To determine constants A and B, two types of boundary conditions need to be considered.

1st : displacement-type of essential B.C.'s
2nd : stress or force-type or natural B.C.'s

With the first type of B.C.'s, (Eq. 3.78) or (Eq. 3) can be applied directly.

With the second type of B.C.'s,
$u \rightarrow \varepsilon_r \ \varepsilon_\theta \ \varepsilon_z$ (strain-displacement relation)
$\varepsilon_s' \rightarrow \sigma_r \ \sigma_z$ (stress-strain relation)

$u = Ar + B/r$
$\varepsilon_r = \partial u/\partial r = du/dr = A - B/r^2$
$\varepsilon_\theta = u/r = A + B/r^2$
$\varepsilon_z = \varepsilon_1$

If material is homogeneous, isotropic and linearly elastic:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} = [C] \begin{Bmatrix} A - B/r^2 \\ A + B/r^2 \\ \varepsilon_1 \end{Bmatrix}$$

$[C]$ is a constant matrix, in terms of E and v. Details are given in (Eq. 3.70) or (Eq. 3.71)

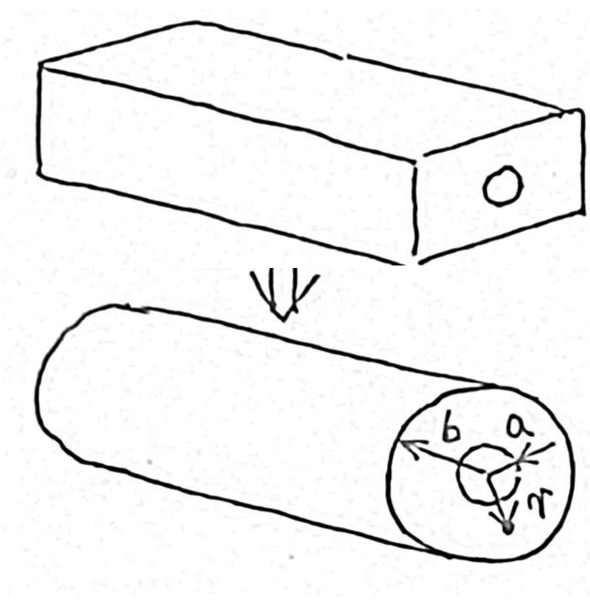
$$\therefore \ \sigma_r = (C_{11} + C_{12})A + \frac{C_{12} - C_{11}}{r^2} B + C_{12}\varepsilon_1 \quad (Eq. 4)$$

$\sigma_z = 2C_{12}A + C_{11}\varepsilon_1 \quad (Eq. 5)$
C_{11}, C_{12} are given by (Eq. 3.72)

3.3.3 Elasticity Approach

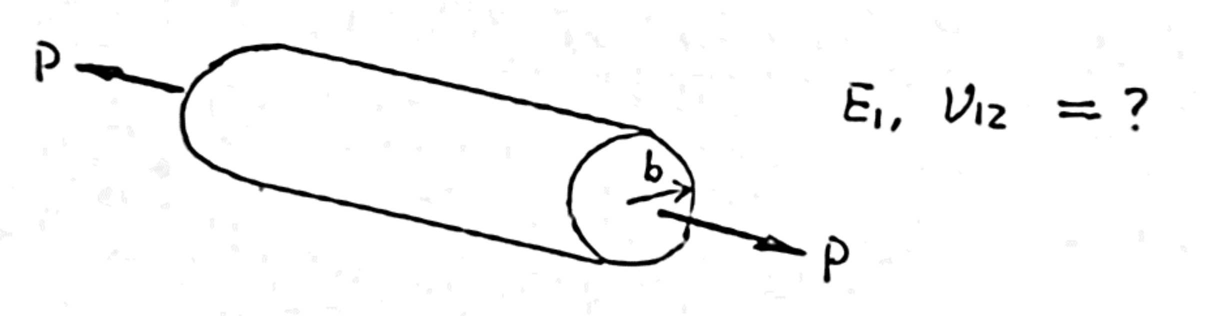
3.3.3.1 Longitudinal Young's Modulus E_1

1) RVE: two concentric cylinders (this RVE is also used with v_{12}) such RVE is known as CCA or CAM
CCA: composite cylinder assembly
CAM: cylindrical assembly model



Fibers: $0 \le r \le a$
Matrix: $a \le r \le b$
$$\therefore V_f = \frac{a^2}{b^2} \quad (Equation\ 3.63)$$
or $a/b = \sqrt{V_f}$

2) A homogenous cylinder representing composite



Assuming that ε_1 is the longitudinal/axial strain developed, after application of P.

$$\therefore\ \sigma_1 = \frac{P}{\pi b^2} \quad (Eq. 3.64)$$
meanwhile, $\sigma_1 = E_1 \varepsilon_1 \quad (Eq. 3.65)$
$$\therefore E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{P}{\pi b^2 \varepsilon_1} \quad (Eq. 3.66)$$
objective is to write P in terms of ε_1 such that E_1 is independent of P.

