 = fix (length (Xr)/2) ;      (look @ half the signal
• freq = (0:(L2-1) / L2 *(fs/2))
```

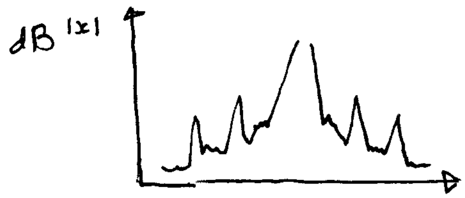
will send code

$X[K] = \sum_n x[n] e^{-j2\pi kn/N}$

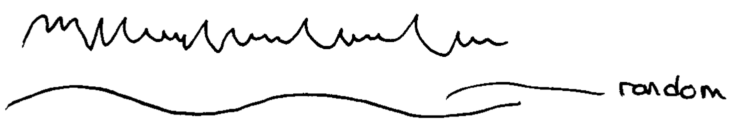
$X(\Omega) =$ DTFT

dB // 1 dB $= 20\log_{10} |A/B|$





Kurtosis,



$KU = \mu_4 / \sigma^4$

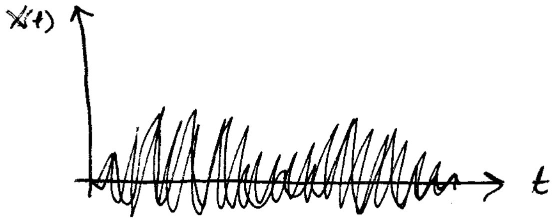
Crest Factor, $CF = X_{max} / \sigma$

For a healthy system, signal → gaussian pdf

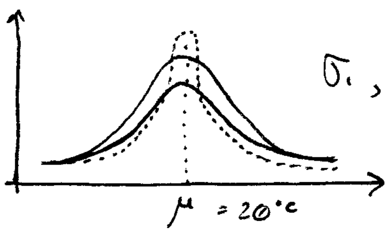
$KU = 3$

Gaussian signal

Peak amplitude probability (%)	CF	KU
4.6	2	3
0.1	3.3	3
0.01	3.9	3

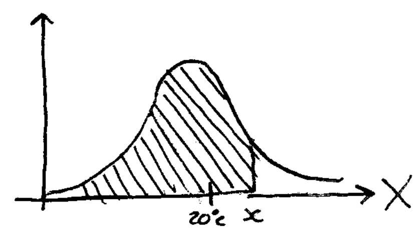


Gaussian p.d.f.



σ_1, σ^2

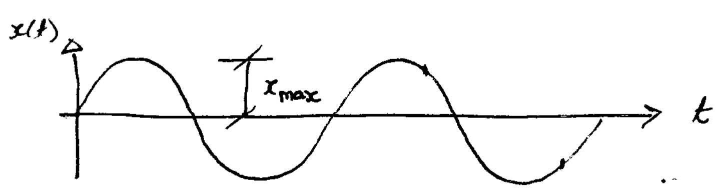
Z-table



$P(x)$ $\leftarrow$ $CF = x_{max}/\sigma$

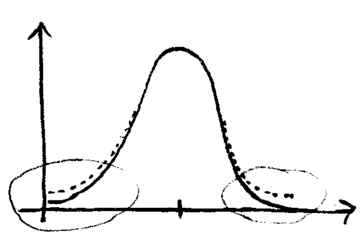
Table 5.2 : The CF and KU for various functions

Signal details	CF	KU
Sine wave pulse train	2	1.5

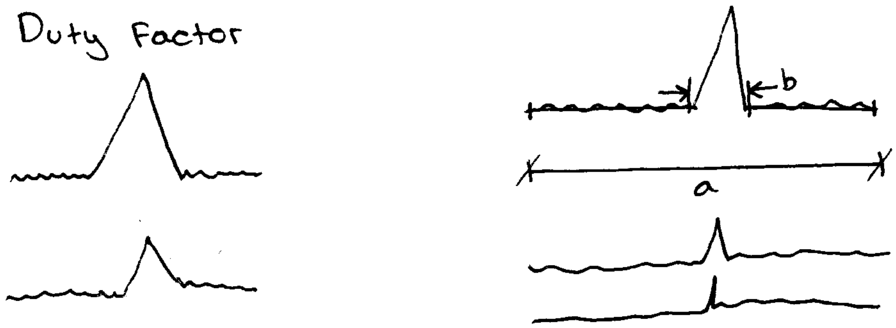


$CF = x_{max}/\sigma$

$KU = \mu_4/\sigma^4$ ~ p.d.f. tail properties

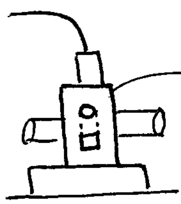


Duty Factor



$\frac{b}{a}$

(*) → can do Q1-4 on A3.



→ bearing housing

time signal

≈ Gaussian

$x[n]$ ADC

→ Anti-aliasing Filtering first

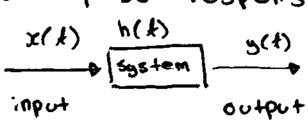
↳ ADC second

→ need to read material for bearing fault detection (Ch.12?)

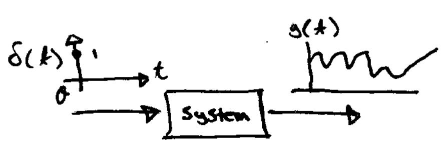
Chapter 4 : Design of Digital Filters

4.1 Analysis of Ideal Filter

1) Impulse response



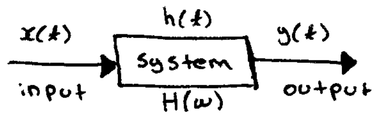
$h(t)$ = impulse (unit pulse) response fn



$$y(t) = h(t) \otimes x(t)$$

$$y[n] = h[n] \otimes x[n], \quad n = 0, 1, 2, \ldots, N-1$$

2) Filters and Filtering



Input : $x(t) = A\cos(\omega_0 t)$

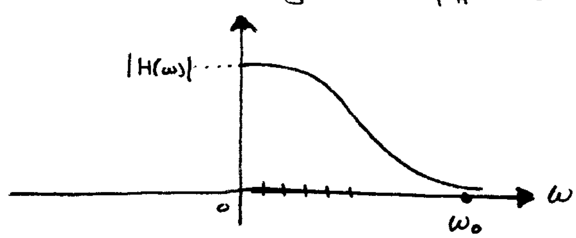
$x(t) \longleftrightarrow X(\omega)$

$Y(\omega) = X(\omega) * H(\omega)$

$Y(\omega) \longleftrightarrow y(t)$

$y(t) = A\,|H(\omega_0)|\cos(\omega_0 t + \phi_H)$

ϕ_H = phase delay : $\phi_H = \arctan \dfrac{Im(H(\omega_0))}{Re(H(\omega_0))}$



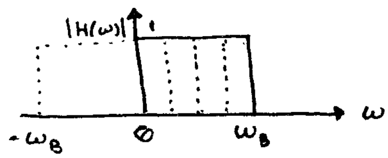
$|H(\omega_0)| = \varnothing$

$y(t) = \varnothing$

Filter, Filtering

3) Ideal Filters

① Ideal low pass Filter



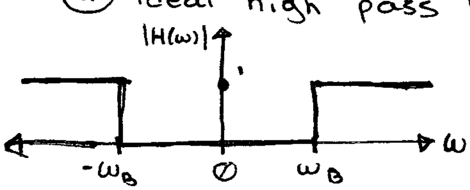
$$|H(\omega)| = \begin{cases} 1, & -\omega_B < \omega < \omega_B \\ \varnothing, & \omega > \omega_B,\quad \omega < -\omega_B \end{cases}$$

$-\omega_B \sim \omega_B$ = pass band

$\omega_B \sim +\infty$, $-\infty \sim -\omega_B$ = stop band

band width $\sim \omega_B$

② Ideal high pass Filter



$$|H(\omega)| = \begin{cases} 1, & \omega \geq \omega_B,\quad \omega \leq -\omega_B \\ \varnothing, & \text{otherwise} \end{cases}$$

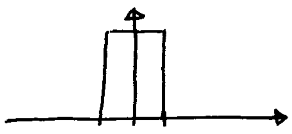
pass bands = $-\infty \sim -\omega_B$

" " = $\omega_B \sim \infty$

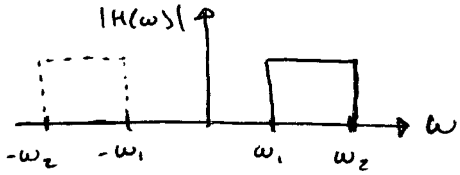
stop band = $-\omega_B \sim \omega_B$

LU Radio Station

102.7 MHz

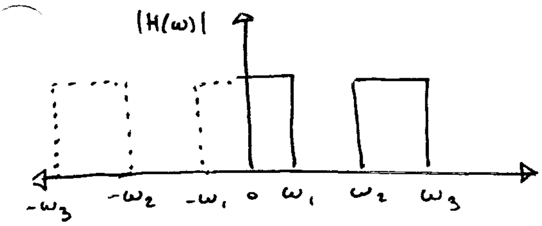


3 Ideal band pass Filter



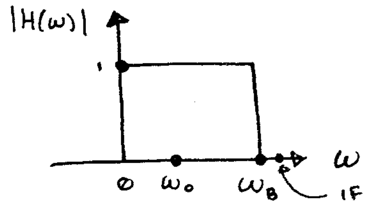
$$|H(\omega)| = \begin{cases} 1 , & \omega_1 \sim \omega_2 , \quad -\omega_2 \sim -\omega_1 \\ 0 , & \omega < -\omega_2 \\ & -\omega_1 \sim \omega_1 , \quad \omega > \omega_2 \end{cases}$$

4 Ideal band stop Filters



$$|H(\omega)| \begin{cases} 1 ; & -\omega_1 \sim \omega_1 , \quad \omega_2 \sim +\infty , \\ 0 ; & -\omega_2 \sim -\omega_1 , \quad \omega_1 \sim \omega_2 \end{cases}$$

LPF :



ω_B = cutoff freq.

