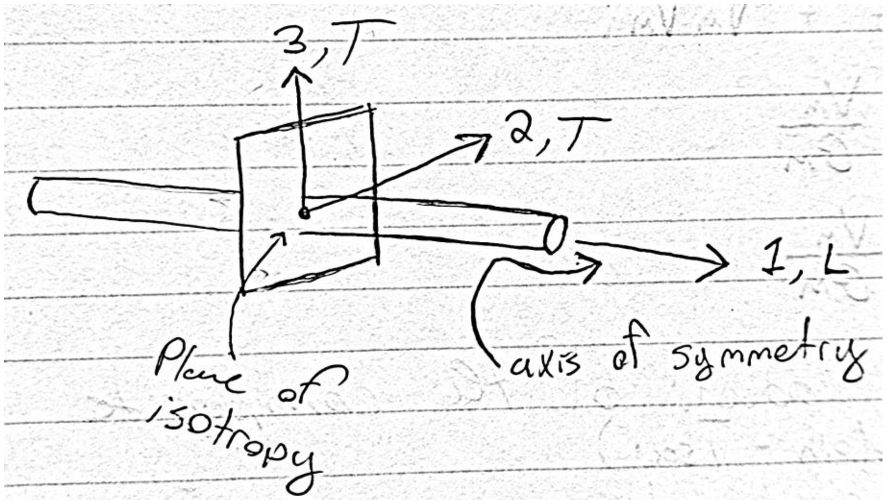
3.3.4 Transversely Isotropic Fibers

- Glass fibers are isotropic
- Carbon/graphite and aramid fibers are transversely isotropic



1) Elastic modulus for the fibers

E_{fL}: Young's Modulus in the Longitudinal direction

E_{fT}: Young's Modulus in the Transverse direction

v_{fL}: Major Poisson's Ratio (that is v_{fLT})

v_{fT}: Minor Poisson's Ratio (or v_{fTL})

G_{fT}: Shear modulus in the L-T plane

2) Elastic moduli of the composite (Mechanics of Materials Approach)

$$E_1 = E_{fL}V_f + E_mV_m$$

$$v_{12} = v_{fL}V_f + v_mV_m$$

$$\frac{1}{E_2} = \frac{V_f}{E_{fT}} + \frac{V_m}{G_m}$$

$$\frac{1}{G_{12}} = \frac{V_f}{G_{f1}} + \frac{V_m}{G_m}$$

3) Elastic moduli of the composite (Halpin-Tsai Method)

E_2: $E_f \leftarrow E_{fT}$

G_{12}: $G_f \leftarrow G_{fT}$

4) Elastic moduli of the composite (Elasticity approach)

E_1: $E_f \leftarrow E_{fL}$; $v_f \leftarrow v_{fL}$

v_{12}: $v_f \leftarrow v_{fL}$; $E_f \leftarrow E_{fL}$

G_{12}: $G_f \leftarrow G_{fT}$

E_2: $v_f \leftarrow v_{fT}$

$G_f \leftarrow G_{fTT}$

$E_f \leftarrow E_{fT}$

G_{fTT} fibers shear modulus in the T-T plane

$$G_{fTT} = \frac{E_{fT}}{2(1 + v_{fT})}$$

Example: Find E_1, E_2, G_{12}, and v_{12} by the three approaches.

Graphite fibers:
$V_f = 0.6$
$E_{fL} = 345\ GPa$
$E_{fT} = 9.66\ GPa$
$G_{fT} = 2.07\ GPa$
$v_{fL} = 0.2$

Epoxy fibers:
$V_m = 0.4$
$E_m = 3.45\ GPa$
$G_m = 1.28\ GPa$
$v_m = 0.35$

Solution:
$v_{fT} = 0.0056$
$G_{fTT} = 4.80\ GPa$

Mechanics of Materials method:
$E_1 = 208.38\ GPa$
$E_2 = 5.6163\ GPa$
$G_{12} = 1.6602\ GPa$
$v_{12} = 0.26000$

