a) determine $[ABD]$ matrix; see Section 4.3

b) determine N_x, N_y, …, M_y, M_{xy}

c) for al plies

- for top and bottom surface of a ply
    - z → global strains → global stresses → local stresses (→ local strains is using max. strain theory)→ SR of the surface.
- SR of a ply

for top and bottom surface of a ply

SR of the laminate; also, the ply (plies) that would fail.

d) FPF load = applied loads · SR of laminate

e) Discount failed ply on plies set load level to FPF load

f) Go back to step a)
For Step c), only loop over all remaining plies for step d)

If SR of laminate ≥ 1

- 2^{nd} ply failure load = FPF load · SR for step e), set load level to 2^{nd} ply failure load

Otherwise

- Keep load level at FPF load discount failed plies (with SR < 1)

g) repeat step f) by progressively updating failure load and discounting failed ply or plies, until termination criterion is met, and UPF load has been determined.

Example 1: A laminate has the layup sequence of [30 / 45]. The top land bottom layers are 0.4 mm and 0.5 mm thick. Both layers have: $E_1 = 170\ GPa$, $E_2 = 20\ GPa$, $G_{12} = 5.5\ GPa$ and $v_{12} = 0.26$.

Given:
$(\sigma_1^T)_{ult} = 2990\ MPa$
$\left(\sigma_1^C\right)_{ult} = 88\ MPa$
$(\sigma_2^T)_{ult} = 45\ MPa$
$\left(\sigma_2^C\right)_{ult} = 148\ MPa$
$(\tau_{12})_{ult} = 21.5\ MPa$
$N_x = N_y = 1000\ N/m$

Determine the failure sequence of the laminate. Termination criterion is when all plies fail. Use maximum stress theory to determine SR. Discount totally the failed ply (or plies). What is the FPF (first ply failure) load? What is the LPF (last ply failure) load?

1) FPF analysis
Steps a), b): see Example in 4.3

Step c)

<u>TOP LAYER</u>

<u>Top surface</u>

$$\text{global stress} = \begin{Bmatrix} 960.6 \\ 1081 \\ -78.91 \end{Bmatrix} (kPa)$$

$$\text{Local stress} = [T] \cdot \begin{Bmatrix} 960.6 \\ 1081 \\ -78.91 \end{Bmatrix} = \begin{Bmatrix} 922.3 \\ 1119 \\ 12.63 \end{Bmatrix} (kPa)$$

$\therefore SR_1 = 3241$

$SR_2 = 40.2$

$SR_6 = 1701$

$\therefore SR = 40.2\ (2T)$

<u>Bottom surface</u>

$$\text{Local stress} = [T] \cdot \begin{Bmatrix} 1291 \\ 1089 \\ -67.12 \end{Bmatrix} (kPa)$$

$\therefore SR_1 = 2316$

$SR_2 = 41.3$

$SR_6 = 320$

$\therefore SR = 41.3\ (2T)$

$\therefore SR$ for top layer is 40.2 (2T)

<u>BOTTOM LAYER</u>

<u>Top surface</u>

$SR = 41.4\ (2T)$

<u>Bottom surface</u>

$SR = 40.1\ (2T)$

∴ SR for bottom later is 40.1 (2T)

$$\text{d) FPF load} = (40.1) \cdot \begin{Bmatrix} 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 40.1 \\ 40.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \frac{\left(\frac{kN}{m}\right)}{\left(\frac{Nm}{m}\right)}$$

e) Only top layer will be included in then next round of analysis; $N_x = N_y = 40.1\ kN/m$

2) UPF analysis

Step b):

$$\{\varepsilon^0\} = \begin{Bmatrix} 8.762 \\ 21.35 \\ -21.78 \end{Bmatrix} \cdot (10^{-3})\ (m/m)$$

$$\{\kappa\} = \begin{Bmatrix} 28.88 \\ 70.38 \\ -71.82 \end{Bmatrix} (1/m)$$

Step c):

<u>TOP LAYER</u>

<u>Top surface</u>

Local stress $= \begin{Bmatrix} -275.7 \\ -275.7 \\ 0 \end{Bmatrix} (MPa)$

$\therefore SR_1 = 0.32$

$SR_2 = 0.54$

$SR_6 => 10^{13}$

$\therefore SR = 0.32\ (1C)$

<u>Bottom surface</u>

Local stress $= [T] \cdot \begin{Bmatrix} 476.2 \\ 476.2 \\ 0 \end{Bmatrix} (MPa)$

$\therefore SR_1 = 6.3$

$SR_2 = 0.095$

$SR_6 => 10^{13}$

$\therefore SR = 0.095\ (2T)$

$\therefore SR$ for top layer is 0.095 (2T)

