2.4 Hooke's Law for a 2D Unidirectional Lamina

1. Plane stress

$\sigma_3 = \tau_{23} = \tau_{31} = 0$

$\gamma_{23} = \gamma_{31} = 0$

However, $\varepsilon \neq 0$ (See Eq. 2.76)

2. [C] and [S] for plane stress situation

$[C]_{6x6} \rightarrow [Q]_{3x3}$ Reduced stiffness/compliance matrix

$[S]_{6x6} \rightarrow [S]_{3x3}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad (Eqn. 2.78 *)$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \quad (Eqn. 2.77 *)$$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q_{22} & 0 \\ sym. & & Q_{66} \end{bmatrix} \quad (Eqn. 2.78)$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ sym. & & Q_{66} \end{bmatrix} \quad (Eqn. 2.77)$$

3. Q_{ij} $(2.93\ a{\sim}d)$– in terms of $E_1\ E_2\ G_{12}\ and\ v_{12}$

$S_{ij}(2.92\ a{\sim}d)$ – in terms of $E_1\ E_2\ G_{12}\ and\ v_{12}$

$i, j = 1, 2, 6$

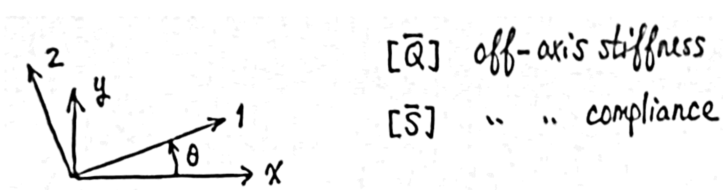
2.5 Hooke's Law for a 2D Unidirectional Angle Lamina (off-axis stiffness and compliance)

In 2.4, we have:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad or \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

[Q], [S]: used with 1-2-3 coordinates, or local coordinates.

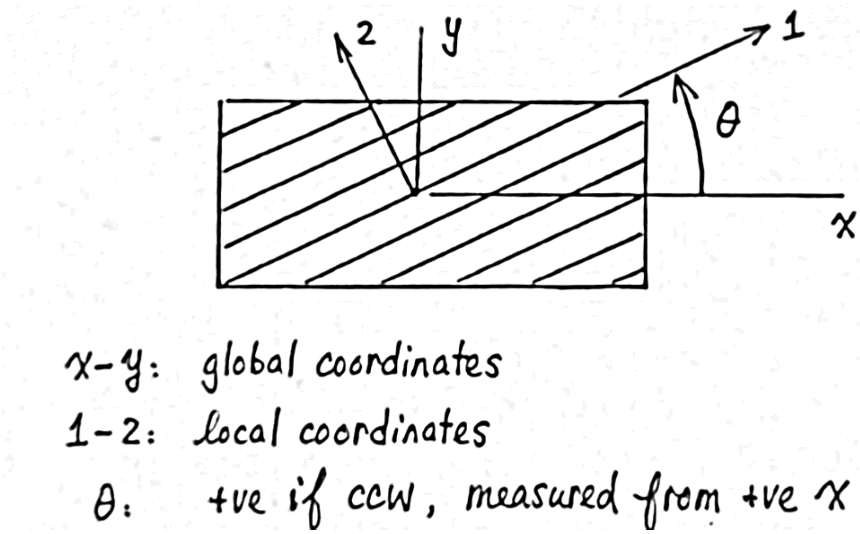
For application, more than 1 lamina will be used; and the laminas are typically placed at various angles, hence angle lamina.