Halpin-Tsai method:
$E_2 = 6.4988\ GPa$
$\rho = 2.2419$
$n = 0.35701$

$G_{12} = 1.7070\ GPa$
$\rho = 1.2419$
$n = 0.21587$

Elasticity method:
$E_1 = 208.42\ GPa$
$v_{12} = 0.25103$
$G_{12} = 1.7019\ GPa$

$A = -38.7795$
$B = 22.3345$
$C = 69.2433$

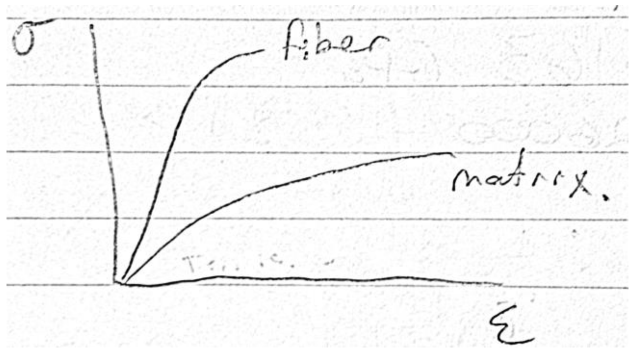
$G_{23} = 2.59971\ GPa$
$v_{23} = 0.275602$

$E_2 = 6.6324\ GPa$

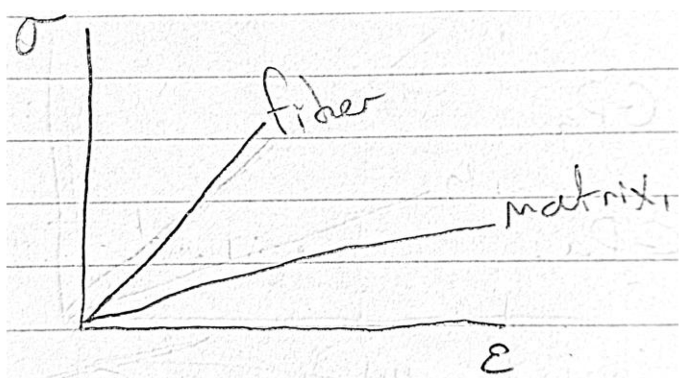
3.4 Ultimate Strength of a Unidirectional Lamina

It's observed that,
1) Fibers behave like ductile materials, but matrix behaves like brittle material

2) Both are not linearly elastic



Within the section, it's assumed that $\sigma - \varepsilon$ plots for both fiber and matrix are linear up to failure (by breakage or by fracture)



What we need before determining ultimate strengths.
Fibers: V_f or V_f'
$(\sigma_f)_{ult}$: strength of fibers in tension and compression
$(\tau_f)_{ult}$: ultimate shear strength of fibers

Matrix: V_m or V_m'
$(\sigma_m)_{ult}$ or $(\sigma_m^T)_{ult}$: ultimate strength of matrix in tension
$(\sigma_m^C)_{ult}$: ultimate strength of matrix under compression
$(\tau_m)_{ult}$: ultimate shear strength of matrix

Composite: E_1, E_2, v_{12}, G_{12}
What we determine:
$(\sigma_1^T)_{ult}$: Ultimate longitudinal tensile strength
$(\sigma_1^C)_{ult}$: Ultimate longitudinal compressive strength
$(\tau_{12})_{ult}$: Ultimate shear strength

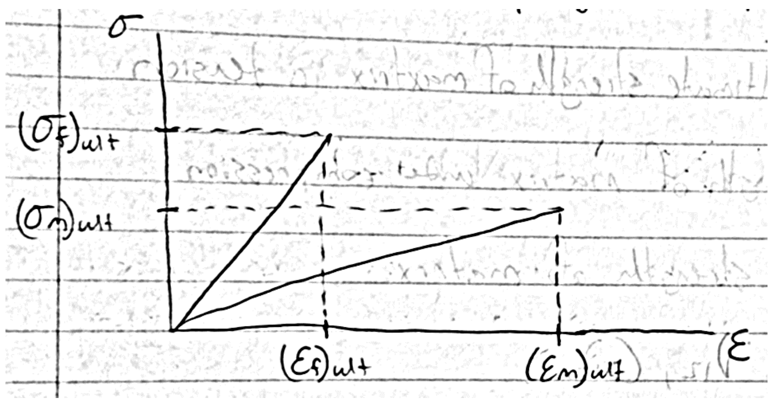
5 sub-sections to determine such ultimate strength

