2) Coordinate setup

$x-y-z$: global coordinates

$x-y$ plan coincides with the mid-plane of the laminate;

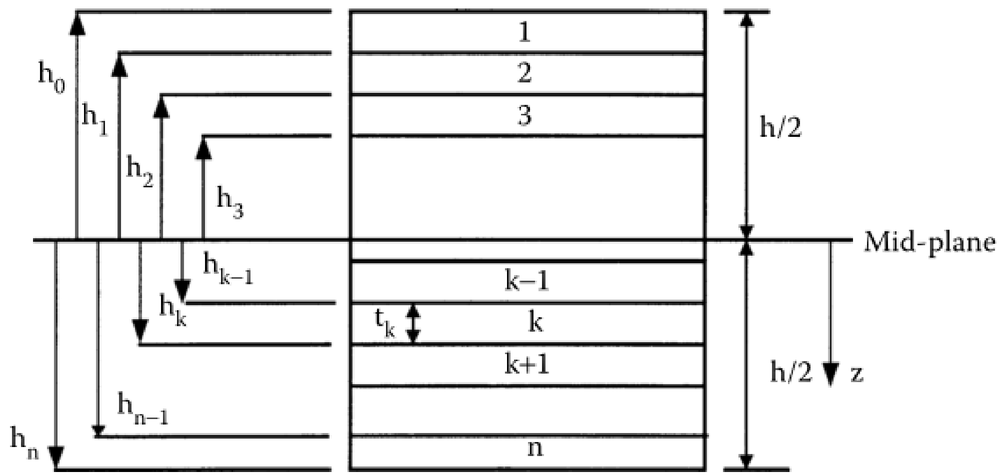


FIGURE 4.6
Coordinate locations of plies in a laminate.

h: thickness of laminate

$h_0 = -h/2$

$h_1 = h_0 + t_1$

$h_2 = h_1 + t_2$

…

$h_N = h/2$

3) Mid-plane Displacements (unknowns to be solved)

$u_o(x, y)$: membrane stretches

$v_o(x, y)$: membrane stretches

$w_o(x, y)$: lateral deflection

4) "Slopes" and Curvatures

$\phi_y(x, y) = \frac{\delta w_o}{\delta x}$: rotation about x

$\phi_x(x, y) = \frac{\delta w_o}{\delta y}$: rotation about y

$\kappa_x(x, y) = -\frac{\delta^2 w_o}{\delta x^2}$: curvature

$\kappa_y(x, y) = -\frac{\delta^2 w_o}{\delta y^2}$: curvature

$\kappa_{xy}(x, y) = -2\frac{\delta^2 w_o}{\delta x \delta y}$: twisting curvature

5) Membrane strain vector and curvature vector

$\{\varepsilon^0\} =$

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \tag{4.14}$$

$\{\kappa\} =$

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}, \tag{4.15}$$

Note: Both ε^0 and κ are based on the midplane.

6 Global displacements and strains at points off the midplane

On midplane where $z = 0$:

$$u_0(x, y), \;\; v_0(x, y), \;\; w_o(x, y)$$

Off the midplane ($z \neq 0$):

$$u = u_0 - z\frac{\partial w_0}{\partial x}. \tag{4.10}$$

$$v = v_0 - z\frac{\partial w_o}{\partial y}. \tag{4.11}$$

$$w(x, y, z) = w_0$$

And:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}. \quad (4.13)$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \quad (4.16)$$

