Experimental as well as analytical work leading to the H-T formulas was conducted at US Wright

Patterson Air Force Base (Dayton, Ohio) during the 1970s.

Halpin and Tsai then published an internal technical report [Environmental Factors in Composite Materials Design, J.C. Halpin and S.W. Tsai, AFML TR-423] summarizing the work, hence Halpin-Tsai formulas or equations.

In the text, Halphin and Haphin are incorrect spelling.

C. State-of-the-art in terms of predicting elastic moduli

Predicting E_2, G_{12}, and G_{23} (G_{23} is required when dealing with transversely isotropic fibers) remains a challenge, especially when new fibers and matrix are considered.

Chamis model, also a semi-empirical model, seems robust.

Bridging model, which is an analytical model, proves to be reliable, albeit not straightforward.

FE models, however detailed, have not proven themselves more accurate than analytical models or semi-empirical models.

A composite's elastic modulus P_c is found by:

$$P_c = \frac{P_m\left(1 + \zeta\eta V_f\right)}{1 - \eta V_f}$$

Where:

P_c can be either E_1, $E2$, or G_{12} for example;

V_f is the volume fraction of fibers;

ζ is a factor that is used to describe the influence of geometry of the fibers;

ζ depends on V_f, and has different values for different moduli of the composite;

$$\eta = \frac{P_f/P_m - 1}{P_f/P_m + \zeta}$$

P_m, P_f are the corresponding moduli of the matrix, and the fibers, respectively.

Fibers should be interpreted as reinforcing phases, within the context of the H-T formulas.

Factor ζ and Recommendation

Unidirectional continuous fibers

E_1	ROM
E_2	$\zeta = 2 + 40V_f^{10}$
G_{12}	$\zeta = 1 + 40V_f^{10}$
v_{12}	ROM

Particulate

E_1	$\zeta = 2 + 40V_f^{10}$
E_2	$\zeta = 2 + 40V_f^{10}$
G_{12}	$\zeta = 1 + 40V_f^{10}$
v_{12}	ROM

Example: Two fiber-reinforced laminas of unidirectional continuous fibers consistent of pitch-based graphite fibers and epoxy, and Kevlar 49 and epoxy respectively. The laminas have the same corrected volume fractions: $V_f' = 58\%$ and $V_m' = 42\%$. The Young's moduli of the fibers can be found in Tables 1.8, 1.9, and 1.10 of the text. Poisson's ratios are, 0.22 for graphite, 0.35 for Kevlar 49, and 0.32 for epoxy, respectively.
For each of the laminas, determine the four elastic moduli by:
(a) The mechanics of materials approach;
(b) The semi-empirical approach; and
(c) The theory of elasticity approach.

From the Tables: $E_{fG} = 55\ Mpsi$, $E_{fK} = 19\ Mpsi$, and $E_m = 0.55\ Mpsi$
Shear moduli of the fibers and epoxy are evaluated: $G_{fG} = 22.5\ Mpsi$, $G_{fK} = 7.04\ Mpsi$m $G_m = 0.208\ Mpsi$.

(b) H-T Formulas (for Kevlar/Epoxy only)
$E_1 = 11.25$ (from ROM approach)
$v_{12} = 0.337$ (from ROM approach)

E_2:
$\zeta = 2 + 40V_f^{10}$
$= 2.172$

$$\eta = \frac{P_f/P_m - 1}{P_f/P_m + \zeta}$$

$$\eta = \frac{(34.5) - 1}{(34.5) + 2.172} = 0.9135$$

$$P_c = \frac{P_m(1 + \zeta\eta V_f)}{1 - \eta V_f}$$

$$E_2 = \frac{E_m(1 + \zeta\eta V_f)}{1 - \eta V_f} = 2.516\ Mpsi$$

(should be higher than number obtained from mechanics of materials approach.)