3) The “fibers” cylinder
(Eq. 3) becomes:
$$u_f = A^f r + \frac{B^f}{r} \quad (0 \le r \le a)$$
B^f must be zero
$$\therefore u^f = A^f \cdot r \quad (0 \le r \le a) \quad (Eq. 3.81)$$

$(Eq. 4)$ and $(Eq. 5)$ become:
$$\begin{Bmatrix} \sigma_r^f = \left(C_{11}^f + C_{12}^f\right) A^f + C_{12}^f \varepsilon_1 \\ \sigma_z^f = 2C_{12}^f + C_{11}^f \varepsilon_1 \end{Bmatrix} (Eq. 3.84)$$
Where $0 \le r \le a$

4) The “matrix” cylinder
$(Eq. 3)$~$(Eq. 5)$ become:
$$u^m = A^m r + \frac{B^m}{r}$$
$$\begin{Bmatrix} \sigma_r^m = (C_{11}^m + C_{12}^m) A^m + \frac{C_{12}^m - C_{11}^m}{r} B^m + C_{12}^m \varepsilon_1 \\ \sigma_z^m = 2C_{12}^m A^m + C_{11}^m \varepsilon_1 \end{Bmatrix} (Eq. 3.86)$$
Where $0 \le r \le a$

So far, unknown constants are:
A^f A^m B^m and ε_1
And ε_1 is related to P.

5) Boundary conditions
5.1) At interface between fibers and matrix cylinders where $r = a$
$u^f = u^m \quad (Eq. 3.88)\sim(Eq. 3.89)$
$\sigma_r^f = \sigma_r^m \quad (Eq. 3.90\,)\sim(Eq. 3.91)$

5.2) At the outer surface of the matrix cylinder
$where\ r = b$
$\sigma_r^m = 0 \quad (Eq. 3.92)\sim(Eq. 3.93)$

The above three boundary conditions are involved with A^f A^m B^m and ε_1;
A^f A^m B^m are then solved in terms of ε_1;

5.3) On any cross-section of CCA or CAM, static equilibrium requires:

$$\int\int_A \sigma_z\, dA = P$$

or $\int\int_{A_f} \sigma_z^f\, dA + \int\int_{A_m} \sigma_z^m\, dA = P \quad (Eq. 3.94)\sim(Eq. 3.97)$

Which results in a relation between P and ε_1.

6) Grand finale
back to (Eq. 3.66)

$$E_1 = \frac{P}{\pi b^2 \varepsilon_1}$$

After lengthy simplification:

$$E_1 = -\frac{2E_m E_f V_f \left(\nu_f - \nu_m\right)^2 \left(1 - V_f\right)}{E_f\left(2\nu_m^2 V_f - \nu_m + V_f \nu_m - V_f - 1\right) + E_m\left(-1 - 2V_f \nu_f^2 + \nu_f - V_f \nu_f + 2\nu_f^2 + V_f\right)} \quad (3.98)$$

3.3.3.2 Major Poisson's Ratio v_{12}
$\left(\frac{u^m}{r}\right)_{r=b}$ is the lateral strain
and:

$$\left(\frac{u^m}{r}\right)_{r=b} = \frac{A^m b + \frac{B^m}{b}}{b} = A^m + \frac{B^m}{b^2}$$

$$\therefore v_{12} = -\frac{A^m + \frac{B^m}{b^2}}{\varepsilon_1}$$

After lengthy simplification:

$v_{12} = v_f V_f + v_m V_m$

$$v_{12} = + \frac{V_f V_m \left(\nu_f - \nu_m\right)\left(2E_f \nu_m^2 + \nu_m E_f - E_f + E_m - E_m \nu_f - 2E_m \nu_f^2\right)}{\left(2\nu_m^2 V_f - \nu_m + \nu_m V_f - 1 - V_f\right)E_f + \left(2\nu_f^2 - V_f \nu_f - 2V_f \nu_f^2 + V_f + \nu_f - 1\right)E_m} \quad (3.103)$$