7) Resultant Forces and Moments

Sign conventions: Figure 4.3

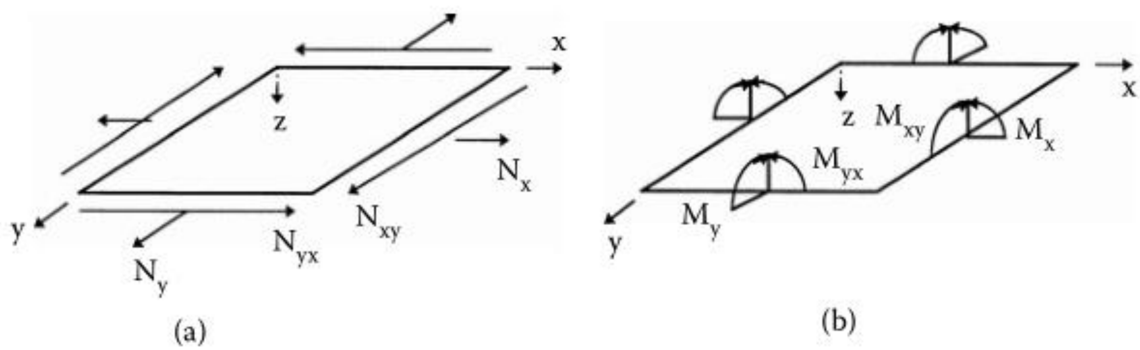


FIGURE 4.3
Resultant forces and moments on a laminate.

Definitions:

Membrane forces (per unit length)

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz, \tag{4.21a}$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y dz, \tag{4.21b}$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz, , \tag{4.21c}$$

Which can be simplified as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz, \tag{4.24a}$$

($*$) Unit: N/m, lb/in, … ($force/length$)

($*$) Integral is broken into sum over layers due to “jumps” at lamina interfaces, as per Figure 4.5.

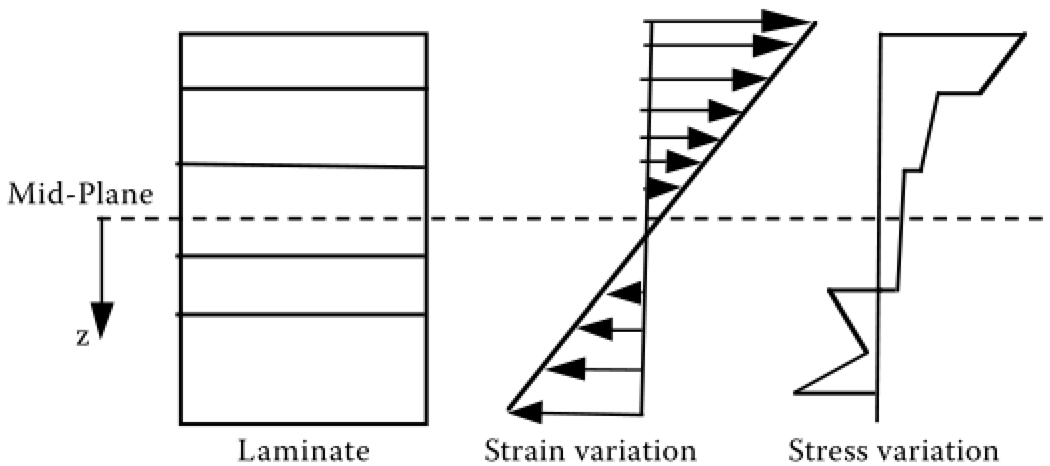


FIGURE 4.5
Strain and stress variation through the thickness of the laminate.

N_x, N_y: Normal forces per unit length
N_{xy}: shear force per unit length

Moments (per unit length)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz, \tag{4.22a}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz, \tag{4.22b}$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz, \tag{4.22c}$$

Which can be simplified as:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k zdz. \tag{4.24b}$$

(∗) Unit: $N \cdot m/m$, $lb \cdot in/in$, … $(moment/length)$

(∗) M_x, M_y: bending moments per unit length
M_{xy}: twisting moment per unit length

<u>8) Stiffness and compliance of a laminated plate</u>

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$[A]$: membrane (extensional) stiffness matrix
$Pa \cdot m$, or $psi - in$ $(stress - length)$
symmetric

$[B]$: membrane-bending coupling stiffness matrix
$Pa \cdot m^2$, or $psi \cdot in^2$ $(stress - length^2)$
symmetric

$[D]$: bending stiffness matrix
$Pa \cdot m^3$, or $psi \cdot in^3$ $(stress - length^3)$
symmetric

Derivation: $pp. 328 \sim 331$

$$A_{ij} = \sum_{k=1}^{n} [(\bar{Q}_{ij})]_k (h_k - h_{k-1}), \quad i = 1,2,6; \quad j = 1,2,6, \tag{4.28a}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\bar{Q}_{ij})]_k (h_k^2 - h_{k-1}^2), \quad i = 1,2,6; \quad j = 1,2,6, \tag{4.28b}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\bar{Q}_{ij})]_k (h_k^3 - h_{k-1}^3), \quad i = 1,2,6; \quad j = 1,2,6. \tag{4.28c}$$

$[ABD]$: “stiffness matrix” of the laminate plate
$[ABD]^{-1}$: “compliance matrix” of the laminate plate obtained numerically (by inversion)
both are symmetric

As long as $[B] \neq 0$, membrane and bending are kinetically coupled.