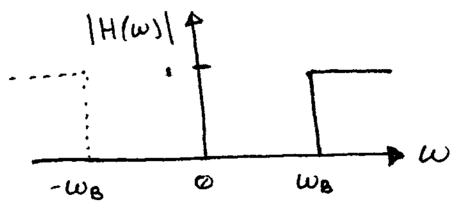
low pass filter

IF this was ω_0 this amplitude becomes zero.

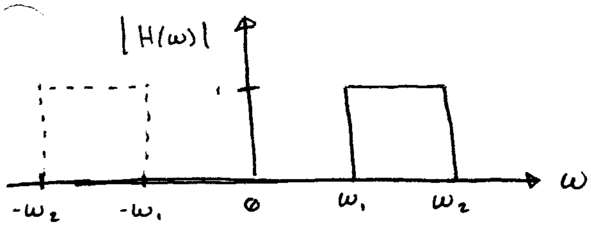
$x(t) = A\cos(\omega_0 t)$

$y(t) = A\,|H(\omega_0)|\cos(\omega_0 t + \phi_n)$

HPL :



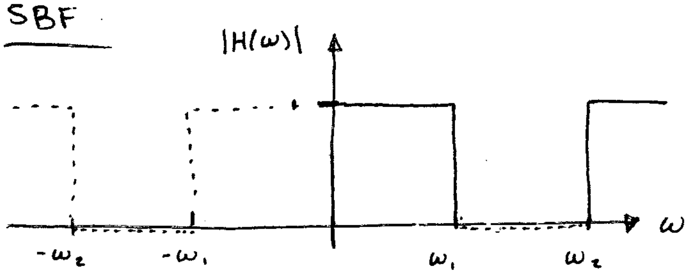
half pass filter



BPF

band pass filter

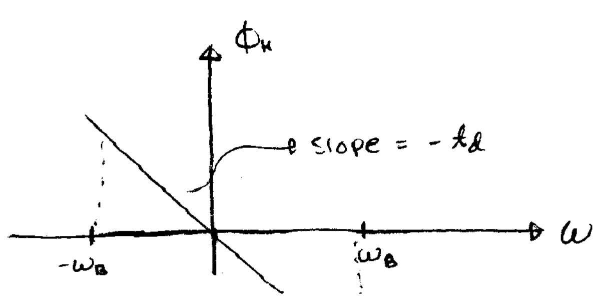
SBF



Stop-band Filtel

4.2 Phase Function of Ideal Filters

A linear phase over the passband



$\phi_H = -\omega t_d$

$x(t) = A\cos(\omega_0 t)$

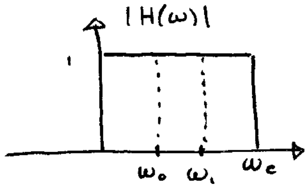
Output :

$y(t) = A\,|H(\omega_0)|\cos(\omega_0 t - \omega_0 t_d)$

$= A \times 1 \times \cos(\omega_0(t - t_d))$

delay

$x = A_0\cos(\omega_0 t) + A_1\cos(\omega_1 t)$

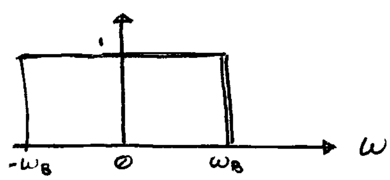


$y(t) = A_0|H(\omega)|\cos(\omega_0 t - \omega_0 t_d) + A_1|H(\omega)|\cos(\omega_1 t - \omega_1 t_d)$

delay

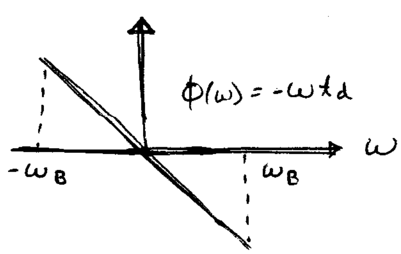
IF ϕ_H is not a linear function of ω over the PB

$\phi_H = c$



CFT

$h(t)$



$$H(\omega) = \begin{cases} 1 & ; \quad -\omega_B \le \omega \le \omega_B \\ \emptyset & ; \quad \text{otherwise} \end{cases}$$

$$\phi_H = \begin{cases} -\omega t_d & ; \quad -\omega_B \le \omega \le \omega_B \\ \emptyset \end{cases}$$

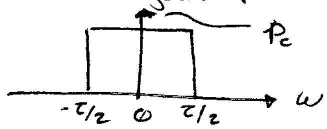
Polar ex.