FPF analysis:

$$Top\ layer: \frac{40.2(2T)}{41.3(2T)}$$

$$Bottom\ later: \frac{41.4(2T)}{40.1(2T)}$$

Example 2 (probably on final): A three-layered laminate is subject to an applied load of $M_x = 10\ N \cdot m/m$. Progressive failure analysis results in the following:

Failure #	Failed Layer(s)	SR	Mode of Failure
1	3	15.9	2T
2	2	1.07	2T
3	1	1.24	1T

(1) What is the UPF load?

$$M_x = (15.9)(10) = 159 \frac{Nm}{m}$$

(2) What is the UPF load, if termination criterion is fiber failure?

$$M_x = (15.9)(1.07)(1.24)(10) = 210.9612\ \frac{Nm}{m}$$

(3) What is the UPF load, if termination criterion is as long as 50% of the plies have failed?

$$M_x = (15.9)(1.07)(10) = 170.13\ \frac{Nm}{m}$$

Example 3: A 12-playered laminate has the following results from the progressive failure analysis:

Failure #	Failed Layer(s)	SR	Mode of Failure
1	1	13.9	1C
2	10	1.08	2T
3	12	0.97	6S
4	11	0.84	2T
	3	0.93	1C
5	8	0.49	2T
	9	0.54	1T

(1) What is the termination criterion used?
1T (the last failure)

(2) What is the SR (with respect to the original load) at UPF?
$(13.9)(1.08) = 15.012$

(3) Physically, in what sequence did layers fail up to the UPF?
Layer 1, 10, followed immediately by 12, 11, 3, 8 and 9;

(4) What is the LPF load?
N/A

Chapter 6

6.1: Introduction

Review of theory of isotropic beams

6.2 Symmetric beams

6.3: Non-symmetric beams

Beams are treated as special cases of plates, mainly, membrane loads are absent. Or:

$N_x = N_y = N_{xy} = 0$

The other simplifications will depend on that problem at hand. e.g.:

Symmetric beams: $\varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = 0$

Nonsymmetric beams: $\varepsilon_x^0 \neq 0,\ \varepsilon_y^0 \neq 0,\ \gamma_{xy}^0 \neq 0$

Nonsymmetric beams won't be discussed.

They bend and warp; they are also stretched or compressed, and sheared.

<u>Symmetric beams:</u>

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$$

Due to symmetry, $[B] = 0$

$\therefore \{N\} = [A]\{\varepsilon^0\}$

$\therefore \{M\} = [D]\{\kappa\}$

That is, because of symmetry, membrane action and bending action are decoupled.

$\because \{N\} = \{0\}$

$\therefore \{\varepsilon^0\} = \{0\}$

<u>Nonsymmetric Beams:</u>

$\{N\} = [A]\{\varepsilon^0\} + \{B\}\{\kappa\} = \{0\}$

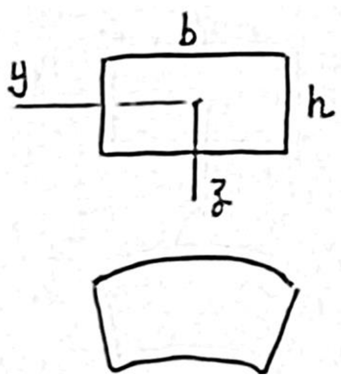
$\therefore \{\varepsilon^0\} = -[A]^{-1}[B]\{\kappa\}$

$\because \{\kappa\} \neq \{0\}$

$\therefore \{\varepsilon^0\} \neq \{0\}$

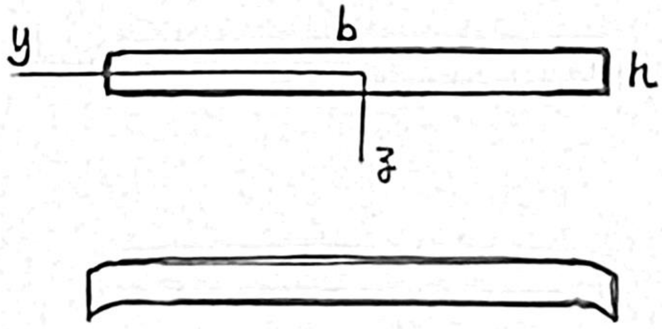
Narrow Beams versus Wide Beams

If cross-sectional dimensions are $b \cdot h$ then $b/h \geq 5$ is considered wide beams.



