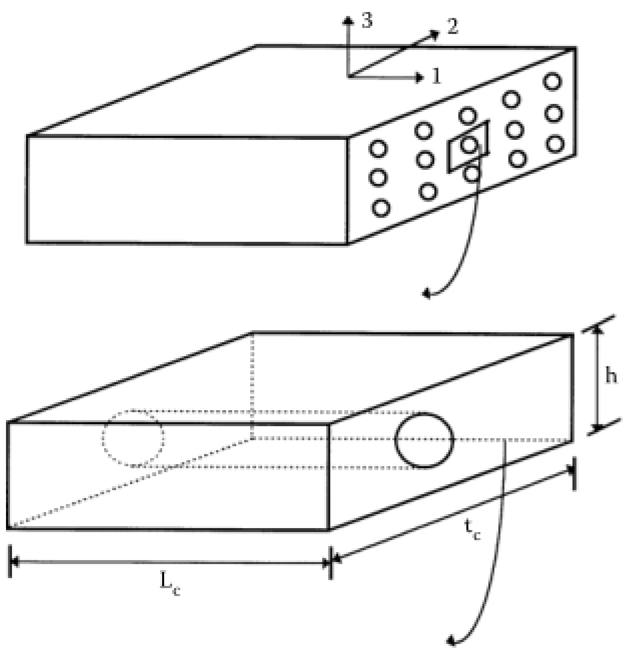
3.3.1 Strengths of Materials Approach

(*) Simple expression for E_1 E_2 v_{12} and G_{12}

(*) Being simple & being accurate

$$\begin{cases} RVE \rightarrow Model? \\ Mathematical\ manipulation? \end{cases}$$

(*) E_1 E_2 v_{12} G_{12} in terms of:
E_f v_f G_f V_f
E_m v_m G_m V_m



RVE for 3.3.1:
Rectangular packing of fibers
d: diameter of fibers
h, t_C: diameter of rectangle
$L_c = l_c$; length of composite

Square packing → Rectangular packing:
Principle is to preserve V_f
$\therefore h = t_c = s$
s being spacing, see 3.2

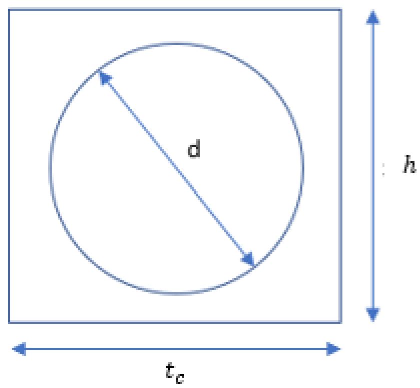
Hexagonal packing → Rectangular packing:
Principle remains to preserve V_f
By section 3.2:

$$V_f = \frac{\pi}{2\sqrt{3}} \frac{d^2}{S^2}$$

Hexagonal packing:

$$V_f = \frac{\pi}{2\sqrt{3}} \frac{d^2}{S^2}$$

Rectangular packing:



(Let l_c be the depth)

$$v_f = \frac{\pi}{4} d^2 \, l_c$$
$$v_c = h \, t_c \, l_c$$
$$V_f = \frac{v_f}{v_c} = \frac{\frac{\pi}{4} d^2 \, l_c}{h \, t_c \, l_c} = \frac{\pi d^2}{4 \, h \, t_c}$$

Equating rectangular packing to hexagonal packing:

$$\therefore \frac{\pi d^2}{4 \, h \, t_c} = \frac{\pi}{2\sqrt{3}} \frac{d^2}{S^2}$$

$h \, t_c = \sqrt{3} \, S^2$ (and further, $t_c = \alpha h$)

3.3.1.1 Longitudinal Young's Modulus E_1

RVE: Fig 3.3 of text, top and middle diagrams in particular.