(Eqn. 4-29) is typically written in the compact form:
$\begin{Bmatrix} N \\ M \end{Bmatrix} = [ABD] \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$

<u>9) Applications</u>
10-step procedure on p. 332
the main steps are:

a. evaluate [ABD] matrix
$k = 1, \ldots, N$
$[Q]_k, \;\; [\bar{Q}]_k$
$h_{k-1}, \;\; h_k$
(Eq. 4.28a,b,c)

b. determine $\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$

find $N_x, N_y, \dots, M_y, M_{xy}$

inverse $[ABD]$

$[ABD]^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix}$

c. evaluate global strains $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$ via (Eqn. 4.16)

for a given z, then evaluate global stresses → local stresses (and local strains is necessary)→ SR

Examples 4.2

Examples 4.3 (the above procedure)

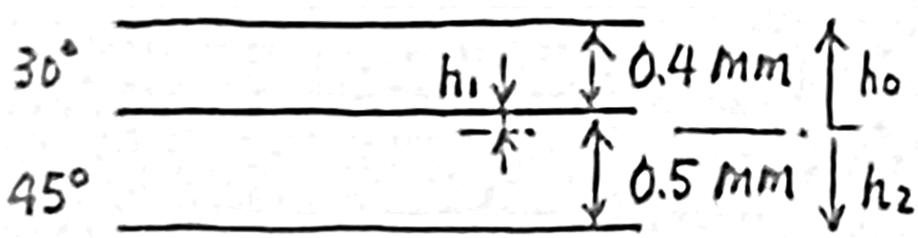
Example: A laminate has the layup sequence of $[30\ /\ 45]$. The top and bottom layers are $0.4\ mm$ and $0.5\ mm$ thick. Both layers have: $E_1 = 170\ GPa$, $E_2 = 20\ GPa$, $G_{12} = 5.5\ GPa$ and $v_{!2} = 0.26$. Determine global stresses σ_x, σ_y, τ_{xy} at the top and bottom surfaces of the layers for the following loading:

$N_x = N_y = 1000\ N/m$; and

Plot stress distributions of σ_x, σ_y, τ_{xy} across the thickness.

Solution:

(1) Need [ABD] matrix, $[ABD]^{-1}$ matrix:



$h_0 = -0.00045\ m$

$h_1 = -0.00005\ m$

$h_2 = 0.00045\ m$

Then,

$[Q] \to [\bar{Q}]_1 \quad (\theta = 30°)$

$[Q] \to [\bar{Q}]_2 \quad (\theta = 45°)$

Applying (Eq. 4.28)

$$[A] = \begin{bmatrix} 6.950 & 3.653 & 3.888 \\ & 3.926 & 2.511 \\ sym & & 3.676 \end{bmatrix} \cdot (10^7) \qquad (Pa \cdot m)$$

$$[B] = \begin{bmatrix} -4.774 & 0.9940 & -1.215 \\ & 2.786 & 2.228 \\ sym & & 0.9940 \end{bmatrix} \cdot (10^3) \qquad (Pa \cdot m^2)$$

$$[D] = \begin{bmatrix} 4.850 & 2.432 & 2.665 \\ & 2.557 & 1.621 \\ sym & & 2.448 \end{bmatrix} \qquad (Pa \cdot m^3)$$

And $[ABD]^{-1} = \begin{bmatrix} A^* & B^* \\ B^* & D^* \end{bmatrix}$

$$[A^*] = \begin{bmatrix} 5.164 & -2..786 & -3.426 \\ & 6.251 & -1.260 \\ sym & & 7.731 \end{bmatrix} \cdot (10^{-8}) \qquad \left(\frac{1}{Pa \cdot m}\right)$$

$$[B^*] = \begin{bmatrix} 3.687 & -1..832 & 3.689 \\ & 0.01186 & -4.574 \\ sym & & -4.680 \end{bmatrix} \cdot (10^{-5}) \qquad \left(\frac{1}{Pa \cdot m^2}\right)$$

$$[D^*] = \begin{bmatrix} 7.468 & -4.037 & -5.257 \\ & 9.260 & -1.641 \\ sym & & 11.68 \end{bmatrix} \cdot (10^{-1}) \qquad \left(\frac{1}{Pa \cdot m^3}\right)$$