3.4.1 Longitudinal Tensile Strength $(\sigma_1^T)_{ult}$

There are 2 scenarios:
Fibers fail first; or matrix fails first

Fibers-fail-first:
Typically takes place for:

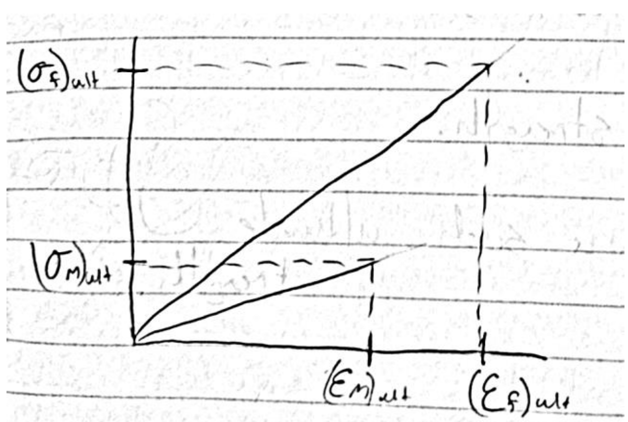
- MMCs
- Thermoplastic polymer composites



$(\varepsilon_f)_{ult} < (\varepsilon_m)_{ult}$ indicates that fibers fail first.

Matrix-fail-first (not in the textbook):
Typically takes place if:

- V_f is low
- PMC



$(\varepsilon_m)_{ult} < (\varepsilon_f)_{ult}$ indicates that matrix fail first.

∵ fibers and matrix taking the load together
∴ σ_1 follows ROM, similar to ROM on E_1
∴ $\sigma_1 = V_f\sigma_f + V_m\sigma_m$
$= V_f E_f \varepsilon + V_m E_m \varepsilon \quad (Eq. A)$

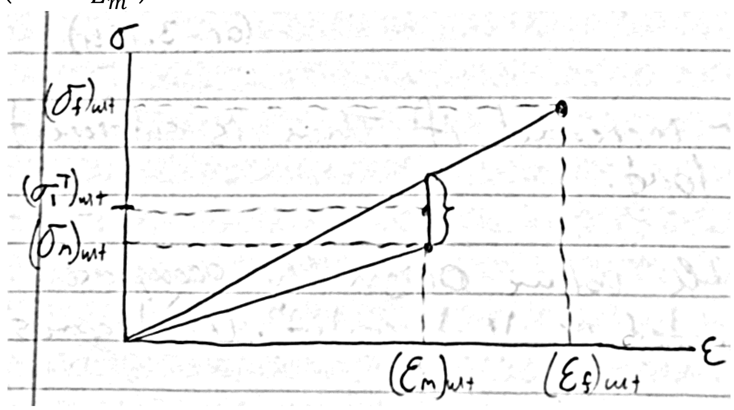
$\because$ matrix-fails-first $(\varepsilon_m)_{ult} < (\varepsilon_f)_{ult}$

$\therefore \varepsilon = (\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m}$

$$\therefore \sigma_1 = (\sigma_m)_{ult}\left(V_m + \frac{V_f E_f}{E_m}\right) \quad (Eq.A1)$$

The full load transfers to the fibers, but due to low V_f, fibers see a large jump in stress and fails immediately.

$$\therefore (\sigma_1^T)_{ult} = (\sigma_m)_{ult}\left(V_m + \frac{V_f E_f}{E_m}\right) \quad (Eq.B)$$



Fibers-fail-first

Fibers and matrix taking the load together

$$\therefore \sigma_1 = V_f E_f \varepsilon + V_m E_m \varepsilon \quad (Eq.A)$$

Load reaches the level that will break the fibers;

$\because (\varepsilon_f)_{ult} < (\varepsilon_m)_{ult}$

$$\therefore \varepsilon = (\varepsilon_f)_{ult} = \frac{(\sigma_f)_{ult}}{E_f}$$

$$\therefore \sigma_1 = (\sigma_f)_{ult}\left(V_f + \frac{V_m E_m}{E_f}\right) \quad (Eq.A2)$$