G_{12}:
$\zeta = 1 + 40V_f^{10}$
$= 1.172$

$$\eta = \frac{(33.8) - 1}{(33.8) + 1.172} = 0.9379$$

$G_{12} = 0.7469\ Mpsi$

	Graphite/Epoxy	Kevlar/Epoxy
$E_1, Mpsi$	32.13	11.25
E_f/E_m	100	34.5
$E_2, Mpsi$	2.79	2.52
v_{12}	0.262	0.337
G_f/G_m	108	33.8
$G_{12}, Mpsi$	0.803	0.747

	Graphite/Epoxy	Kevlar/Epoxy
E_f/E_m	100	34.5
$E_{2(IROM)}/E_{2(H-T)}$	46%	50%
G_f/G_m	108	33.8
$G_{12(IROM)}/G_{12(H-T)}$	61%	56%

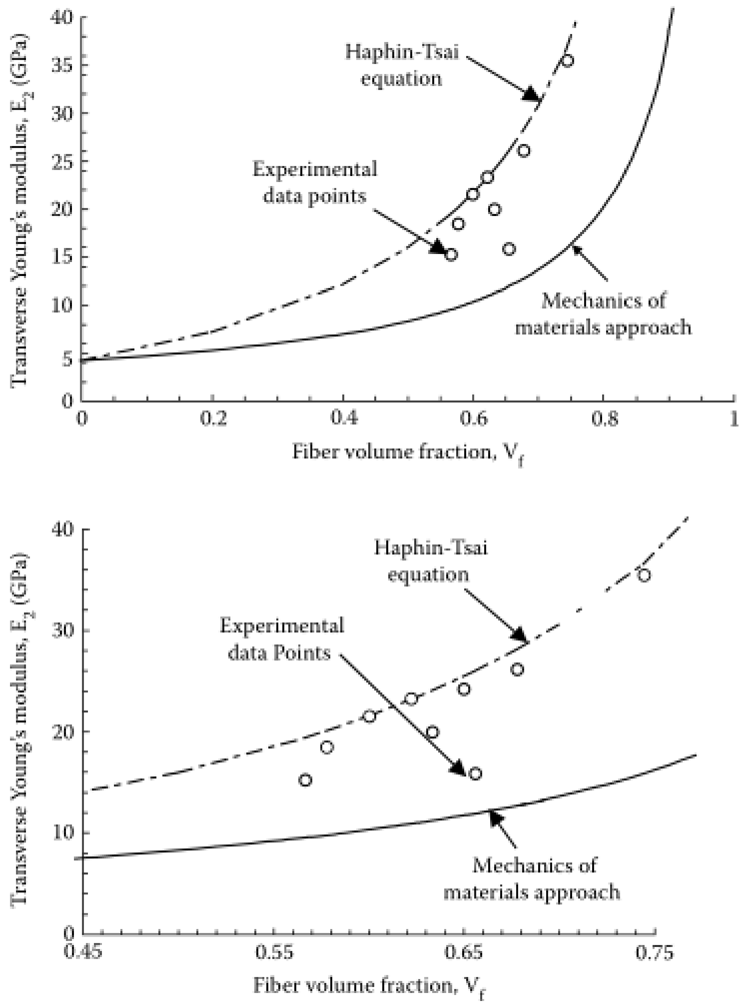


FIGURE 3.15
Theoretical values of transverse Young's modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