$H(\omega) = 1 \times e^{-j\omega t_d} = \cos(-\omega t_d) + j\sin(-\omega t_d)$ $\quad |e^{j\theta}| = \cos\theta + j\sin\theta$

$$H(\omega) = P_{2B}\, e^{-j\omega t_d}$$

P_{2B} = rectangular function $\quad -\omega_B \sim \omega_B$

P_τ = rectangular pulse with τ



$\rightarrow \quad \tau \,\mathrm{sinc}\left(\dfrac{\tau t}{2\pi}\right) \longleftrightarrow 2\pi P_\tau$

$\rightarrow \quad \dfrac{\tau}{2\pi}\,\mathrm{sinc}\left(\dfrac{\tau t}{2\pi}\right) \longleftrightarrow P_\tau$

$\rightarrow \quad \tau = 2\omega_B$

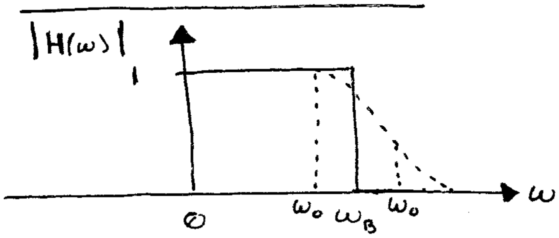
$\rightarrow \quad \dfrac{\cancel{2}\omega_B}{\cancel{2}\pi}\,\mathrm{sinc}\left(\dfrac{\cancel{2}\omega_B t}{\cancel{2}\pi}\right) \longleftrightarrow P_{2B}$

$\delta(t-c) \longleftrightarrow e^{-j\omega c}$

$c = t_d$

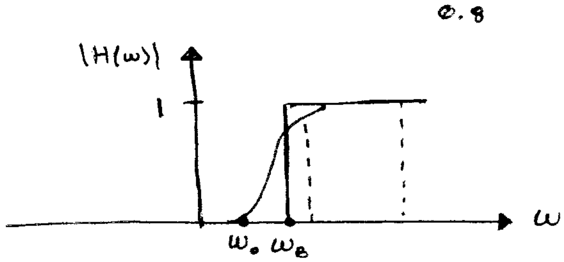
$$\underbrace{\left(\frac{\omega_B}{\pi}\right)\mathrm{sinc}\left(\frac{\omega_B(t-t_d)}{\pi}\right)}_{h(t)\ \rightarrow\ \text{non-causal}} \longleftrightarrow P_{2B}\, e^{-j\omega t_d}$$

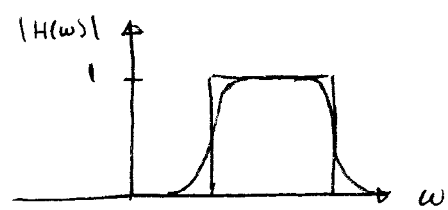
4.2 Causal Filters



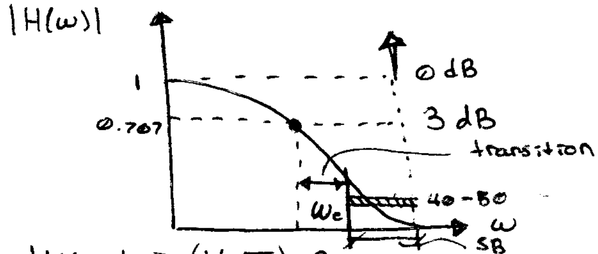
Input : $\quad x(t) = A\cos(\omega_0 t)$

Output : $\quad y(t) = A\underbrace{|H(\omega)|}_{0.8}\cos(\omega_0 t + \phi_H)$



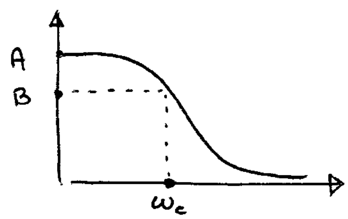


Passband of LPF



$|H(\omega)| \geq (1/\sqrt{2}) \approx 0.707 \sim 3dB$ (magic number)

$dB = 20 \log_{10} |H(\omega)|$



$3dB = 20 \log (A/B) = 20 \log A/A/\sqrt{2}$

$B = A/\sqrt{2}$

ω_c = cut-off freq. of the LPF

Pass-band : $0 \sim \omega_c$

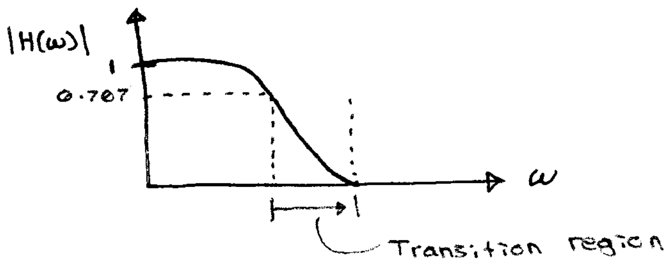
- Stop band : drop $40 \sim 50$ dB
- transition region should be as narrow as possible
- Butterworth Filters

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\zeta > 0$ damping ratio

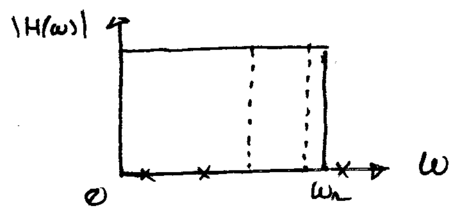
$s = j\omega$

$$H(\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$



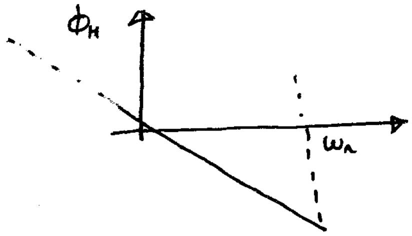
Def'n of passband of a lowpass filter.

Ideal Filter (LPF)



Freq. domain

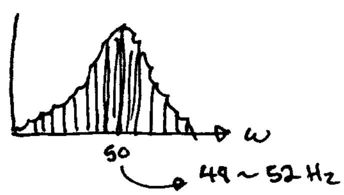
time domain

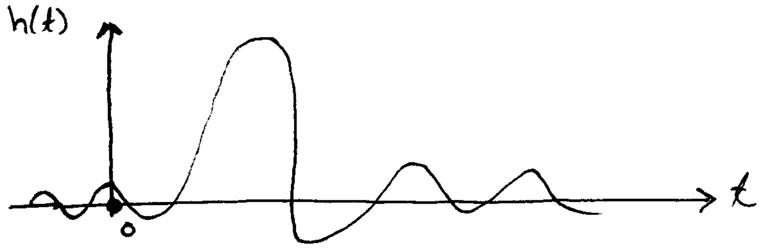


$\phi_H = -\omega t_d$

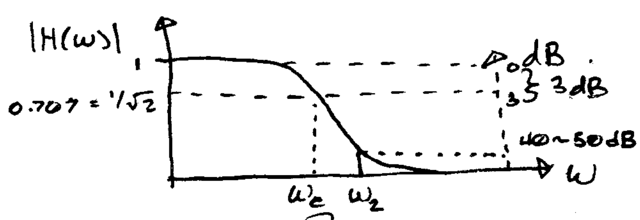
Input : $x(t) = A\cos(\omega_0 t)$

Output : $y(t) = A\,|H(\omega)| * \cos(\omega_0 t + \phi_H)$

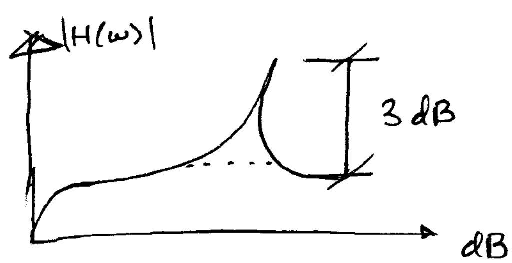


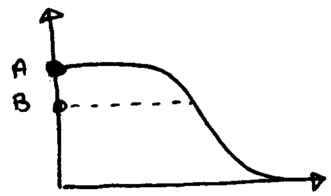


no initial energy



where ω_c = cutoff freq.