Determining E_1:

l_c: length of RVE

A_c: cross-section of RVE

$\therefore v_c = A_c l_c$

$\therefore v_f = V_f v_c$

$= (V_f A_c) l_c$

$V_f A_c$: cross-section of fibers

$v_m = V_m v_c$

$= (V_m A_c) l_c$

$V_m A_c$: cross-section of matrix

Under F:

Axial loading, statically indeterminate

$$\delta = \frac{PL}{EA}$$
$$k = \frac{EA}{L}$$
$$k_f = \frac{E_f V_f A_c}{l_c}$$
$$k_m = \frac{E_m V_m A_c}{l_c}$$
$$k_c = \frac{E_1 A_c}{l_c}$$

$\therefore k_c = k_f + k_m$
$\therefore \frac{E_1 A_c}{l_c} = \frac{E_f V_f A_c}{l_c} + \frac{E_m V_m A_c}{l_c}$
$\therefore E_1 = E_f V_f + E_m V_m$ (Rule of mixture)

Rule of Mixture
It refers to the method of estimating composite's property by volume-weighted average of like properties of the constituents.

E_1:
RVE: Fig 3.3, middle diagram
Statically indeterminate problem
Calculation: Example 3.3
Comparing with experimental results: Fig 3.6
E_1 is by rule-of-mixture
E_1 is accurate