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad or \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\bar{S}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$[Q] \rightarrow [\bar{Q}]? \quad [S] \rightarrow [\bar{S}]?$$

Step 1: Transformation Matrix [T]



Global stresses: $\sigma_x, \sigma_y, \tau_{xy}$
Local stresses: $\sigma_1, \sigma_2, \sigma_6 = \tau_{12}$

Define $c = cos\theta,\ \ s = sin\theta$
Then:

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}$$

$[T]$ is orthogonal, i.e., $[T]^{-1} = [T(-\theta)]$

$$\therefore \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$or \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

Where:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T^{-1}][Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

Step 2: [T] is applicable to tensorial strains, i.e.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\varepsilon_6 \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

Step 3: Reuter's matrix

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\varepsilon_6 \end{Bmatrix}$$

$$= [R][T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

$$= [R][T][R^{-1}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Step 4: Subs into (A)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Where:
$[\bar{Q}] = [T]^{-1}[Q][R][T][R]^{-1}$

And it can be shown that:
$[\bar{S}] = [R][T]^{-1}[R]^{-1}[S][T]$
$[\bar{S}] = [\bar{Q}]^{-1}$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q_{22} & 0 \\ sym. & & Q_{66} \end{bmatrix}$$

$[\bar{Q}]$ = full matrix symmetric ($Eqn. 2.104\ a{\sim}f$)

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ sym. & & S_{66} \end{bmatrix}$$

$[\bar{S}]$ = full matrix symmetric ($Eqn. 2.104\ a{\sim}f$)

2.6 Engineering Constants of an Angle Lamina

Elastic moduli in the x and y directions:

$$E_x = 1/\overline{S_{11}}$$
$$E_y = 1/\overline{S_{22}}$$

Shear modulus in the $x-y$ plane:

$$G_{xy} = 1/\overline{S_{66}}$$

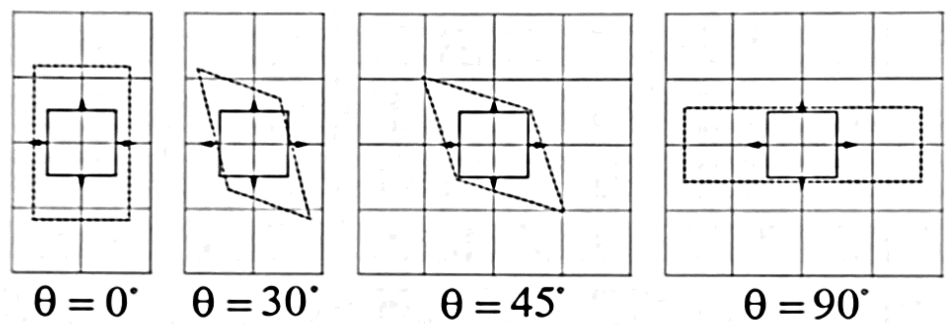
Poisson's ratios:

$$v_{xy} = -\overline{S_{12}}/\overline{S_{11}}$$
$$v_{yx} = -\overline{S_{12}}/\overline{S_{22}}$$

Shear coupling factors:
Unlike isotropic materials, an angle lamina may develop shear strains when subject to normal stresses; or when being stretched/compressed, shear stress may be developed.

$m_x = -\overline{S_{16}} \cdot E_1$ (non-dimensional quantity)
relating ε_x to τ_{xy}, or σ_x to γ_{xy}

$m_y = -\overline{S_{26}} \cdot E_1$ (non-dimensional quantity)
relating ε_x to τ_{xy}, or σ_y to γ_{xy}



No shear strain (deformation) if angle is 0° or 90°

2.8 Strength Failure Theories of an Angle Lamina
7 sub-sections

Overview:
2.8.7 - Comparing theories, and with experimental data

2.8.2 – Strength ratio
2.8.3 – Failure envelopes

2.8.1 – 4 theories (Note: τ_{12} and σ_6 are interchangeable, so are γ_{12} and ε_6)
2.8.4
2.8.5
2.8.6

Overview on Strength Theories

A) Purpose of strength theories

Similar to isotropic materials such as metals, strength theories are to allow for determination of when failure occurs if a component is in 2- or 3- dimensional state of stress.