See example 3.10 for numerical application

3.3.3.3 Transverse Young's Modulus E_2

- CCA model gives lower and upper bounders of E_2
- 3-phase model gives an exact solution for G_{23} which will lead to $E_2 = 2(1 + v_{23})G_{23}$
- Example 3.11 for detailed steps

a) (missed note)

b) CCA model was used, together with energy method (which is different from the classical method of solving PDEs/ODEs)

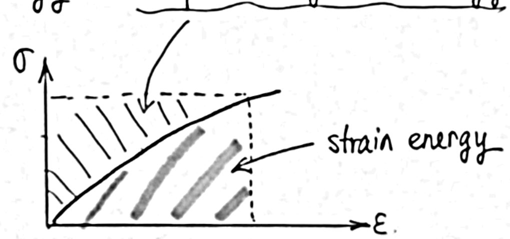
c) Upper bound: maximum potential energy principle

actual strain energy (which is completely unknown due to complexity of problem) $\leq$ strain energy by trial functions meeting certain conditions

d) Lower bound: minimum complementary energy principle

actual complementary strain energy (unknown) $\leq$ complementary energy by trial functions meeting certain conditions

e. strain energy vs. complementary strain energy



Strain energy: in terms of strains and elastic moduli

Complementary strain energy: in terms of stresses and compliances (compliances are the inverse of elastic moduli)

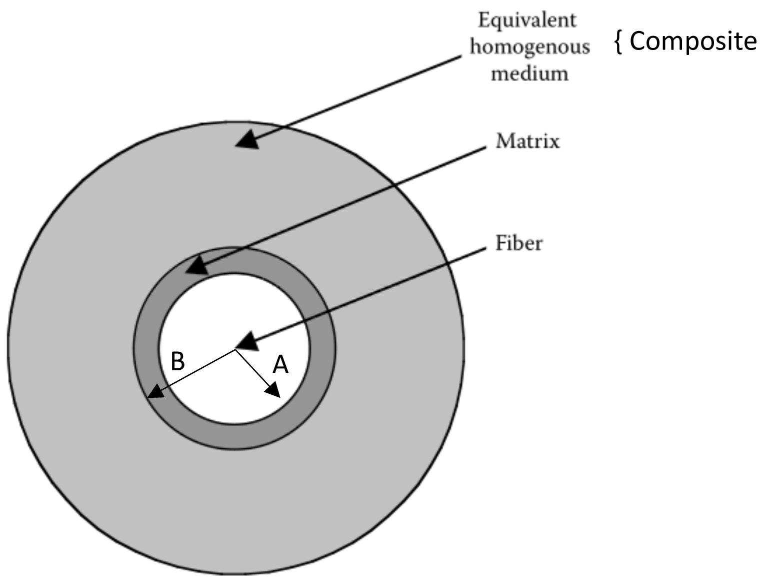
f) the Principle of Minimum Potential Energy:

Of all displacement fields satisfying the prescribed displacement boundary conditions, the field which satisfied stress equilibrium minimizes the stored elastic energy of the system.

g) the Principle of Minimum Complementary Energy:
Among those stress distributions that satisfy the stress equilibrium condition at each point, and that are in equilibrium with the external loads acting on the body, the true stress distribution minimizes the strain energy.

B) 3-phase model (Fig 3.20)
Exact solution for G_{23}: work by R.M. Christensen and K.H. Lo, "Solutions for Effective Shear Properties in Three Phase Sphere and Cylinder Models", J. Mech. And Phys. of Solids, 1979.

Then: $E_2 = 2(1 + v_{23})G_{23}$



C) Example 3.11

3.3.3.4 Axial Shear Modulus G12 (or In-Plane Shear Modulus)
1) RVE:
CCA or CAM, see Fig. 3.19
But it's no longer an axisymmetric case

2) Coordinates
Rectangular coordinates: x_1, x_2, x_3
Displacements: u_1 – axial displacement
u_2, u_3 – displacement along 2- and 3- axis.
Cylindrical coordinates: r, θ, z

3) Solution method:
Simi-inverse method
A certain form of displacement solution is assumed, typically with parameters to be determined so that equilibrium and/or boundary conditions can be met.