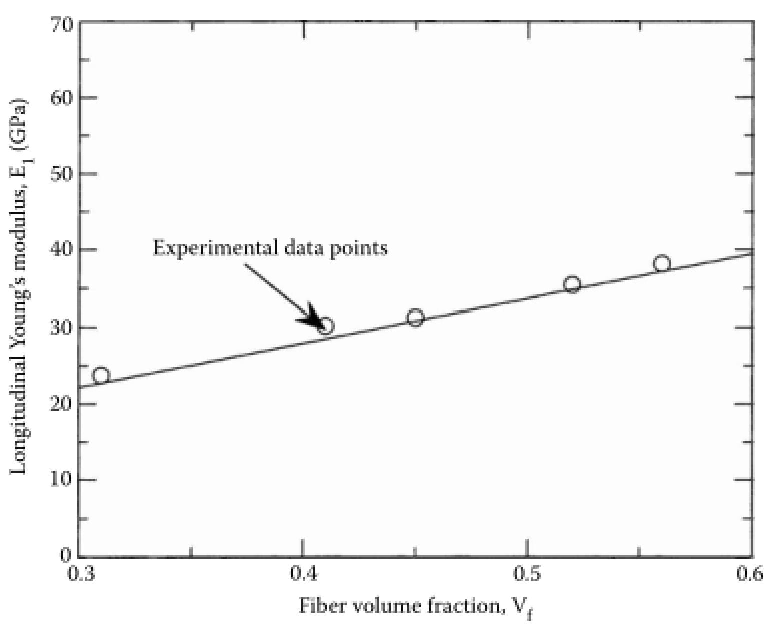
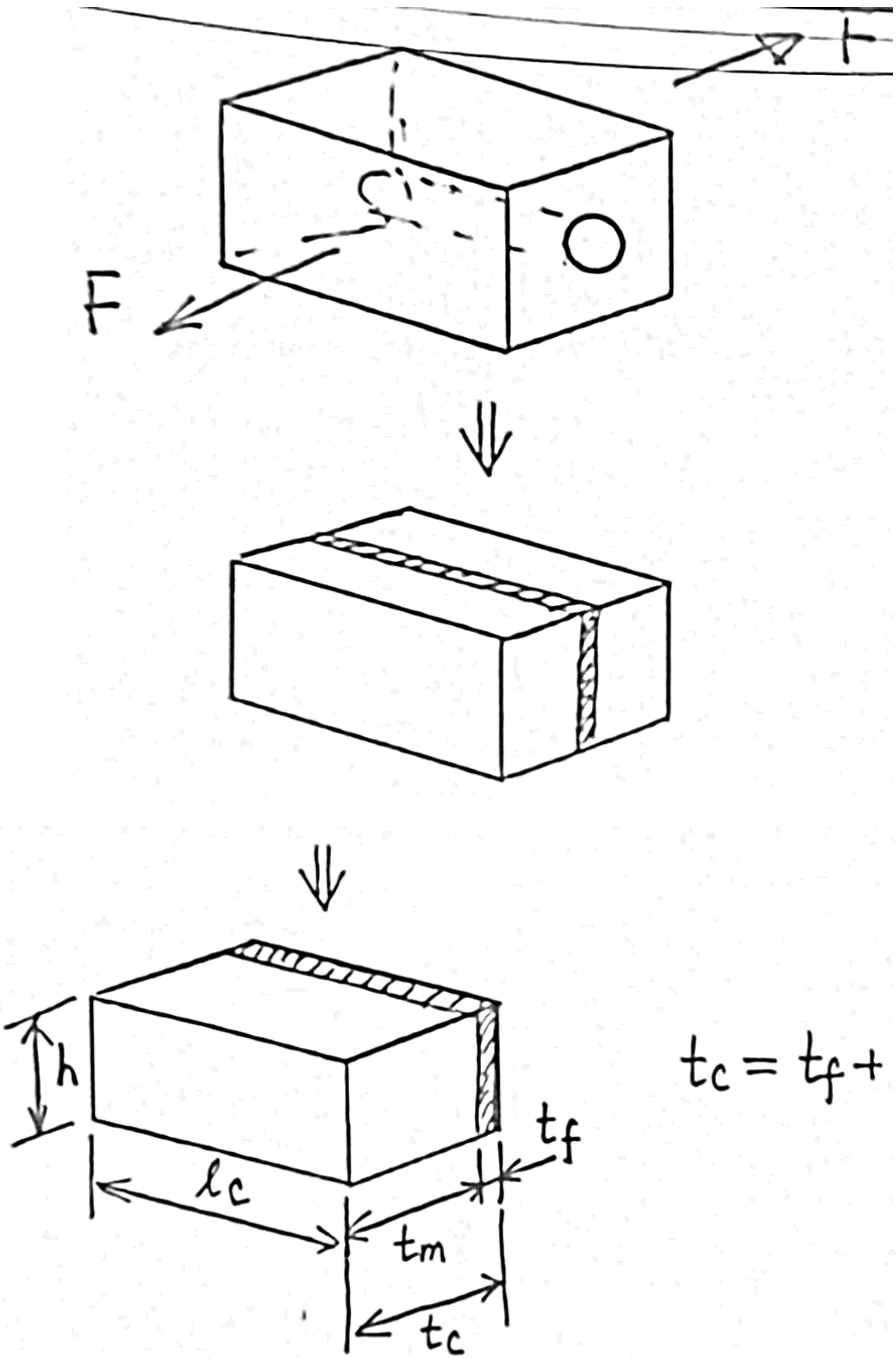


Figure 3.6

3.3.1.2 Transverse Young's Modulus, E_2

This time, RVE is loaded in the 2-direction.



$$t_c = t_f + t_m$$

E_2:
Axial loading

$$k = \frac{EA}{L}$$

$$k_f = \frac{E_f(h\,l_c)}{t_f}$$

$$k_m = \frac{E_m(h\,l_c)}{t_m}$$

$$k_c = \frac{E_2(h\,l_c)}{t_c}$$

$$\therefore \frac{1}{k_c} = \frac{1}{k_f} + \frac{1}{k_m}$$

$$\therefore \frac{t_c}{E_2(h\,l_c)} = \frac{t_f}{E_f(h\,l_c)} + \frac{t_m}{E_m(h\,l_c)}$$

$$\therefore \frac{1}{E_2} = \frac{t_f}{E_f\,t_C} + \frac{t_m}{E_m\,t_c}$$

$\therefore \frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$ (Inverse rule of mixture)