$3\,dB = 20\log_{10}(A/B)$

$B = A/\sqrt{2}$

(or $B = 0.707A$)

(2) Butterworth Filter

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ζ = damping ratio (internal impedance)

$s = j\omega$

$$H(\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

<u>Let $\omega_n = 1$</u>

Poles : $-\omega_n/\sqrt{2} + j\omega_n/\sqrt{2}$

Zeroes : none

$$H(\omega) = \frac{\omega_n^2[\quad]}{[(\omega_n^2-\omega^2) + j(2\zeta\omega_n\omega)][(\omega_n^2-\omega^2) - j\omega]}$$

$= Re + jIm$

$$|H(\omega)| = \sqrt{Re^2 + Im^2} = \frac{\omega_n^2}{\sqrt{(\omega_n^2-\omega^2) + 4\zeta^2\omega_n^2\omega}}$$

$\zeta = 0 \sim 1$

$\zeta = 1/\sqrt{2} = 0.707 \approx 0.7$

$\Rightarrow$ PB <u>Maximal flat</u>

(Butterworth) <u>BW</u>

$$|H(\omega)| = \frac{\omega_n^2}{\sqrt{(\omega^2-\omega_n^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

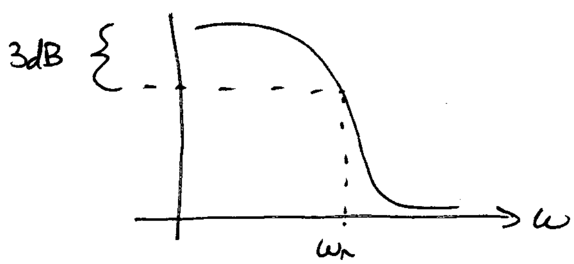
$$= \frac{\omega_n^2}{\sqrt{(\omega^2-\omega_n^2)^2 + 2\omega_n^2\omega^2}} \longrightarrow |H(\omega)| = \frac{1}{\sqrt{\frac{(\omega^2-\omega_n^2)^2}{\omega_n^4} + \frac{2\omega_n^2\omega}{\omega_n^4}}}$$

then $|H(\omega)| = \dfrac{1}{\sqrt{1+(\omega/\omega_n)^4}}$

if $\omega = \omega_n$

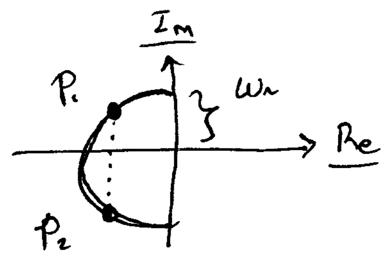
$|H(\omega)| = 1/\sqrt{2} \sim 3\,dB$

ω_n = cutoff freq. of LP BW Filter



$$|P_{1,2}| = \sqrt{(-1/\sqrt{2}\,\omega_n)^2 + (\omega_n/\sqrt{2})^2} = \omega_n$$

$-1/\sqrt{2}\,\omega_n \pm j\,1/\sqrt{2}\,\omega_n$



two poles

B.W. Filters

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

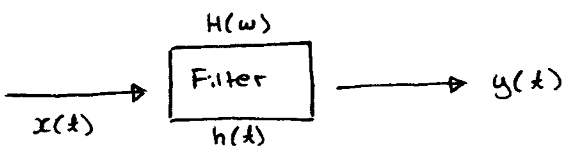
$$P_{1,2} = \frac{-\omega_n}{\sqrt{2}} \pm j\frac{\omega_n}{\sqrt{2}}$$

$$s = j\omega$$

$$|H(\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

$\omega = 0 \longrightarrow 10\,\omega_n$

ζ = damping ratio



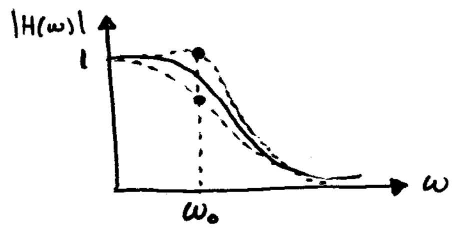
$A\cos(\omega_0 t)$

$$y(t) = A|H(\omega)|\cos(\omega_0 t + \Phi_H)$$

$$\zeta = (1/\sqrt{2}) \approx 0.7$$

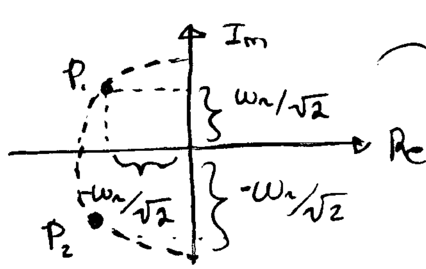
P.B. ~ maximal flat

Butterworth



$$|P_1| = \sqrt{(-\omega_n/\sqrt{2})^2 + (\omega_n/\sqrt{2})^2} = \omega_n$$

$$|P_2| = \qquad + \qquad = \omega_n$$



semi circle w/ radium ω_n

If $\zeta = 1/\sqrt{2}$

$$|H(\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 2\omega_n^2\omega^2}}$$

$$= \frac{1}{\sqrt{1 + (\omega/\omega_n)^4}}$$

If $\omega = \omega_n$

$|H(\omega)| = 1/\sqrt{2}$, ω_n = cutoff Freq.

3) N^{th} order BW Filters

3^{rd} order BW:

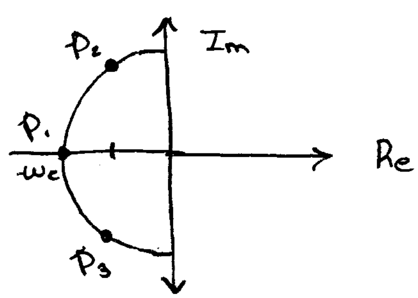
$$H(s) = \frac{\omega_c^3}{(s+\omega_c)(s^2+\omega_c s+\omega_c^2)}$$

no zeroes

$P_1 = -\omega_c$

$P_{2,3} = -\omega_c/2 \pm j\sqrt{3}/2\ \omega_c$

$|P_2| = \sqrt{(-\omega_c/2)^2 + (\sqrt{3}/2\,\omega_c)^2} = \omega_c$



$s = j\omega$

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_n)^{2\times 3}}}$$

N-pole BW Filter

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_n)^{2N}}}$$

ω_n = cutoff Freq.

MATLAB:

$\omega_n = 1$ rad/s

buttap()

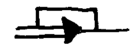
→ Running BW-1

$a = [1 \quad 1.4142 \quad 1]$

$b = [0 \quad 0 \quad 1]$

$$H(s) = \frac{\emptyset s^2 + \emptyset s + 1}{s^2 + 1.4142 s + 1}$$

; K = gain



op. amp

(*) Assignment 4 due next Tuesday (Nov. 12)

(4) Chebyshev Filters

BW : monotonic Fxn

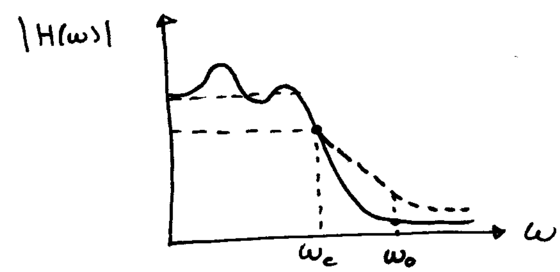
transition bend is wide

CV : ~ not monotonic

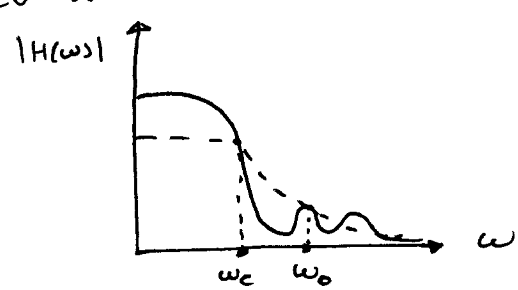
narrow transition

CV - type I filter

CV - 1



CV - 2



N - Pole CV-1

$$|H(\omega)| = \frac{1}{\sqrt{1 + \xi^2 T_N(\omega/\omega_1)}}$$

$T_N(x) = 2x\,T_{N-1} - T_{N-2}(x)$

$T_0 = 1$, $T_1(x) = x$

$T_2 = 2x\,T_1 - T_0$

$= 2x^2 - 1 \quad \ldots$

$$T_3 = 2xT_2 - T_1$$
$$= 2x(2x^2-1) - x = 4x^3 - 3x$$

$$T_4 = 2xT_3 - T_2$$
$$= 2x(4x^3 - 3x) - (2x^2 - 1)$$
$$= 8x^4 - 8x^2 + 1$$

$N = 2$

$$|H(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 T^2}} = \frac{1}{\sqrt{1+\varepsilon^2[2(\omega/\omega_1)^2 - 1]^2}}$$

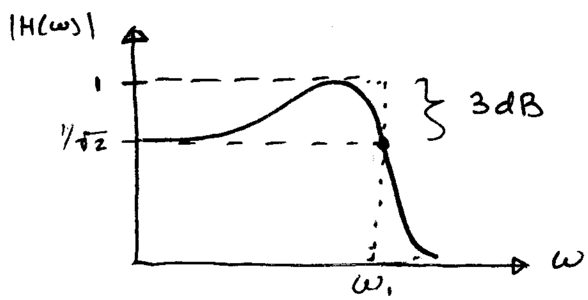
$\omega = 0 \sim \omega_1$

$$|H(\omega)| = \begin{cases} \dfrac{1}{\sqrt{1+\varepsilon^2}} & : \omega = 0 \\ \dfrac{1}{\sqrt{1+\varepsilon^2}} & : \omega = \omega_1 \\ 1 & : \text{if } (\omega/\omega_1)^2 = 1/2 \end{cases}$$

If $\varepsilon = 1$; if $\omega = \omega_1$

if $\omega = 0$, $|H(\omega)| = 1/\sqrt{2}$

if $\omega = \omega_1$, $|H(\omega)| = 1/\sqrt{2}$



ω_1 = cutoff freq.