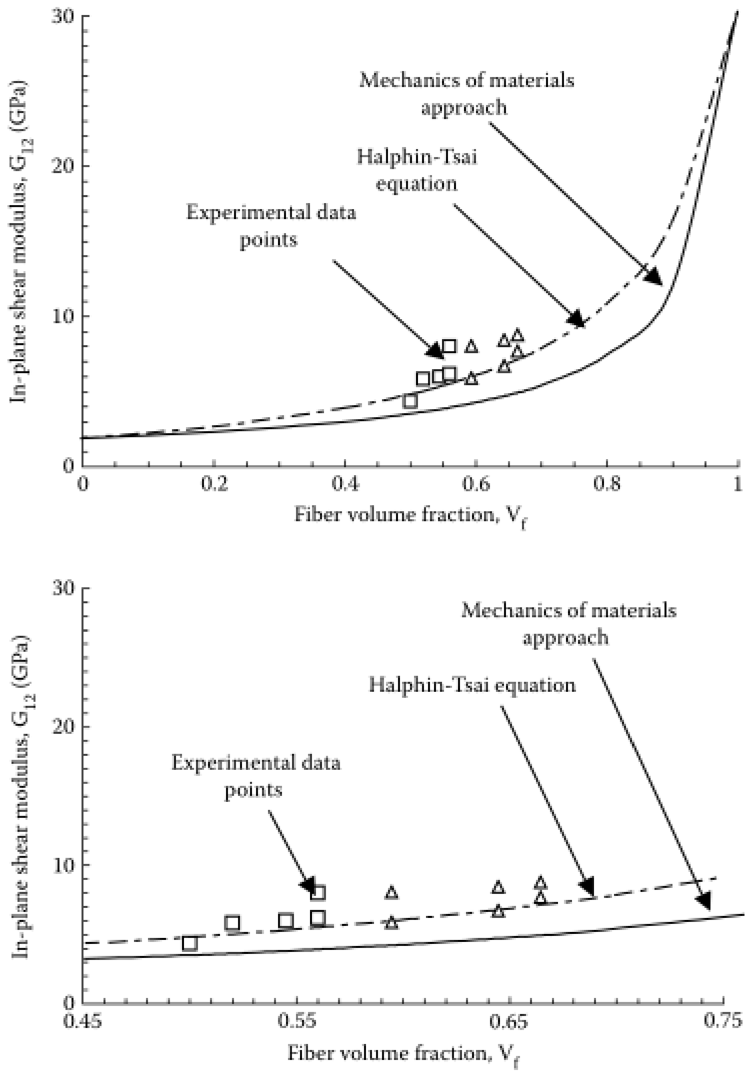


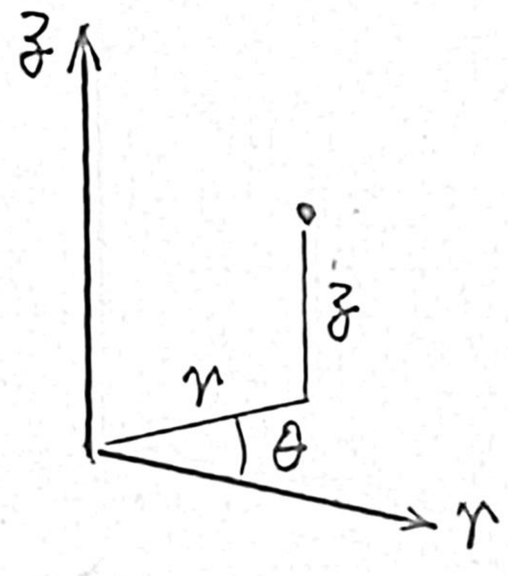
FIGURE 3.17
Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

Theory of Elasticity Using the Cylindrical Coordinates

The following elasticity theory is for homogeneous, isotropic and linearly elastic materials. Therefore, it is applicable to isotropic fiber as a cylinder, and matrix as a cylinder.

As a reminder, composites reinforced with unidirectional continuous fibers are not isotropic. Carbon or graphite fibers, and aramid fibers are transversely isotropic.

1. Cylindrical Coordinates (r, $\boldsymbol{\theta}, \boldsymbol{z}$)



Typically, z represents direction 1; r and θ form the 2-3 plane.

2. Unknowns

The 15 unknowns are:

Displacements $u_r\ u_\theta\ u_z$
Stresses $\sigma_r\ \sigma_\theta\ \sigma_z\ \tau_{r\theta}\ \tau_{\theta z}\ \tau_{zr}$
Strains $\varepsilon_r\ \varepsilon_\theta\ \varepsilon_z\ \gamma_{r\theta}\ \gamma_{\theta z}\ \gamma_{zr}$

3. Relations Governing the 15 Unknowns

There are 3 sets of relations.

(A) The strain displacement relations:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$
$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$
$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$
$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

Where $u_r\ u_\theta$ and u_z are displacements in the r, θ, and z directions, respectively.

(B) The stress-strain relations (or the Hooke's law, or the consecutive relations)

$$\{\sigma\} = [\sigma_r\ \sigma_\theta\ \sigma_z\ \tau_{r\theta}\ \tau_{\theta z}\ \tau_{zr}]^T$$
$$\{\varepsilon\} = [\varepsilon_r\ \varepsilon_\theta\ \varepsilon_z\ \gamma_{r\theta}\ \gamma_{\theta z}\ \gamma_{zr}]^T$$

$$\{\sigma\} = [C]\{\varepsilon\}$$

Where $[C\}$ is the 6x6 matrix:

$$[C] = \begin{bmatrix} C_1 & C_2 & C_2 & 0 & 0 & 0 \\ C_2 & C_1 & C_2 & 0 & 0 & 0 \\ C_2 & C_2 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}$$

Where C_1 and C_2 are constants in terms of E and v

(C) The equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{zr}}{\partial z} + \bar{R} = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{\partial \tau_{\theta z}}{\partial z} + \bar{\theta} = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r} + \bar{Z} = 0$$

Where terms with over-bar are body forces in the r, θ, and z directions, respectively. Body forces are forces per unit volume.