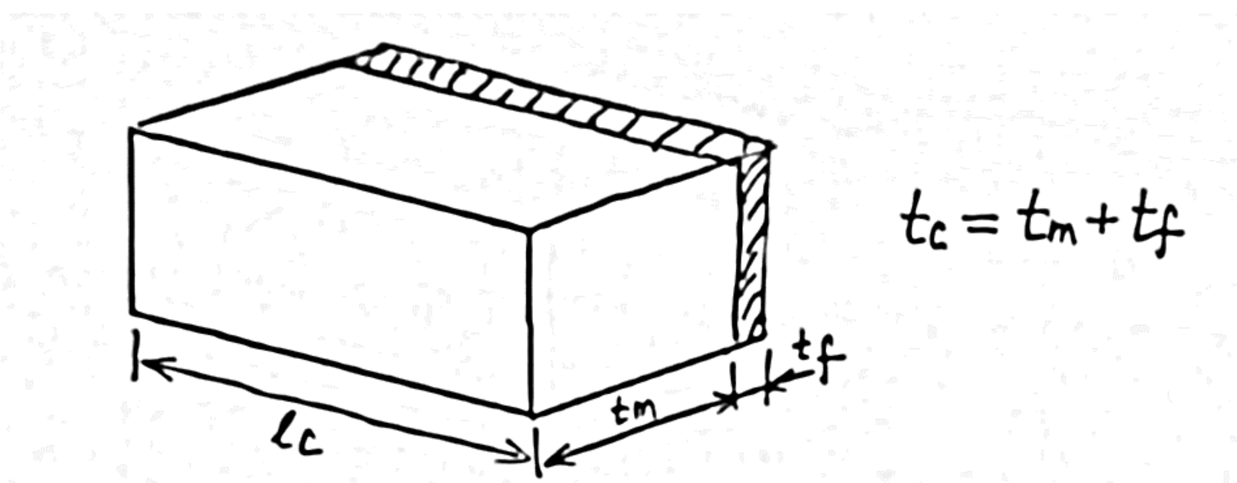
E_2:
RVE: Figure 3.7
Calculation: Example 3.4
Comparing with experimental results: Fig 3.10 inverse ROM results in lower-bound solution.
Solutions may be as low as 40%~50% of where values should be, depending on the difference between E_f and E_m (or E_f/E_m) and V_f

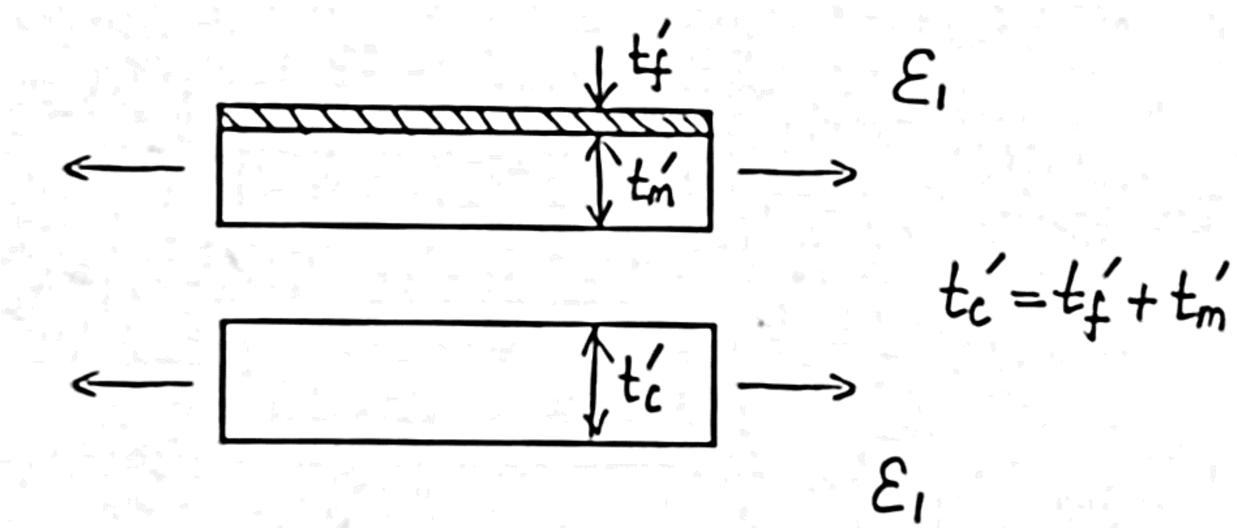
Typically, the higher the ratio E_f/E_m, and/or the higher the V_f value, the higher the deduction.