Nov. 7/19

Lab on Monday : Given 3 vectors
(Monday @ 10:30)

- healthy bearing (f_N)
- outer raceway damage (f_{od})
- inner raceway damage (f_{id})

→ FFT (amplitude) → windowing function (hanning, hamming)

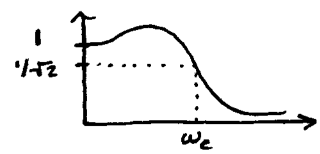
→ Kurtosis

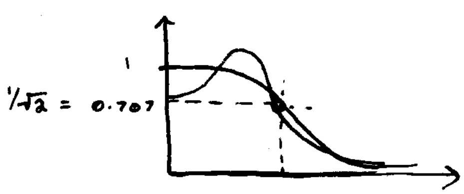
→ characteristic Freq. (f_N)

4) CV Filter

CV-1 ~ ripples in the pass band

CV-2 ~ ripples in the stop band





$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 T_N^2(\omega/\omega_1)}}$$

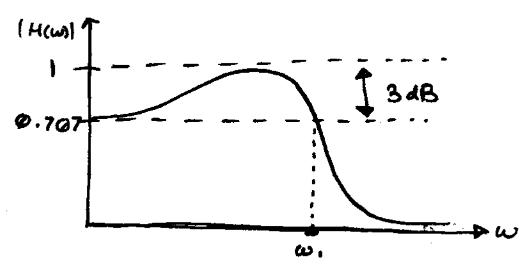
$T_0 = 1$, $T_1 = x$

$N = 2$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2\left[2(\omega/\omega_1)^2 - 1\right]^2}}$$

$$|H(\omega)| \begin{cases} 1/\sqrt{2} & ; \quad \omega = 0 \\ 1/\sqrt{2} & ; \quad \omega = \omega_1 \\ 1 & ; \quad 2(\omega/\omega_1)^2 - 1 = \pm 1 \end{cases}$$

⇒



If $\varepsilon = 1$; 3 dB ripple in pass band.

$$H(s) = \frac{0.251\,\omega_1^3}{s^3 + 0.597\omega_1 s^2 + 0.928\omega_1^2 s + 0.251\omega_1^2}$$

$$H(s) = \frac{(s - z_1)}{(s-p_1)(s-p_2)(s-p_3)}$$

LP $\quad \omega_c = 1$ rad/s

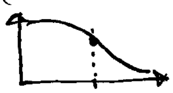
5) Freq. transformation

LP, (BW, CV ~ 3dB)

with $H(s)$, and cutoff frequency ω_1

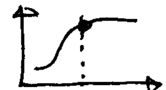
→ LP with cutoff freq of ω_2

$(s\,\omega_1/\omega_2)$



HP with cutoff Freq of ω_2

$(\omega_1\omega_2/s)$



BP with BP of $\omega_2 \sim \omega_3$

$\omega_1 \left(\dfrac{s^2 + \omega_2\omega_3}{s(\omega_3 - \omega_2)} \right)$

→ BS with SB $\omega_2 - \omega_3$

$$\omega_1 \frac{s(\omega_3 - \omega_2)}{s^2 + \omega_2\omega_3}$$

Example 2

3-pole BW Filter

$$H(s) = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$$

ω_c = cutoff freq., $\omega_1 = \omega_c$

BP : $\omega_2 = 3$, $\omega_3 = 5$ rad/s

$$s = \omega_c \frac{s^2 + 3 \times 5}{s(5-3)} = \omega_c \left(\frac{s^2 + 15}{2s} \right)$$

$$H(s) = \frac{\omega_c^3}{\left(\omega_c \frac{s^2+15}{2s}\right)^3 + 2\omega_c \left(\omega_c \frac{s^2+15}{2s}\right)^2 + 2\omega_c^2 \left[\omega_c \frac{s^2+15}{2s}\right] + \omega_c^3}$$

$$= \frac{8s}{s^6 + 4s^5 + 53s^4 + 128s^3 + 795s^2 + 900s + 3375}$$

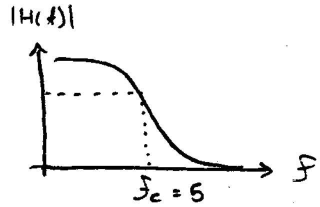
Example 4.2

- Corresponding LPF (BW, CV-I order) - H(s)
  - ω_c = 1 rad/s
  - Get H(s)
  - Transform H(s) $\longrightarrow$ desired Filter

    Frequency transform

$\boxed{\text{Example 4.2}}$ Design a three-pole Butterworth low-pass filter with a bandwidth of 5 Hz.

Solution :

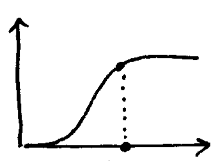


Given: $$H(s) = \frac{(s-z_1)}{(s-p_1)(s-p_2)(s-p_3)}$$

$$H(s) = \frac{2s+1}{s^3+2s^2+s+4}$$ . . . MATLAB

$\boxed{\text{Example 4.3}}$ Design a 3-pole high-pass filter with cutoff frequency $\omega = 4$ Hz

Solution :



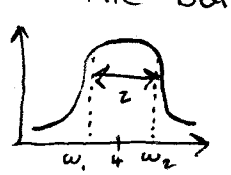
ω_c =

$f_c = 4$ Hz

$\omega_c = 2\pi \times 4$ (rad/s)

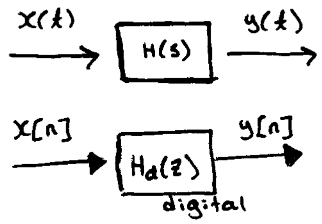
$\boxed{\text{Example 4.4}}$ In example 4.2, a three-pole Butterworth lowpass filter was transformed to a bandpass filter with the passband centered at $\omega = 4$ Hz. The bandwidth is equal to 2 Hz.

Solution :



4.3 - Design of Digital Filters

1) Digital Filter



- DTFT. (discrete time Fourier transform)

$X(\Omega) = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$; where $-\pi \le \Omega \le \pi$

- DFT (discrete Fourier transform)

$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi k n}{N}\right)}$; where $\Omega = \frac{2\pi k}{N} \sim 2\pi f\left(\frac{1}{N}\right) \sim \omega$

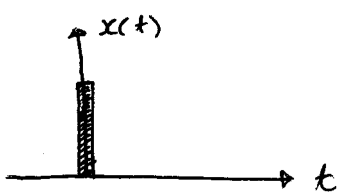
$\Omega = \omega T$

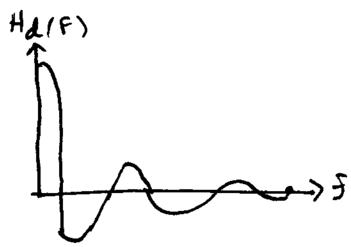
- ZT (z-transform)

$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}$; where $z = e^{j\Omega} = e^{j\omega T} = e^{sT}$

$z = e^{sT}$

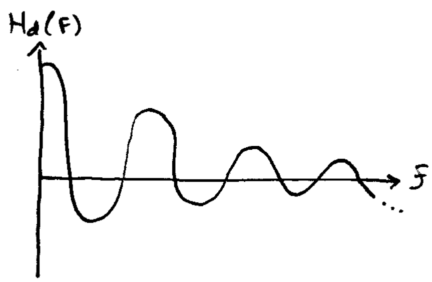
- Input impulse





Finite number of steps

FIR: Finite impulse response fil



infinite number of steps

IIR : infinite input response

2) Design of IIR Filters

- analog Filter
- digitization , $H_d(z)$

$z = e^{sT}$

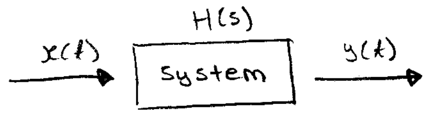
$\ln z = sT$

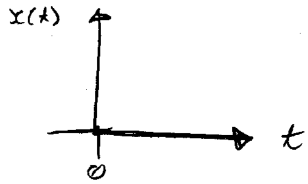
$s = 1/T \ln(z)$

SIGNALS

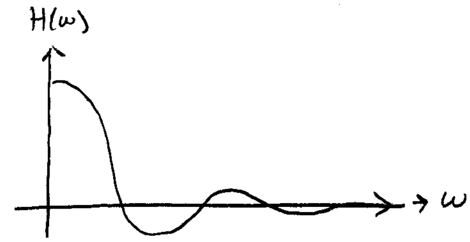
1

Nov. 15/19

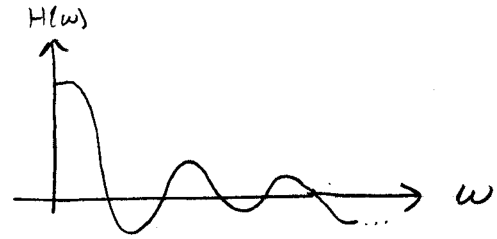




FIR



IIR



Design of IIR Filters

- analog filter prototype
  $H(s)$, $H(\omega)$
- transform analog prototype
  $\longrightarrow$ digital filter : $H_d(z)$

$s = 1/T \ln(z)$

T = time data sample interval $1/f_s$

Taylor : $\ln z = 2\left(z + z^3/3 + z^5/5 + \cdots\right)$

Bilinear transformation

$S = 1/T \ln(z) \approx (2/T)(z-1/z+1)$

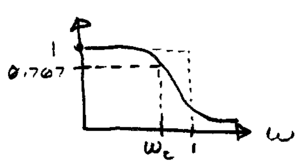
$H_d(z) = H(z) = H\left(\frac{2}{T}\frac{z-1}{z+1}\right)$

$H(s) \sim h(t)$ (approximate)

3) Warping Errors

$H_d(z)$

LPF with cutoff freq. ω_c



Digital Filter

$\Omega_c = \omega_c T$

$H_d(z)$ has cutoff freq.