$\therefore \kappa_y \neq 0$
$\kappa_{xy} \neq 0$
Poisson ratio effect
(Observed in isotropic beams too)

pp. 433~440
(Beginning of 6.2 → end of Example 6.1)



$\therefore \kappa_y = 0$
$\kappa_{xy} = 0$
Edge effect
(Less in isotropic beams)

pp. 440~444
(Example 6.2)

Wide Beams
Loading: $M_x = \pm M/b$ (sign convention)
Curvatures: $\kappa_y = \kappa_{xy} = 0$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \kappa_x \\ 0 \\ 0 \end{Bmatrix}$$

$\because M_x = D_{11}\kappa_x$

$\therefore \kappa_x = \dfrac{M_x}{D_{11}}$

$M_y = D_{12}\kappa_x = \dfrac{D_{12}}{D_{11}} M_x$

$M_{xy} = D_{16}\kappa_x = \dfrac{D_{16}}{D_{11}} M_x$

Global strains at ζ:

$$[\varepsilon\} = \zeta \begin{Bmatrix} \kappa \\ 0 \\ 0 \end{Bmatrix}$$

Global stresses at ζ of layer κ:
$\{\sigma\}_\kappa = [\bar{Q}]_\kappa \{\varepsilon\}$

Flexural modulus of beam:

$$E_x^{wide} = \frac{12 D_{11}}{h^3}$$

$E_x^{wide} I$ replaces EI in deflection and/or slope determination

Narrow Beams
Loading: $M_x = \pm M/b$ (sign convention)
Curvatures: $M_y = M_{xy} = 0$

$$\therefore \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = [D]^{-1} \begin{Bmatrix} M_x \\ 0 \\ 0 \end{Bmatrix}$$

Or:
$\kappa_x = D_{11}^* M_x$
$\kappa_y = D_{12}^* M_x$
$\kappa_{xy} = D_{16}^* M_x$

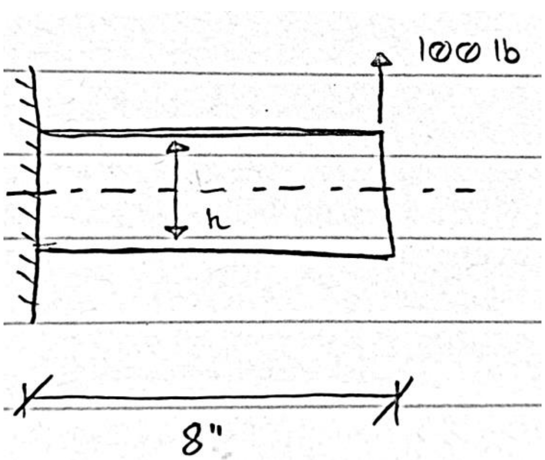
Global strains at ζ:
$\{\varepsilon\} = \zeta\{\kappa\}$

Global stresses at ζ of layer κ
$\{\sigma\}_\kappa = [\bar{Q}]_\kappa \{\varepsilon\}$

Flexural modulus of beams (same as E_x^f in 4.4):
$$E_x^{narrow} = \frac{12}{h^3 D_{11}^*}$$

$E_x^{narrow} I$ replaces EI in deflection and/or slope determination

Example: A cantilever beam has the layup sequence of [0 / 90 / 0]s. The beam is 8" long and 3" wide. A load of 100 lb is applied at the free end of the beam. Each lamina is 0.1" thick, with $E_1 = 5.5\ Mpsi$, $E_2 = 1.5\ Mpsi$, $G_{12} = 0.95\ Mpsi$, and $v_{12} = 0.275$.
Determine,
(1) The maximum stress developed in the laminated beam;
(2) The deflection and slope at the free end of the beam.



Bending moment at the clamped end:
$M = 800\ lb \cdot in$
$M_x = M/b = 266.67\ lb \cdot in/in, (positive)$