(2) Loading is $N_x = N_y = 1000\ N/m$

$$\left\{\frac{\varepsilon^0}{\kappa}\right\} = [ABD]^{-1} \begin{Bmatrix} 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 23.78 \cdot 10^{-6}\ m/m \\ 34.65 \cdot 10^{-6}\ m/m \\ -46.86 \cdot 10^{-6}\ m/m \\ 18.55 \cdot 10^{-3}\ 1/m \\ -18.20 \cdot 10^{-3}\ 1/m \\ -8.855 \cdot 10^{-3}\ 1/m \end{Bmatrix}$$

Normal forces can cause shear deformation, bending and twisting.

<u>Top layer:</u> $[\bar{Q}] = [\bar{Q}] \uparrow$

Top surface $\zeta = h_0 = -0.00045\ m$

$$\{\varepsilon\} = \{\varepsilon^0\} + \zeta[\kappa] = \begin{Bmatrix} 15.43 \\ 42.83 \\ -42.88 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = [\bar{Q}]\{\varepsilon\} = \begin{Bmatrix} 960.6 \\ 1081 \\ -78.91 \end{Bmatrix} kPa$$

Bottom surface $\zeta = h_1 = -0.00005\ m$

$$\{\varepsilon\} = \begin{Bmatrix} 22.85 \\ 35.56 \\ -46.42 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = \begin{Bmatrix} 1298 \\ 1265 \\ 53.70 \end{Bmatrix} kPa$$

<u>Bottom layer:</u> $[\bar{Q}] = [\bar{Q}] \downarrow$

Top surface $\zeta = h_1 = -0.00005\ m$

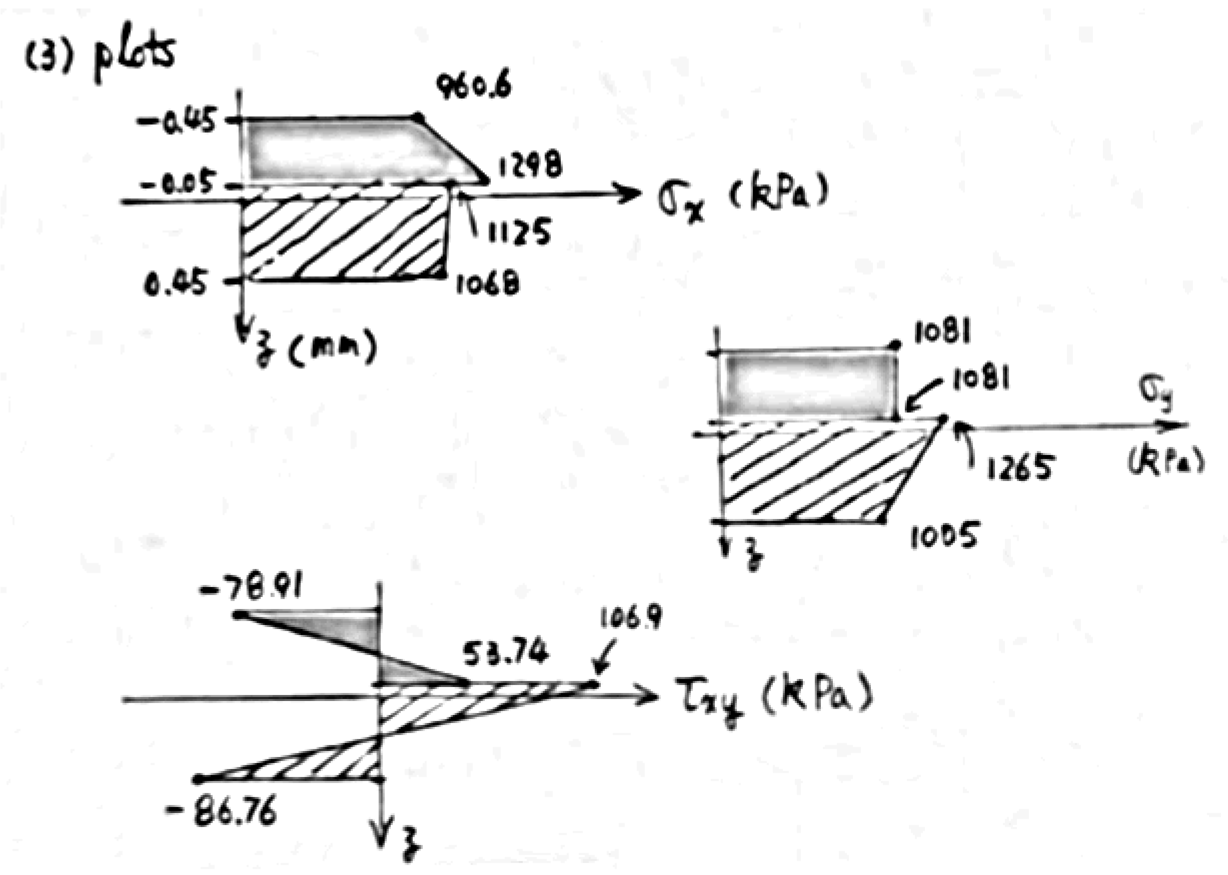
$$\{\varepsilon\} = \begin{Bmatrix} 22.85 \\ 35.56 \\ -46.42 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = \begin{Bmatrix} 1125 \\ 1265 \\ 106.9 \end{Bmatrix} kPa$$