For example, if the cylinder is heavy and z takes the vertical direction, then $\bar{Z} = -\rho g$with ρ being the mass density per unit volume and g being the gravitational acceleration.

Centrifugal force is another example of body force: $\bar{R} = \rho r \omega^2$. Again, ρ is the mass density per unit volume.

Note that sets (A) and (C) are differential. Set (B) is linear.

4. Axisymmetric Problems

It means symmetry about the z-axis, as a result, (a) the unknowns are now functions of r and z only; and (b) $u_\theta = 0$.

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_\theta = \frac{u_r}{r} + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}$$
$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \cancel{\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}} = 0$$

$$\gamma_{\theta z} = \cancel{\frac{1}{r}\frac{\partial u_z}{\partial \theta}} \cancel{+} \cancel{\frac{\partial u_\theta}{\partial z}} = 0$$

$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

As a result, $\gamma_{r\theta}$ and $\gamma_{\theta z} = 0$. Because of homogenous, isotropic, and linearly elastic material assumption, $\tau_{r\theta} = \tau_{\theta z} = 0$. Vectors $\{\sigma\}$ and $\{\varepsilon\}$ are reduced to 4x1; [C] is a 4x4 matrix.

5. Additional simplifications

5.1 All body forces are zero.

- Second equilibrium equation is satisfied, automatically:

$$\cancel{\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}} \cancel{+} \cancel{\frac{\partial \tau_{r\theta}}{\partial r}} \cancel{+} \cancel{\frac{2\tau_{r\theta}}{r}} \cancel{+} \cancel{\frac{\partial \tau_{\theta z}}{\partial z}} \cancel{+ \bar{\theta}} = 0$$

5.2 $\tau_{zr} = 0$, or no shear deformation on any $z - r$ plane.

- The third equilibrium equation becomes:

$$\frac{\partial \sigma_z}{\partial z} \cancel{+} \cancel{\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}} \cancel{+} \cancel{\frac{\partial \tau_{zr}}{\partial r}} \cancel{+} \cancel{\frac{\tau_{zr}}{r}} \cancel{+ \bar{Z}} = 0$$

That is:

$$\frac{\partial \sigma_z}{\partial z} = 0$$

Which means σ_z is either a constant, or a function of r only. The option $\sigma_z = constant$ is chosen.

- The first equilibrium equation simplified to:

$$\frac{\partial \sigma_r}{\partial r} \cancel{+} \cancel{\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}} + \frac{\sigma_r - \sigma_\theta}{r} \cancel{+} \cancel{\frac{\partial \tau_{zr}}{\partial z}} \cancel{+ \bar{R}} = 0$$

Or:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (Eq. 1)$$

- $\gamma_{zr} = 0$ due to homogeneous, isotropic, and linearly elastic material assumption. **

6. PDE to ODE

Making use of strain-stress relations (the inverse of stress-strain relations) such that equation $(Eq. 1)$ is in terms of strains ε_r, ε_θ, and ε_z.

From strain-displacement relations,

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{u_r}{r} + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} = 0$$
$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} = 0$$
$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0$$

Making use of the following in particular

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_\theta = \frac{u_r}{r}$$
$$\varepsilon_z = constant$$

$$\sigma_z = constant$$

Then the final form of $(Eq. 1)$ is in terms of displacements.

By now, there is only one displacement, u_r, that is involved. Also, u_r is no longer a function of z.

Therefore, the subscript r in u_r can be dropped (i.e., u now denotes the radial displacement), and the partial derivatives become ordinary derivatives. The final form of $(Eq\ 1)$ is:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \quad (Eq. 2)$$

The solution to $(Eq. 2)$ is:

$$u = Ar + \frac{B}{r} \quad (Eq. 3)$$

Where A and B are constants to be determined by boundary conditions.

$(Eq. 2)$ and $(Eq. 3)$ are numbered as $(Eq. 3.73)$ and $(Eq. 3.78)$, respectively, in the text.