Load transfers to the matrix, causing increase in stress in the matrix, and fracture in matrix, leading to failure of composite.

$$\therefore (\sigma_1^T)_{ult} = (\sigma_f)_{ult}\left(V_f + \frac{V_m E_m}{E_f}\right) \quad (Eq.C)$$

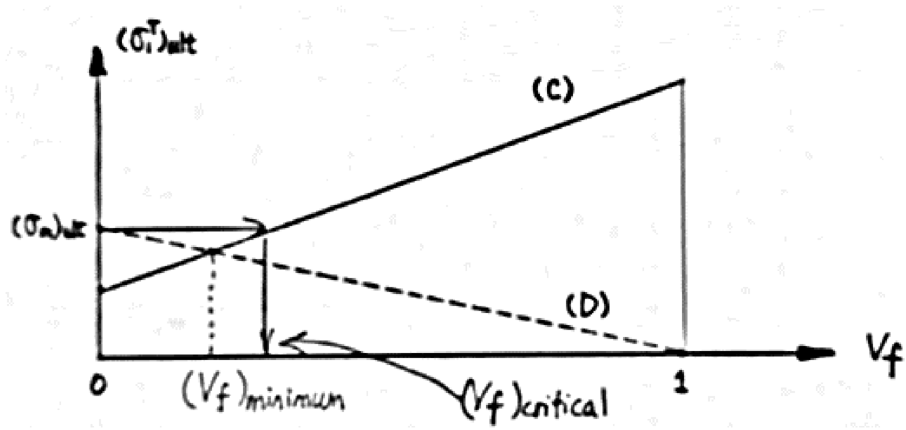
Loads can be further increased if there is sufficient matrix to take the load.

Once fibers break, the volume originally occupied by the fibers is regarded as void content. (Eq. A) becomes:

$\sigma_1 = V_m \sigma_m$

$(\sigma_1^T) = V_m (\sigma_m)_{ult} \quad (Eq.D)$

Now the question is which $(\sigma_1^T)_{ult}$ to use, (Eq. C) or (Eq. D)?



$$(V_f)_{minimum} = \frac{(\sigma_m)_{ult} - E_m \cdot \frac{\sigma_f}{E_f}}{(\sigma_f)_{ult}\left(1 - \frac{E_m}{E_f}\right) + (\sigma_m)_{ult}}$$

If $V_f < (V_f)_{minimum}$ use $(Eq. D)$
If $V_f \geq (V_f)_{minimum}$ use $(Eq. C)$

However it is practically impossible if $(\sigma_1^T)_{ult}$ by $(Eq. C)$ or $(Eq. D)$ is less than $(\sigma_m)_{ult}$

$$(V_f)_{critical} = \frac{(\sigma_m)_{ult} - E_m \cdot \frac{(\sigma_f)_{ult}}{E_f}}{(\sigma_f)_{ult} - (\sigma_f)_{ult}\frac{E_m}{E_f}}$$

If $V_f < (V_f)_{critical}$ $(\sigma_1^T)_{ult} = (\sigma_m)_{ult}$
If $V_f \geq (V_f)_{critical}$ use $(Eq. C)$

3.4.2 Longitudinal Compressive Strength $(\sigma_1^C)_{ult}$

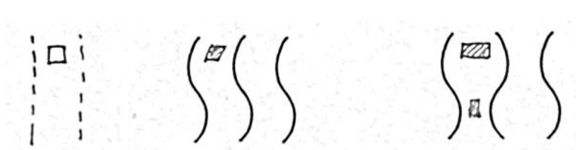
1. Modes of failure (3 or 4)

1) Tensile Failure:

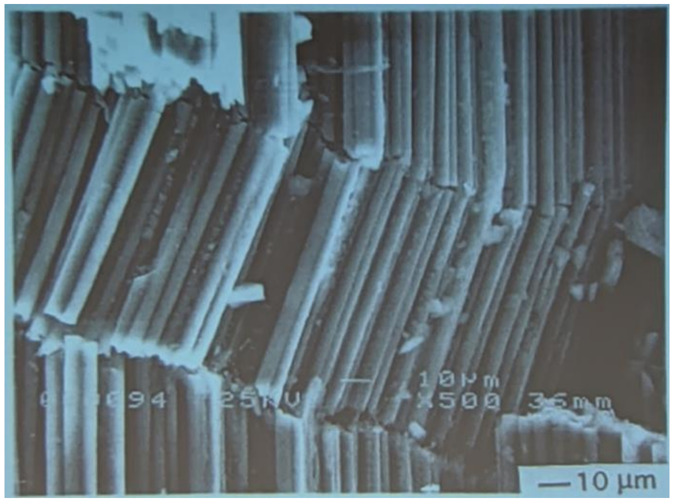
- Excessive tensile strain in the matrix
  - Matrix fractures or fiber-matrix bonding fractures
- Common with thermoset

2) Microbuckling: refers to fibers buckling "inside" the matrix

- Common with Kevlar
- Two possible microbucklng modes
  - In-phase or shear mode
  - Out-of-phase or extensional mode



3) Kinging/Shearing: refers to direct shear failure of fibers, or kinking of fibers if not sheared.



2. Tensile failure

$$(\sigma_1^c)_{ult} = \frac{E_1(\varepsilon_2^T)_{ult}}{\nu_{12}}. \qquad (3.169)$$

3. Microbuckling

Extensional mode:

$$S_1^c = 2\left[V_f + (1-V_f)\frac{E_m}{E_f}\right]\sqrt{\frac{V_f E_m E_f}{3(1-V_f)}}, \qquad (3.173a)$$

Shear mode:

$$S_2^c = \frac{G_m}{1-V_f}. \qquad (3.173b)$$

4. Shearing/kinking:

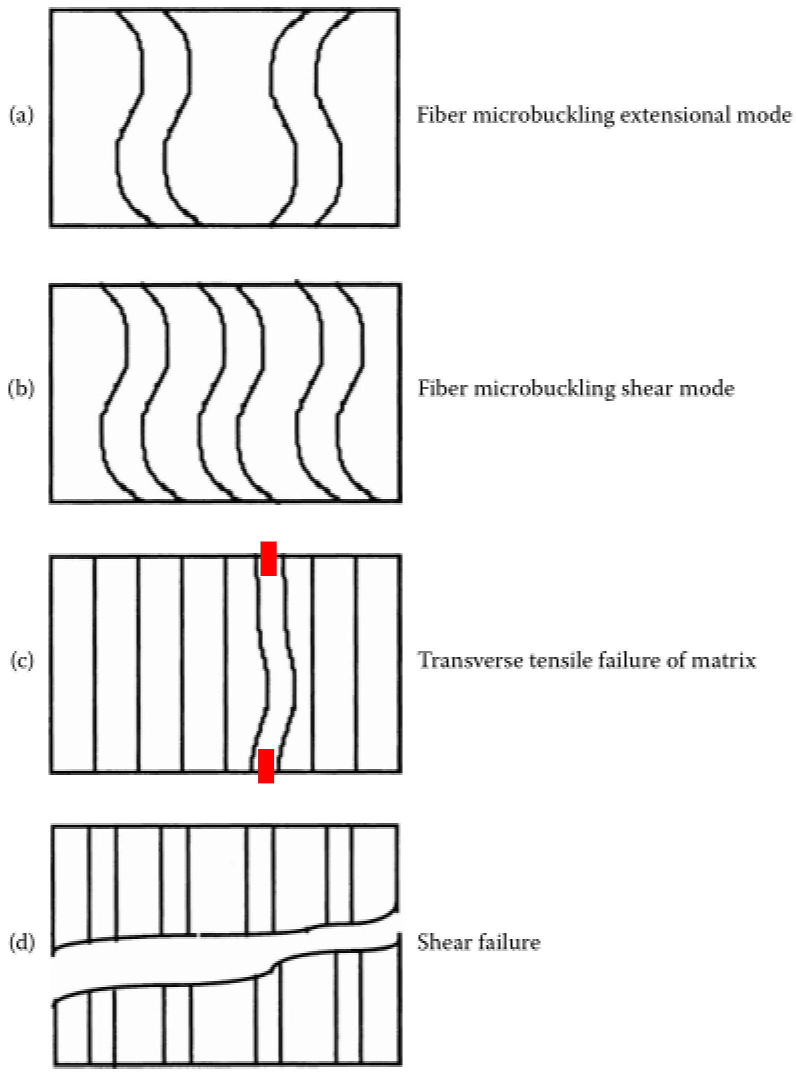
$$(\sigma_1^c)_{ult} = 2\left[(\tau_f)_{ult} V_f + (\tau_m)_{ult} V_m\right]. \qquad (3.175)$$