4) Assumed displacements

$$u_1 = -\frac{\gamma_{12}^0}{2} x_2 + F(x_2, x_3)$$

$$u_2 = \frac{\gamma_{12}^0}{2} x_1$$

$$u_3 = 0\ , \qquad (3.113a, b, c)$$

Where, F(x_2, x_3) is a unknown function to be determined;
γ_{12}^0 is imposed shear strain, similar to the ε_1 that is assumed in 3.3.3.1 and 3.3.3.2 in order to evaluate E_1 and v_{12}. Here, γ_{12}^0 is imposed so as to evaluate G_{12}.

5) Roadmap to G_{12}
- Condition $F(x_2, x_3)$ must meet: (Eq. 3.117)
- Transformation F(x_2, x_3) to $F(r, \theta)$: (Eq. 3.118 ~ Eq. 3.126)
- Solution of $F(r, \theta)$: (Eq. 126), $F(r, \theta) = \left(Ar + \frac{B}{r}\right) cos\theta$
- $u_1(r, \theta)$, $\tau_{1r}(r, \theta)$
- u_1, τ_{1r} for fiber cylinder (A_1, γ_{12}^0)
- u_1, τ_{1r} for matrix cylinder $(A_2, B_2, \gamma_{12}^0)$
- 4 boundary conditions
$r = a$, 2 conditions
$r = b$, 2 conditions
A_1, A_2, B_2 in terms of γ_{12}^0
$\therefore\ \tau_{12}^m$ in terms of γ_{12}^0

$$G_{12} = \frac{\tau_{12}^m|_{r=b,\ \theta=0}}{\gamma_{12}^0}$$

Final expression:

$$G_{12} = G_m \left[\frac{G_f(1+V_f) + G_m(1-V_f)}{G_f(1-V_f) + G_m(1+V_f)} \right]. \qquad (3.160)$$

6) Example 3.12 for numerical applications of (3.160).

Example: Two fiber-reinforced laminas of unidirectional continuous fibers consistent of pitch-based graphite fibers and epoxy, and Kevlar 49 and epoxy respectively. The laminas have the same corrected volume fractions: $V_f' = 58\%$ and $V_m' = 42\%$. The Young's moduli of the fibers can be found in Tables 1.8, 1.9, and 1.10 of the text. Poisson's ratios are, 0.22 for graphite, 0.35 for Kevlar 49, and 0.32 for epoxy, respectively.
For each of the laminas, determine the four elastic moduli by:
(a) The mechanics of materials approach;
(b) The semi-empirical approach; and
(c) The theory of elasticity approach.

(a) The mechanics of materials approach

	Graphite/Epoxy	Kevlar/Epoxy
$E_1, Mpsi$	32.13	11.25
E_f/E_m	100	34.5
$E_2, Mpsi$	1.29	1.26
v_{12}	0.262	0.337
G_f/G_m	108	33.8
$G_{12}, Mpsi$	0.489	Revised: 0.477

(b) The semi-empirical approach
- from previous notes

(c) The theory of elasticity approach
- from MATLAB

<u>Summary of results:</u>
Graphite/Epoxy

	$Mech\ of\ Mat'ls$	$H-T$	$Elasticity$
$E_1, Mpsi$	32.13	32.13	32.13
E_f/E_m	100	100	100
$E_2, Mpsi$	1.29	2.79	2.01
v_{12}	0.262	0.262	0.255
G_f/G_m	108	108	108
$G_{12}, Mpsi$	0.489	0.803	0.759

Kevlar Epoxy

	$Mech\ of\ Mat'ls$	$H-T$	$Elasticity$
$E_1, Mpsi$	11.25	11.25	11.25
E_f/E_m	34.5	34.5	34.5
$E_2, Mpsi$	1.26	2.52	1.88
v_{12}	0.337	0.337	0.340
G_f/G_m	33.8	33.8	33.8
$G_{12}, Mpsi$	0.477	0.747	0.711

Summary re: Elasticity approach

1) CCA or CAM, and 3-phase model
2) "Exact" solutions: E_1 v_{12} G_{12}, $G_{23} \rightarrow E_2$
3) Values typically fall between those by mech of materials , and by H-T.

Fig 3.21 comparing E_2 by 3 approaches

Fig 3.22 comparing G_{12} by 3 approaches

4) Derivations seem lengthy, so do some final expressions
5) It introduced artificial voids.