$\Omega_c = 2\tan^{-1}(\omega_c T/2) \neq \omega_c T$ $\sim$ warping error

approx. bilinear transformation

gay

pre-warping

$\Omega_c = 2\tan^{-1}(\omega_c T/2)$

$\Omega_c/2 = \tan^{-1}(\omega_c T/2)$

$\tan(\Omega_c/2) = (\omega_c T/2)$

$\omega_c = (2/T)\tan(\Omega_c/2)$

cut-off freq. of analog prototype

$\omega_p - (2/T)\tan(\Omega_c/2) = (2/T)\tan(\omega_c T/2)$

to replace $\omega_c \rightsquigarrow \Omega_c = \omega_c T$

$\boxed{\text{Example 4.8}}$ Consider the two-pole Butterworth Filter with transfer Function:

$$H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Filter with $\omega_c = 2$, and $T = 0.2$

$\longrightarrow f_s = 1/T = 5$ Hz

digital Filter

$$H_d(z) = H(s)\big|_{s=(2/T)(z-1/z+1)}$$

$$= \frac{\omega_c^2}{\left((2/T)(z-1/z+1)\right)^2 + \sqrt{2}\omega_c\left((2/T)\left(\frac{z-1}{z+1}\right)\right) + \omega_c^2}$$

$$= \frac{0.0302z^2 + 0.0605z + 0.0302}{z^2 - 1.4514z + 0.5724}$$

$\Omega_c = (2/T)\tan^{-1}(\omega_c T/2) = 0.3948$

Desired : $\Omega_c = \omega_c T = 2 \times 0.2 = 0.4$

$\omega_p = (2/T)\tan(\Omega_c/2) = (2/0.2)\tan(0.4/2)$

$= 2.027$ rad/s

$\omega_p \longrightarrow \omega_c$ (prewarping)

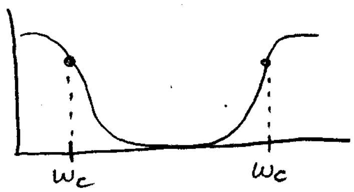
$$H_d = \frac{0.0309(z^2 + 2z + 1)}{z^2 - 1.444z + 0.5682}$$

$\Omega_c = \omega_c T = 0.4$

Solution $\quad H_d(z) = \Omega \quad (-\pi \sim \pi)$

Design of IIR Filters in MATLAB

- bilinear

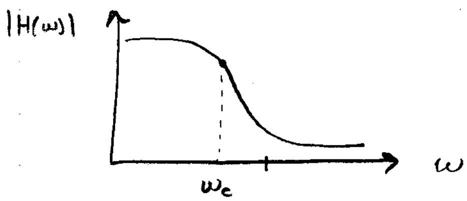


- butter (prototype, Freq. transformation, pre-warping)
- cheby ($\Omega_c = \omega_c T/\pi$
- Filter

Solution

$x(t) = 1 + \cos t + \cos(5t)$

$\omega = 1$ $\qquad \omega = 5$



$1 < \omega_c < 5$

$\boxed{\text{Example}}$ two-pole butterworth w/ transfer Fxn:

$$H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\,\omega_c s + \omega_c^2} \quad ; \quad \text{with} \quad \omega_c = 2 \text{ rad/s}, \quad T = 0.2$$

→ If $\omega_c = 2$ rad/s

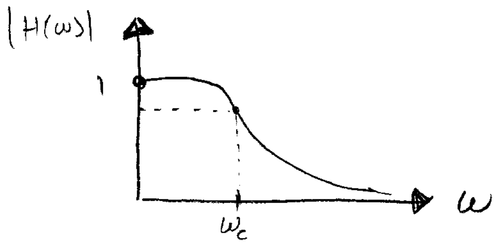
↳ $\omega_p = 2.027$ rad/s

$$H_d(z) = \frac{0.0309\,(z^2 + 2z + 1)}{z^2 - 1.444z + 0.5682}$$

$\boxed{\text{Example}}$

$$x = 1 + \underbrace{\cos(t)}_{\substack{\omega = 1 \text{ (rad/s)} \\ f = 1/2\pi \text{ (Hz)}}} + \underbrace{\cos(5t)}_{\substack{\omega = 5 \text{ (rad/s)} \\ f = 5/2\pi \text{ (Hz)}}}$$

Remove $\cos(5t)$ using 2-pole Butterworth



Choose $\omega_c = 2$ rad/s

$f_{max} = 5/2\pi$

$f_s = 2 f_{max} = 5/\pi \approx 1.59$ Hz

$f_s = 5$ Hz, $T = 1/f_s = 0.2$

In MATLAB; Filter

butter

5.4 Fault Detection in Rolling Element Bearing

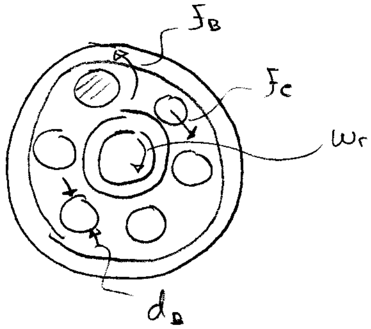
Bearings → main cause of defects in rotating machinery

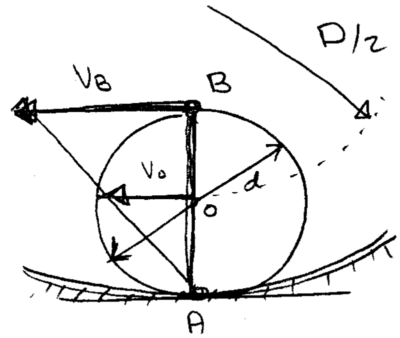
Small / medium size machines (75 %)

Types of defects:

- distributed defects (wear, ...)
- localized

bearing materials are subjected to dynamic loading





$$V_B = \omega_r\left(\frac{D}{2} - d/2\right)$$

$$V_o = \frac{V_B}{2} = \frac{\omega_r}{2}\left(\frac{D}{2} - \frac{d}{2}\right)$$

cage speed

$$\text{Cage Frequency} = \frac{V_o}{D/2} = \frac{\omega_r}{D}\left(\frac{D}{d} - \frac{d}{2}\right)$$

Cage Freq.