5. Conclusion:

4 values of $(\sigma_1^C)_{ult}$ by $(Eq. 3.169)$, $(Eq. 3.173a)$, $(Eq. 3.173b)$, and $(Eq. 3.175)$

Choose the smallest value.

Example 3.14 for detail.



3.4.3 Transverse Tensile Strength $(\sigma_2^T)_{ult}$

$$(\sigma_2^T)_{ult} = E_2(\varepsilon_2^T)_{ult}, \qquad (3.182)$$

Example 3.15 for detail.

3.4.4 Transverse Compressive Strength $(\sigma_2^C)_{ult}$

$$(\sigma_2^C)_{ult} = E_2(\varepsilon_2^C)_{ult}, \qquad (3.183)$$

Example 3.17 for detail.

3.4.5 In-Plane Shear Strength $(\tau_{12})_{ult}$

$$(\tau_{12})_{ult} = G_{12}(\gamma_{12})_{ult}$$

$$= G_{12}\left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_{mult}. \tag{3.191}$$

Example 3.17 for detail.

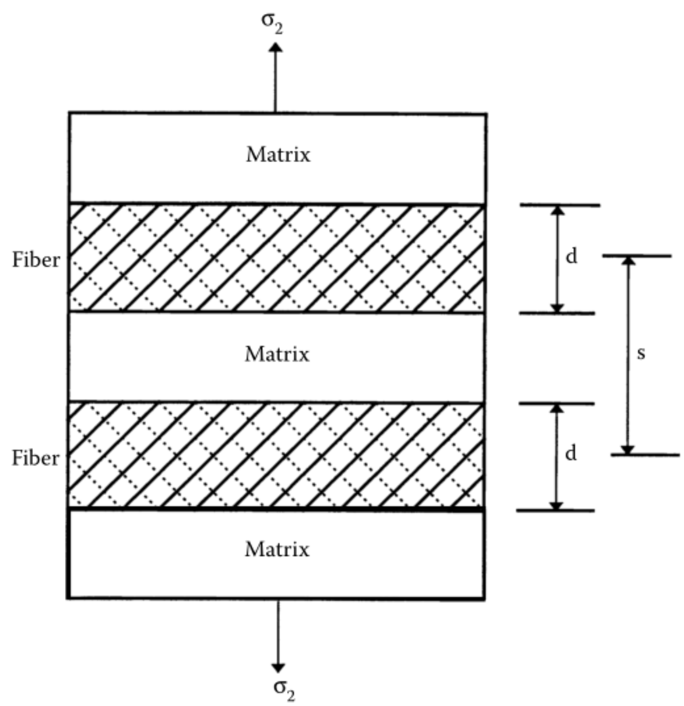
A few notes regarding equation in subsections 3.4.2 ~ 3.4.4:

1) S_1^C and S_2^C are by buckling analysis (or eigenvalue analysis)
2) All other equations are mostly results of doing simple mechanics of materials analyses; the RVEs are similar or identical to those used in 3.3.1
3) Where $(\varepsilon_2^T)_{ult}$ is needed, it should be the lesser of

- Empirical formula, $(Eq. 3.170)$
- Mechanics of materials approach $(Eq. 3.171)$

4) ROM is applied a few times.