$I = \frac{bh^3}{12} = 0.054\ in^4$

$\delta_{tip} = \frac{PL^3}{3EI}$

$\theta_{tip} = \frac{PL^2}{2EI}$

$b = 3.0"$

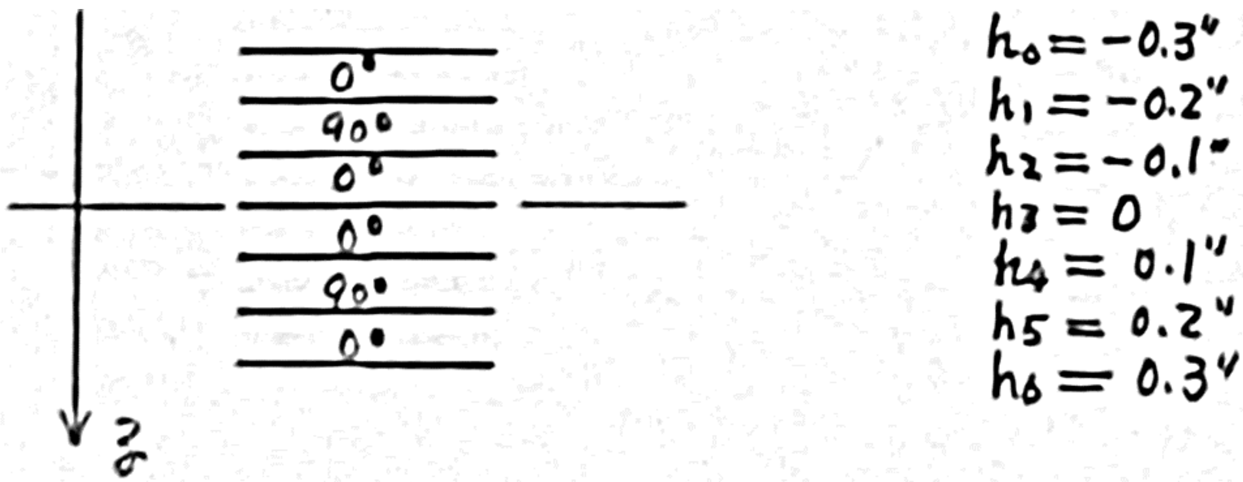
$h = 0.6"$

∴ narrow beam ($[D]^{-1}$ needed)

or wide beam ([D] needed)

$[Q]$

$[\bar{Q}]$ for 0° & 90°



$$[Q] = \begin{bmatrix} 5.616 & 0.4212 & 0 \\ & 1.523 & 0 \\ sym. & & 0.9500 \end{bmatrix} \cdot 10^6\ (psi)$$

$$[D] = \begin{bmatrix} 82.03 & 7.581 & 0 \\ & 46.63 & 0 \\ sym. & & 11.71 \end{bmatrix} \cdot 10^3\ (psi \cdot in^3)$$

$$[D]^1 = \begin{bmatrix} 1.238 & -0.2012 & 0 \\ & 2.177 & 0 \\ sym. & & 5.848 \end{bmatrix} \cdot 10^{-5} \left(\frac{1}{psi \cdot in^3}\right)$$

$D_{11}^* = 1.238 \cdot 10^{-5} \left(\frac{1}{psi \cdot in^3}\right)$

$E_x^{narrow} = \frac{12}{h^3 D_{11}^*} = 4.488 \cdot 10^6\ (psi)$

$\kappa_x = D_{11}^* M_x = 0.003301\ (1/in)$

$\kappa_y = D_{12}^* M_x = -0.0005365\ (1/in)$

$\kappa_{xy} = D_{16}^* = D_{16}^* M_x = 0$

$$[\kappa\} = \begin{Bmatrix} 3.301 \\ -0.5365 \\ 0 \end{Bmatrix} \cdot 10^{-3} \ \left(\frac{1}{in}\right)$$

Bottom surface of laminate, $\zeta = 0.3"$

$$\{\varepsilon\} = \begin{Bmatrix} +0.9903 \\ -0.1610 \\ 0 \end{Bmatrix} \cdot 10^{3} \ \left(\frac{in}{in}\right)$$

$$\{\sigma\}_6^b = \begin{Bmatrix} +5.494 \\ +0.1705 \\ 0 \end{Bmatrix} \ (ksi)$$

$$\delta_{tip} = \frac{PL^3}{3 \cdot E_x^{narrow} I} = 0.0704"$$

$$\theta_{tip} = \frac{PL^2}{2 \cdot E_x^{narrow} I} = 0.0132 \ rad$$

$D_{11} = 82.03 \cdot 10^3 \ (psi \cdot in^3)$

$E_x^{wide} = \frac{12 D_{11}}{h^3} = 4.557 \cdot 10^6 \ (psi)$

$$\kappa_x = \frac{M_x}{D_{11}} = 0.003251 \ \left(\frac{1}{in}\right)$$

$$\{\kappa\} = \begin{Bmatrix} 3.251 \\ 0 \\ 0 \end{Bmatrix} \cdot 10^{-3} \ \left(\frac{1}{in}\right)$$

Bottom surface of laminate, $\zeta = 0.3"$

$$\{\varepsilon\} = \begin{Bmatrix} +0.9753 \\ 0 \\ 0 \end{Bmatrix} \cdot 10^{3} \ \left(\frac{in}{in}\right)$$

$$\{\sigma\}_6^b = \begin{Bmatrix} +5.494 \\ +0.4108 \\ 0 \end{Bmatrix} \ (ksi)$$

$$\delta_{tip} = \frac{PL^3}{3 \cdot E_x^{wide} I} = 0.0693"$$

$$\theta_{tip} = \frac{PL^2}{2 \cdot E_x^{wide} I} = 0.0130 \ rad$$