Bottom surface $\zeta = h_2 = 0.00045\ m$

$$\{\varepsilon\} = \begin{Bmatrix} 31.13 \\ 26.46 \\ -50.84 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = \begin{Bmatrix} 1068 \\ 1005 \\ -86.76 \end{Bmatrix} kPa$$



4.4 In-Plane and Flexural Modulus of a Laminate

$[ABD] = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$

Which is a symmetric matrix. ([A], [B] and [D] are inverse as well)

$[ABD]^{-1} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}$

Where $[A^*]$, $[D^*]$ are symmetric
and $[B^*]$ may not be symmetric
and $[C^*] = [B^*]^T$

In-plane constants:

$$E_x = \frac{1}{hA_{11}^*}; \quad Effective\ in-plane\ longitudinal\ modulus$$

$$E_y = \frac{1}{hA_{22}^*}; \quad Effective\ in-plane\ transverse\ modlus$$

$$G_{xy} = \frac{1}{hA_{66}^*}; \quad Effective\ in-plane\ shear\ modulus$$

$$v_{xy} = -\frac{A_{12}^*}{A_{11}^*}; \quad Effective\ in-plane\ Poisson's\ ratio$$

$$v_{yx} = -\frac{A_{12}^*}{A_{22}^*}; \quad Effective\ in-plane\ Poisson's\ ratio$$

Note:
The larger Poisson's ratio is the <u>major</u>, and the other one is the <u>minor</u>

$$\frac{v_{xy}}{E_x} = \frac{v_{yx}}{E_y}$$

Flexural constants:

$$E_x^f = \frac{12}{h^3 D_{11}^*}; \quad effective\ flexural\ longitudinal\ modulus$$

$$E_y^f = \frac{12}{h^3 D_{22}^*}; \quad effective\ flexural\ transverse\ modulus$$

$$G_{xy}^f = \frac{12}{h^3 D_{66}^*}; \quad effective\ flexural\ shear\ modulus$$

$$v_{xy}^f = -\frac{D_{12}^*}{D_{11}^*}; \quad effective\ flexural\ Poisson's\ ratio$$

$$v_{yx}^f = -\frac{D_{12}^*}{D_{22}^*}; \quad effective\ flexural\ Poisson's\ ratio$$

Note:
The larger Poisson's ratio is the <u>major</u>, and the other one is the <u>minor</u>

$$\frac{v_{xy}^f}{E_x} = \frac{v_{yx}^f}{E_y}$$

- Example 4.4 goes through the steps above

Consider that you were given a completed [ABD] matrix, then you can use the following tools to analyze it further.