$(\varepsilon_2^T)_{ult}$: mechanics of materials formula:



Transverse-direction deformations
Fiber: $\delta_f = \varepsilon_f \cdot d$
Matrix: $\delta_m = \varepsilon_m \cdot (s - d)$
Composite: $\delta_c = \varepsilon_c \cdot s$

But, $\delta_c = \delta_f + \delta_m$

$\therefore\ \varepsilon_c \cdot s = \varepsilon_f \cdot d + \varepsilon_m \cdot (s - d)$

$\varepsilon_c = \frac{d}{s}\varepsilon_f + \left(1 - \frac{d}{s}\right)\varepsilon_m$

$E_f \varepsilon_f = E_m \varepsilon_m\ (same\ stress\ in\ fibers\ and\ in\ matrix)$

$\therefore \varepsilon_c = \varepsilon_m \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right]$

If ε_m reaches $(\varepsilon_m^T)_{ult}$, ε_c reaches its ultimate value $(\varepsilon_2^T)_{ult}$; that is:

$(\varepsilon_2^T)_{ult} = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right](\varepsilon_m^T)_{ult}$

Where,

$(\varepsilon_m)_{ult}$ = ultimate tensile strain of the matrix

d = diameter of the fibers

s = center-to-center spacing between fibers

$\frac{d}{s}$ depends on packing and V_f

And, $(\varepsilon_2^T)_{ult}$ the empirical formula

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult}(1 - V_f^{1/3}), \quad (3.170)$$

State-of-the-art in terms of predicting or evaluating ultimate strengths

a) With the availability of new (and newer) materials, there are composites with combinations of brittle fibers and brittle matrix, brittle fibers and ductile matrix, in addition to ductile fibers plus brittle matrix as discussed in class.
b) In terms of the combination of ductile fibers plus brittle matrix, the trend seems to be moving away from assuming linear $\sigma - \varepsilon$ up to failure for matrix; Various approaches are seen to deal with the non-linearity, and to various degree of success.
c) It is known that predicting elastic moduli remains a challenge. It is an even bigger challenge for predicting ultimate strengths.

Chapter 2: Macromechanical Analysis of a Lamina

2.1

2.2

2.3 – done before chapter 3 (previously covered)

2.4

2.5

2.6

2.7 – stiffness matrix, compliance matrix, and their applications

2.8 – failure theories of a lamina

2.9 – hydrothermal situation (not covered)

Basics

Contacted notation, $[C]$ and $[S]$

Stress vector

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Strain vector

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Stiffness matrix $[C]$:

$$\{\sigma\} = [C]\{\varepsilon\}$$

Compliance matrix $[S]$:

$$\{\varepsilon\} = [S]\{\sigma\}$$

And:

$$[S] = [C]^{-1}$$
$$[C] = [S]^{-1}$$

Shear strains are the so-called engineering shear strains instead of torsional shear strains

Example 1:

A fiber-reinforced lamina of unidirectional continuous fibers consists of the high-strength graphite fiber and epoxy. The lamina has $V_f = 0.45$ and zero void content. Find $(\sigma_1^T)_{ult}$ of the lamina, given the following:

$E_f = 280\ GPa$
$\left(\sigma_f\right)_{ult} = 5700\ MPa$
$E_m = 3.45\ GPa$
$(\sigma_m)_{ult} = 60\ Mpa$

$$\left(\varepsilon_f\right)_{ult} = \frac{\left(\sigma_f\right)_{ult}}{E_f} = 0.020$$
$$(\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m} = 0.017$$

Therefore, matrix fails first, and Eq. (B) is used to determine $(\sigma_1^T)_{ult}$:

$$(\sigma_1^T)_{ult} = (\sigma_m)_{ult}\left(V_m + V_f\frac{E_f}{E_m}\right) = 2{,}224\ MPa$$

Example 2:

A fiber-reinforced lamina of unidirectional continuous fibers consists of the high-modulus graphite fiber and epoxy. The lamina has $V_f = 0.45$ and zero void content. Find $(\sigma_1^T)_{ult}$ of the lamina, given the following:

$E_f = 530\ GPa$
$\left(\sigma_f\right)_{ult} = 1900\ MPa$
$E_m = 3.45\ GPa$
$(\sigma_m)_{ult} = 60\ Mpa$

$$\left(\varepsilon_f\right)_{ult} = \frac{\left(\sigma_f\right)_{ult}}{E_f} = 0.0036$$
$$(\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m} = 0.017$$

Therefore, fiber fails first, and we need to decide what equation to use.
$(Eq. 3.165)$: $V_{minimum} = 0.0244 = 2.44\%$
$(Eq. 3.166)$: $V_{critical} = 0.0252 = 2.52\%$