$$f_c = \frac{f_r}{D}\left(\frac{D}{2} - \frac{d}{2}\right) = \frac{f_r}{D}\cdot\frac{D}{2}\left(1 - \frac{d}{D}\right)$$

Ball rotating freq.

$$\omega_B = \frac{V_B}{d} = \frac{\omega_r}{\cancel{2}d}\left(\frac{D}{2} - \frac{d}{2}\right)$$

$$f_b = \frac{f_r}{\cancel{2}d}\left(\frac{D}{2} - \frac{d}{2}\right)$$

Cage $\quad f_c = \dfrac{f_r}{2}\left(1 - d/D \cos\alpha\right)$

Ball (rotating) $\quad f_b = \dfrac{f_r D}{2d}\left(1 - \dfrac{d^2}{D^2}\cos^2\alpha\right)$

Outer race: $\quad f_{od} = \dfrac{Z f_r D}{2d}\left(1 - \dfrac{d}{D}\cos\alpha\right)$

Inner race : $\quad f_{id} = \dfrac{Z f}{2}\left(1 + \dfrac{d}{D}\cos\alpha\right)$

Inner race defect :

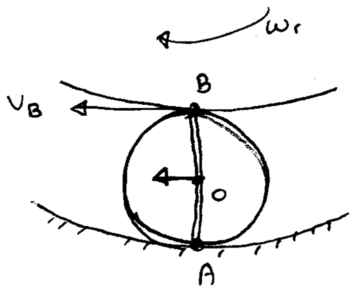
$\omega_{id} = Z(\omega_r - \omega_c)$

$f_{id} = Z(f_r - f_c)$

Rolling element damage

$\omega_{rd} = 2\omega_b$

$f_{rd} = 2f_b = \dfrac{f_r D}{d}\left(1 - \dfrac{d^2}{D^2}\cos^2\alpha\right)$



$$V_B = \omega_r\left(\frac{D}{2} - \frac{d}{2}\right)$$

$$V_o = \frac{V_B}{2} = \frac{\omega_r}{2}\left(\frac{D}{d} - \frac{d}{2}\right)$$

$$\omega_c = \frac{V_o}{D/2} = \frac{\omega_r}{2}\left(1 - \frac{d}{D}\right)$$

Cage: $$f_c = \frac{f_r}{2}\left(1 - \frac{d}{D}\cos\alpha\right)$$

Rolling Element: $$f_b = \frac{D f_r}{2d}\left(1 - \frac{d^2}{D^2}\cos^2\alpha\right)$$

Outer race: $$\omega_{od} = Z\omega_c$$

$$f_{od} = \frac{Z f_r}{2}\left(1 - \frac{d}{D}\cos\alpha\right)$$

Inner race defect: $$\omega_{id} = Z(\omega_r - \omega_c)$$

$$= Z\left[\omega_r - \frac{\omega_r}{2}\left(1 - \frac{d}{D}\cos\alpha\right)\right]$$

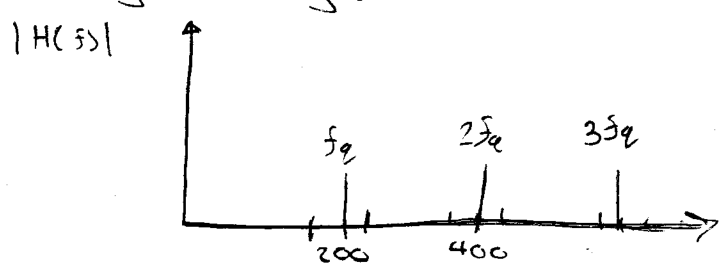
$$= \frac{Z\omega_r}{2}\left[2 - 1 + \frac{d}{D}\cos\alpha\right]$$

$$f_{id} = \frac{Z f_r}{2}\left(1 + \frac{d}{D}\cos\alpha\right)$$

Rolling element $\omega_{od} = 2\omega_b$

$$f_{od} = 2 f_b = \frac{D f_r}{d}\left(1 - \frac{d^2}{D^2}\cos^2\alpha\right)$$

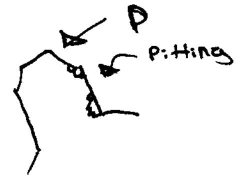
Healthy Bearing: f_r



$f_{od} = 30$ Hz

5.3 Gear System Monitoring

1) Damage :



dynamic loading : fatigue

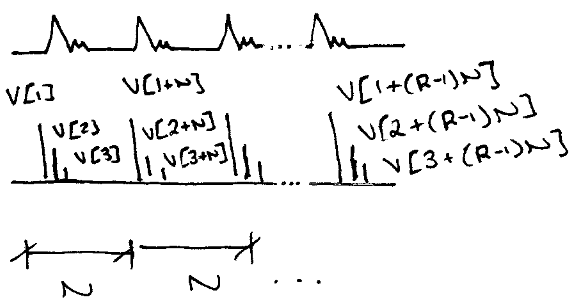
contact force : pitting

tensile : breakage

dynamics , stiffness

Gear signal is periodic

2) Time syncronous filtering



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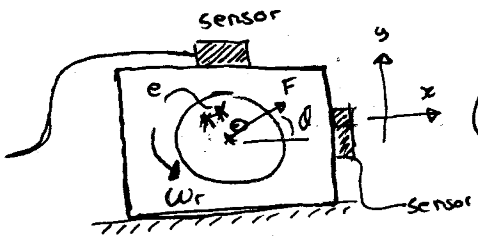
Nov. 25/19



For a rotor :

((center of mass (should be the same)
center of rotation

but in reality, that's not how it works.

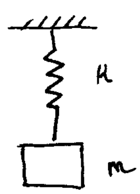
Mass = m

centrifugal force $\Rightarrow$ $F = m\omega_r^2$

$\theta = \omega_r t$

$y = e\sin\theta = e\sin(\omega_r t)$

mass - spring



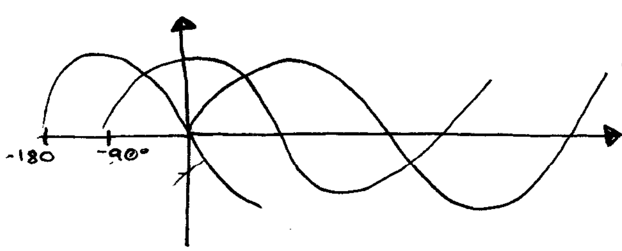
$\zeta = 1/\sqrt{2} = 0.707$

Natural Freq: $\omega_n = \sqrt{K/m}$

1 displacement : $y = e\sin(\omega_r t)$

2 velocity : $V = \dot{y} = \dfrac{dy}{dt} = e\omega_r\cos(\omega_r t)$

$= e\omega_r\sin(\omega_r t + \pi/2)$



$y = e\sin(\omega_r t)$

$y'(t) = \lim\limits_{\Delta t \to 0} \dfrac{y(t+\Delta t) - y(t)}{\Delta t}$

non-causal

3 acceleration : $a = \dfrac{dv}{dt} = e\omega_r^2\cos(\omega_r t + \pi/2)$

$= e\omega_r^2\sin(\omega_r t + \pi)$

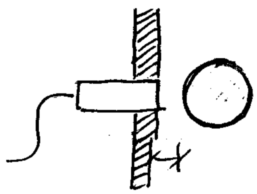
- Impedence

$\omega_n = \sqrt{K/m}$ harmonies

- Sensors measure response to the vibrating Forces.
- Oil Film

Displacement transducers (sensors)

magnetic disp. sensor



optical sensor

output ~ distance

- velocity transducers

  output ~ velocity of the vibrations

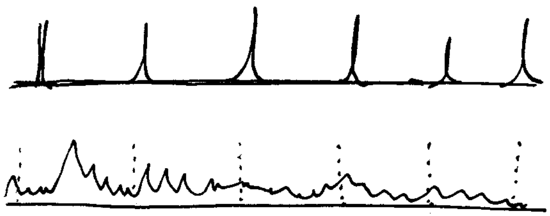
- accelerometer

  output ~ acceleration

Gear signal vibration is periodic

T.S.A.