B) Strength theories available to isotropic materials

Ductile materials

- Max. shear stress theory
- Distortion energy theory (von Mises theory)

Brittle materials

- Max. normal stress theory
- Coulomb-Mohr theory
- Modified Coulomb-Mohr theory

C) Challenges when dealing with unidirectional laminas

- They are direction/orientation dependent
- Tensile and compressive strengths are different in both the longitudinal and transverse directions; e.g., $(\sigma_1^T)_{ult} > \left(\sigma_1^C\right)_{ult}$, but $(\sigma_2^T)_{ult} < \left(\sigma_2^C\right)_{ult}$
- They retain part of ductile behavior; at the same time , they retain part of brittle behavior

D) List of strength theories

Unidirectional Laminas	Isotropic Materials
Max. stress	Max. normal stress
Max. strain	Max. normal strain
Tsai-Hill	Distortion energy
Tsai-Wu (Quadratic)	Total strain energy

2.8.7 Comparison of Experimental Results with Failure Theories

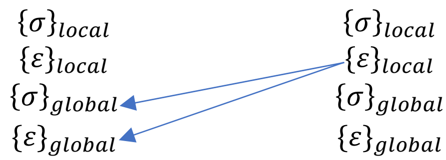
1) Max. stress & max strain theories don't compare well with experimental results.

2) Tsai-Hill and Tsai-Wu theories don't compare well with experimental results.

3) Tsai-Hill or Tsai-Wu theory, however, doesn't indicate the specific mode of failure, which max. stress and max. strain theories do.

Failure Mode	Shorthand Notation
1: Tensile failure in longitudinal direction (or fiber direction)	1T
2: Compressive failure in longitudinal direction	1C
3: Tensile failure in transverse failure	2T
4: Compressive failure in the transverse direction	2C
5: in-plane shear	6S

Transformation between local (1-2-3) axes and global (x-y-z) axes:



Example 1:

A unidirectional graphite/epoxy lamina ($\theta = 50°$) is subject to $\sigma_x = 0$, $\sigma_y = -3\ MPa$, $\tau_{xy} = 4\ MPa$. Find the local stresses and local strains. Given, for the lamina, $E_1 = 181\ GPa$, $E_2 = 10.3\ GPa$, $v_{12} = 0.28$, and $G_{12} = 7.2\ GPa$.

Solution:

Global stresses → $(via\ [T])$ Local stresses → $(via\ [S])$ Local strains

$$[T] = \begin{bmatrix} 0.4132 & 0.5868 & 0.9848 \\ 0.5868 & 0.4131 & -0.9848 \\ -0.4924 & 0.4924 & -0.1736 \end{bmatrix}$$

$$[S] = \begin{bmatrix} 0.5525 & -0.1547 & 0 \\ & 9.709 & 0 \\ sym. & & 13.89 \end{bmatrix} (10^{-9}) \left(\frac{1}{Pa}\right)$$

Local stress $= [T]$ ∗global stress $= \begin{Bmatrix} 2.179 \\ -5.179 \\ -2.172 \end{Bmatrix} (MPa)$

Local strain $= [S]$ ∗global stress $= \begin{Bmatrix} -0.0200 \\ -0.505 \\ -0.302 \end{Bmatrix} (10^{-3})$

Example 2:

A unidirectional graphite/epoxy lamina ($\theta = 50°$) is subject to $\sigma_x = \sigma_1$, $\sigma_y = -\sigma$, and $\tau_{xy} = 0$ (where σ is in Pa). Find the local stresses and local strains in terms of σ. Given, for the lamina, $E_1 = 181\ GPa$, $E_2 = 10.3\ GPa$, $v_{12} = 0.28$, $G_{12} = 7.2\ GPa$.

Solution:

Global stresses → $(via\ [T])$ Local stresses → $(via\ [S])$ Local strains

Local stress $= [T]$ ∗global stress

$$[T]\begin{Bmatrix} \sigma \\ -\sigma \\ 0 \end{Bmatrix} = \sigma \begin{Bmatrix} -0.1736 \\ 0.1736 \\ -0.9848 \end{Bmatrix}$$

Local strain $= [S]$ ∗local stress

$$\sigma[S]\begin{Bmatrix} -0.1736 \\ 0.1736 \\ -0.9848 \end{Bmatrix} = \sigma \begin{Bmatrix} -0.001228 \\ 0.01713 \\ -0.1368 \end{Bmatrix} (10^{-9})$$