So, use Eq. (C) or $(Eq. 3.164)$

$$(\sigma_1^T)_{ult} = \left(\sigma_f\right)_{ult}\left(V_f + V_m\frac{E_m}{E_f}\right) = 861.8\ MPa$$

Example 3:

For the lamina in Example 2, evaluate $(\sigma_1^T)_{ult}$, $(\sigma_1^C)_{ult}$, $(\sigma_2^T)_{ult}$, $(\sigma_2^C)_{ult}$ and $(\tau_{12})_{ult}$, given the following:

$E_f = 530\ GPa$
$v_f = 0.23$
$G_f = 215\ GPa$
$(\sigma_f)_{ult} = 1900\ MPa$
$(\tau_f)_{ult} = 36\ MPa$

$E_m = 3.45\ GPa$
$v_m = 0.30$
$G_m = 1.33\ GPa$
$(\sigma_m)_{ult} = 72\ MPa$
$(\tau_m)_{ult} = 34\ MPa$

$E_1 = 240.4\ GPa$
$E_2 = 11.66\ GPa$
$v_{12} = 0.2685$
$G_{12} = 3.472\ GPa$
$Hexagonal\ packing$
Solution:
$(\sigma_1^T)_{ult} = 861.8\ MPa$
$(\sigma_1^C)_{ult} = 69.80\ MPa$
$(\sigma_2^T)_{ult} = 56.87\ MPa$

$(\sigma_2^C)_{ult} = 103.5\ MPa$
$(\tau_{12})_{ult} = 26.63\ MPa$

Textbook changes (errors in equations):

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^2),$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4). \qquad (2.104a–f)$$

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$$= \frac{1}{2}\tan^{-1}\left(-\frac{2\times10^6 + 3\times10^6}{2(4\times10^6)}\right)$$

$$= 16.00^0$$

8. The principal strains are given by[4]

Bottom line should be -16.00°

from Equation (3.2) as

$$w_f + w_m = w_c \cdot$$

From the definition of the density of a single material,

$$w_c = r_c v_c,$$
$$w_f = r_f v_f, \text{ and} \qquad (3.3a\text{–}c)$$
$$w_m = r_m v_m.$$

Bottom line should be $w_m = \rho_m v_m$

FIGURE 3.23
Longitudinal and transverse direction in a transversely isotropic fiber.

$$E_1 = E_{fL} V_f + E_m V_m,$$

$$\frac{1}{E_2} = \frac{V_f}{E_{fT}} + \frac{V_m}{E_m},$$

$$\nu_{12} = \nu_{fT} V_f + \nu_m V_m,$$

Bottom line should be $v_{12} = v_{fL} V_f + v_m V_m$

$$+\frac{1}{2}\begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix}(10^9)[(0.0075)^2-(0.0025)^2]$$

$$[B]=\begin{bmatrix} -3.129\times10^6 & 9.855\times10^5 & -1.972\times10^6 \\ 9.855\times10^5 & 1.158\times10^6 & -1.972\times10^6 \\ -1.072\times10^6 & -1.072\times10^6 & 9.855\times10^5 \end{bmatrix} Pa\text{-}m^2 .$$

Highlighted areas should be zero.

Midterm Review

Chapter 1
Definition of composite materials:
Reinforcing phase: purpose, shapes, types of fibers
matrix: purpose, materials choices for matrix

Manufacture of Fibers

Applications

Chapter 2
2.3: Independent mechanical properties vs. Types of materials
(and why we use those constants as well)
e.g. orthotropic materials, 9 transversely isotropic materials (what do we need to determine 9 transversely isotropic materials, what is the plane of symmetry), resulting 5…

Chapter 3
3.2: V_f, V_m, W_f, W_m, void content
a few fibers + a few matrices + voids
$V_f', V_m'; V_{fmax}, RVE$
When an equation in the text is only valid for zero void content

3.3: Isotropic fibers + isotropic matrix
transversely isotropic fibers + isotropic matrix
mech. of mat'ls
Halpin-Tsai
elasticity (E_1, E_2, v_{12} won't appear on midterm, too long – but G_{12} could)