Inverse rule of mixture is simple, but not accurate, because it's not true axial attention.

Midterm: October 31st (Thursday) during class time.

3.3.1.3 Major Poisson's Ratio v_{12}





$Poisson's\ ratio$:

$$v = -\frac{lateral\ strain}{longitudinal\ strain} = -\frac{lateralstrain}{\varepsilon_1}$$

$$Normal\ strain\ \varepsilon = \frac{\Delta l}{l_o} = \frac{l' - l_o}{l_o}$$

$\therefore Fibers$: $lateral\ strain$

$$= \frac{t_f' - t_f}{t_f}$$

$$v_f = -\frac{t_f' - t_f}{t_f\ \varepsilon}$$

$$t_f' = t_f - v_f t_f \varepsilon_1\ (\boldsymbol{Eq.1})$$

$Matrix$:

$$t_m' = t_m - v_m t_m \varepsilon_1\ (\boldsymbol{Eq.2})$$

$Composite$:

$$t_c' = t_c - v_{12} t_c \varepsilon_1\ (\boldsymbol{Eq.3})$$

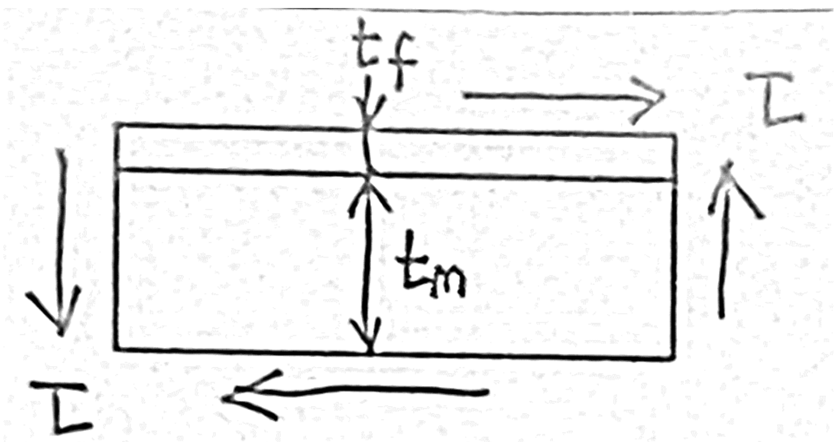
$$(\boldsymbol{Eq.1}) + (\boldsymbol{Eq.2}): t_f' + t_m' = t_f + t_m - (v_f t_f)\varepsilon_1 = (v_m t_m)\varepsilon_1\ (\boldsymbol{Eq.4})$$