Signal average



Gear ratio = 1.2

Interpolation

GR = 2

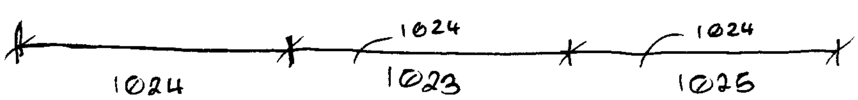
$Z_1 = 17$ 19, 21, 23 ...

→ The number of revolutions, R, can be determined according to noise reduction requirements.

→ due to shaft speed variations, the number of samples may not be equal from one revolution to another. Interpolation should be carried out to resample the data revolution by revolution.

$f_s = 20000\ Hz$

20000 samples / sec



Interpolation

→ resample, m

5.3 Amplitude and Phase Demodulation

Z_1 = # of teeth

f_r = rotation speed

Mesh Freq.

$f_m = kZ_1 f_1$; where $k = 1, 2, 3, \ldots$

$$x[n] = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) + \dots + A_K\cos(\omega_K t + \phi_K)$$
$$= \sum_{k=1}^{K} A_k \cos(\omega_k t + \phi_k)$$

A_k = amplitude corresponding to the k^{th} mesh freq.

ϕ_k = phase

$\omega_k = 2\pi f_k = 2\pi k \times z_1 f_1$

Signal average :

$$x[n] = \sum_{k=1}^{K} A_k \cos(2\pi k z_1 f_1 t + \phi_k) \qquad n = 1, 2, 3, \dots, N-1$$

$$x[n] = \sum_{k=1}^{K} (A_k + a_k)\cos(2\pi k z_1 f_1 t + \overbrace{a_k + \phi_k}^{\alpha_k})$$

Amplitude demodulation :

Phase ———

- Analytical signal

  Hilbert transform

  Add an imaginary part

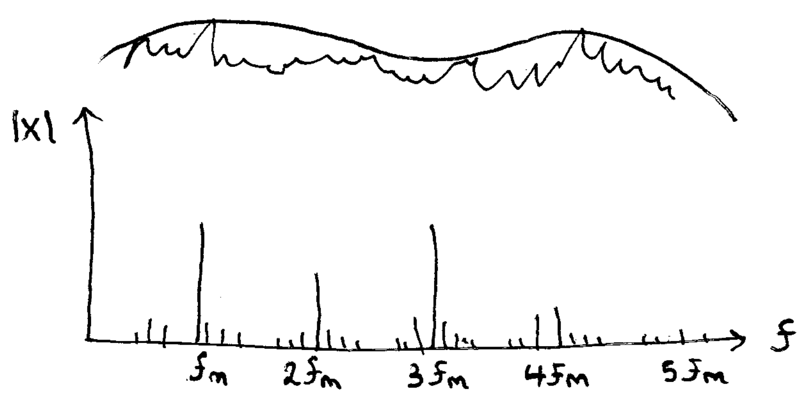
$$\underbrace{y[n]}_{\text{complex valued signal}} = \underbrace{x[n]}_{\text{real}} - \underbrace{jH(x[n])}_{\text{imag. part}}$$

$$y_{amp}[n] = \sqrt{Re^2[y[n]] + Im^2[y[n]]}$$

Phase : $y_{phase}[n] = \arctan\left(\dfrac{Im(y[n])}{Re(y[n])}\right)$

amplitude modulation

$\longrightarrow$ envelope



- Overall residual signal

multiple bandstop filter to remove gear mesh Freq. & it's harmonics

$X[n]$ ~ signal average (TSA)

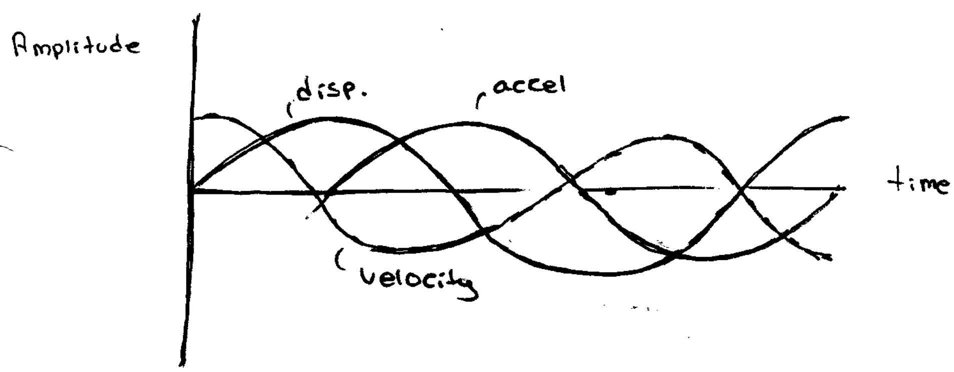
Analytical Signal :

$y_0[n] = x_0[n] - jH(x_0[n])$

amplitude modulation :

$y_{0,amp} = \sqrt{Re^2[y_0] + Im^2[y_0]}$

$y_{0,phase} = \arctan\left(\frac{Im[y_0]}{Re[y_0]}\right)$



$V = 90°(d)$

$a \sim 90°(v)$

$= \sim 180°(d)$

Displacement Transducers

a) Displacement transducers measure relative motion between the shaft and the output

b) Typically, the actual useful frequency range of proximity probes is up to 500 Hz

c) Shaft surface scratches, out-of-roundness, and variation in electrical properties will produce signal errors. Surface treatment and run-out subtraction can be used to solve these problems.

disp. ~ 500 Hz

Velocity (~ 2000 Hz)

accelerometer

$$\omega_n = \sqrt{k/m}$$

Piezoelectric accelerometer

charge $\longrightarrow$ $F = ma$

Selecting the right transducer for an application:

a) The parameter of interest
- disp x
- velocity $v = \dot{x}$
- acceleration $a = \dot{v} = \ddot{x}$

b) Mechanical impedence considerations

c) Frequency considerations.

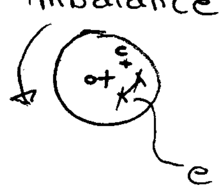
If the freq. of the vibration is $> 1000\,Hz$, you must use accelerometer. If $10 \to 1000\,Hz$, either velocity or acceleration transducers can be used.

Bearings & Gears

↳ most common source of defects (rolling element)

5.5 Fault Diagnosis in other Machinery Systems

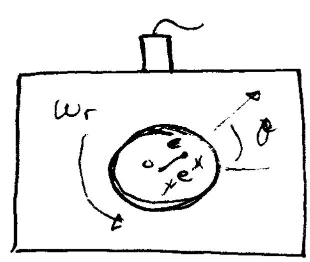
i) Imbalance



O : centre of rotation

c : centre of mass

e : difference between the two.



$F = m\omega_r^2 r$

$F_y = F\sin\theta = m\omega_r^2 \sin(\omega_r t)$

once per revolution

Characteristics of imbalance :

→ it is Sinusoidal at a Freq. of once per revolution

→ it is a rotating vector

→ its amplitude increases with speed because the imbalance force

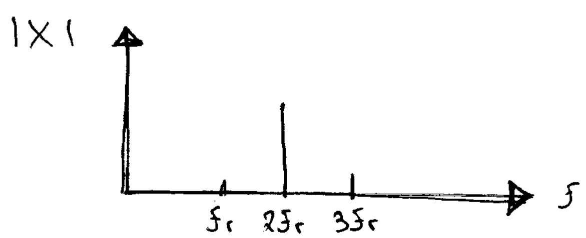
→ phase plays a key role in detecting & analyzing imbalance

2) Misalignment

Characteristics of Misalignment

→ It is usually characterized by a $2 \times f_r$ Frequency component, with large number of harmonics.

→ It has high axial vibration levels



→ the ratio of $1 \times f_r$ to $2 \times f_r$ component levels can be used as an indicator of misalignment severity.

3) Resonance

Force Frequencies or it's harmonics

≈ structure natural freq.

Natural Freq. in machinery vibration analysis

1) resonance of the structure can cause changes in vibration level

2) the dynamics of rotating shafts change significantly near natural Freq.

3) resonance of transducers will limit the operating frequency range of measurement.