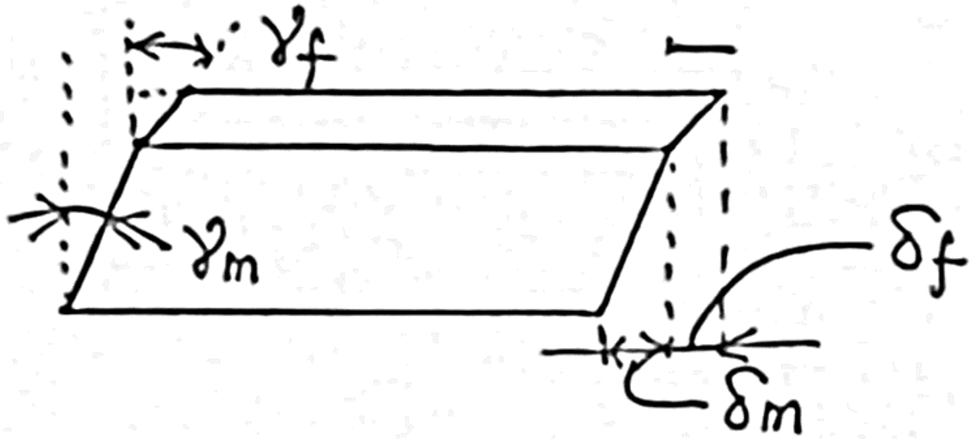
$$(\boldsymbol{Eq.3}) + (\boldsymbol{Eq.4}): \therefore -v_{12} t_c \varepsilon_1 = -v_f t_f \varepsilon_1 - v_m t_m \varepsilon_1$$

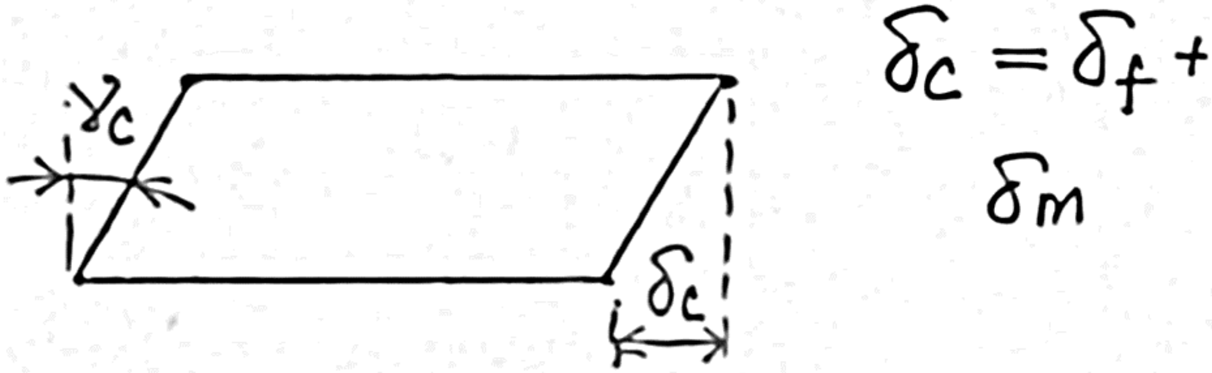
$$\therefore v_{12} = v_f V_f + v_m V_m\ (rule\ of\ mixture)$$

v_{12}:
RVE: Fig. 3.11
Calculation: Example 3.5
v_{12} by rule-of-mixture is accurate

3.3.1.4 In-plane Shear Modulus G_{12}







G_{12}:

$Matrix\colon v_m = \tan^{-1}\frac{\delta_m}{t_m} \approx \frac{\delta_m}{t_m}$

$(\tan^{-1} x \approx x \ \ for\ very\ small\ angles)$

But,

$$\gamma_m = \frac{\tau}{G_m}$$

$$\therefore \frac{\delta_m}{t_m} = \frac{\tau}{G_m}$$
$$\therefore \delta_m = \frac{\tau\, t_m}{G_m}$$

$$Fibers: \delta_f = \frac{\tau\, t_f}{G_f}$$

$$Composite\ \delta_c = \frac{\tau\, t_c}{G_{12}}$$

And:
$$\frac{\tau\, t_c}{G_{12}} = \frac{\tau\, t_m}{G_m} + \frac{\tau\, t_f}{G_f}$$
$$\therefore \frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}\ (inverse\ rule\ of\ mixture)$$

G_{12}:
RVE: Fig. 3.12
Calculation: Example 3.6
Comparing with experimental results: Fig. 3.13
G_{12} is by inverse rule-of-mixture expression which gives lower bound solution.
Discussions pertaining E_2 are mostly applicable with the exceptions of smaller deduction (25~40%), and the ratio G_f/G_m instead of E_f/E_m

4 sub-sections
3.3.1 Strength of Materials Approach (Mechanics of Materials)
3.3.2 Sem-Empirical Approach
3.3.3 Theory of Elasticity Approach
3.3.4 Transversely Isotropic Fibers (c- and k- fibers are transversely isotropic)

3.3.2 Semi-Empirical Models
A. Some historical notes

- Mechanics of Materials approach yields simple expressions, rule of mixture (ROM) or inverse ROM; General observation is, ROM-based expressions give good accuracy, but the inverse ROM-based ones are far from satisfactory.
- A series of formal approaches (As they are known as) took place after realizing shortcomings of the inverse ROM. Such approaches were based on theory of elasticity, in one form or the others. The differences lie in the methodologies of solving the PDEs involved.
- Basic assumptions behind the formal approaches are, noting that some of the basic assumptions have been used in 3.3.1
    - Fibers and matrix are homogeneous and isotropic;
    - The resulting composite is homogeneous and orthotropic;
    - Void content is zero;

- There is perfect bonding between constituents;
  - Constituents and resulting composite are linearly elastic;
  - Composite is initially stress-free;
  - Fibers are regularly spaced and aligned.
- The formal approaches include the following methods, to list just a few
  - Classical or exact method (See 3.3.3)
  - Variation methods or energy methods that are either analytical or numerical. The former gives rise to bounds on elastic moduli; and the latter typically leads to finite difference method and finite element method.
  - Mori-Tanaka models (or inclusion models). The key aspect is to assume that the average strain of the inclusion (i.e., the fibers) is related to the average strain of the matrix by a to-be-determined fourth-order tensor.
  - Self-consistent models (suitable for composites having particulate or short fibers as reinforcing phase) .....
- To meet the desire of engineers to have simple yet accurate formulas, efforts were taken, in the 1960' to 1970', to interpolate existing theoretical as well as experimental results.
  - Experimental data to best-fit ROM- or inverse ROM-based formulas with modifying factors (wasn't successful);
  - Experimental data to best-fit re-arranged formulas from self-consistent models, and simplified by introducing factors (was successful);
  - Halpin-Tsai formulas/equations are the best-known outcome of such effort; self-consistent models with factors.

Example: Two fiber-reinforced laminas of unidirectional continuous fibers consistent of pitch-based graphite fibers and epoxy, and Kevlar 49 and epoxy respectively. The laminas have the same corrected volume fractions: $V_f' = 58\%$ and $V_m' = 42\%$. The Young's moduli of the fibers can be found in Tables 1.8, 1.9, and 1.10 of the text. Poisson's ratios are, 0.22 for graphite, 0.35 for Kevlar 49, and 0.32 for epoxy, respectively.
For each of the laminas, determine the four elastic moduli by:
(a) The mechanics of materials approach;
(b) The semi-empirical approach; and
(c) The theory of elasticity approach.

From the Tables: $E_{fG} = 55\ Mpsi$, $E_{fK} = 19\ Mpsi$, and $E_m = 0.55\ Mpsi$
Shear moduli of the fibers and epoxy are evaluated: $G_{fG} = 22.5\ Mpsi$, $G_{fK} = 7.04\ Mpsi$m $G_m = 0.208\ Mpsi$.

(a) The mechanics of materials approach:

	Graphite/Epoxy	Kevlar/Epoxy
$E_1, Mpsi$	32.13	11.25
E_f/E_m	100	34.5
$E_2, Mpsi$	1.29	1.26
v_{12}	0.262	0.337
G_f/G_m	108	33.8
$G_{12}, Mpsi$	0.489	0.420

(b)