<u>Measures used to gauge how close a laminate is to an equivalent orthotropic material:</u>
In terms of membrane action:

$$r_N = \sqrt{\left(\frac{A_{26}}{A_{11}}\right)^2 + \left(\frac{A_{26}}{A_{22}}\right)^2}$$

In terms of bending action:

$$r_M = \sqrt{\left(\frac{D_{16}}{D_{11}}\right)^2 + \left(\frac{D_{26}}{D_{22}}\right)^2}$$

It is desired that $r_N \to 0, r_M \to 0$

Measures used to gauge symmetry of a laminate:

$$r_B = \frac{3}{(A_{11} + A_{22} + A_{66})h} \sqrt{\sum_i \sum_j \left(B_{ij}\right)^2}$$

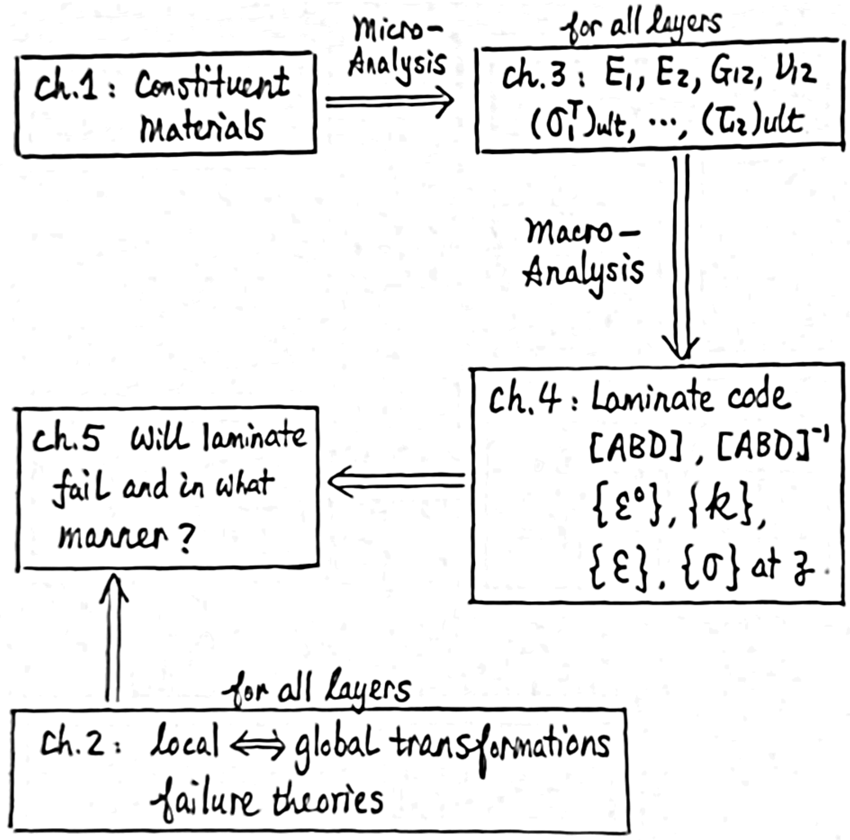
The closer r_B is to zero, the more symmetry there is

Summary:

$r_N,\ r_M,\ r_B$: the closer they are to zero, the more accurate it is to use E_x, E_y, …, G_{xy}^f and v_{xy}^f to represent the entire laminate, and to treat the laminate as an orthotropic material.

Chapter 5: Failure, Analysis and Design of Laminates

5.1: Introduction



5.2: Special Cases of Laminates

In 4.2, laminate codes were introduced. From mainly the perspective of layer orientations, cross-ply laminates, angle-ply laminates, balanced laminates, symmetric laminates, and anti-symmetric laminates were defined.

In this section, the above laminates will be defined from the perspectives of constituents and mixtures, and thicknesses, in addition to layer orientations. The effect on the $[ABD]$ matrix will be dealt with as well.

1. Symmetric Laminates

These are laminates in which fiber orientations, constituents and mixtures, and thicknesses of the top half of the laminate are mirror image of the bottom half.

A symmetric laminate can have even or odd number of layers.
→ $[B] = [0]$; as a result, membrane and bending actions are uncoupled, kinetically.

2. Cross-Ply Laminates

They are laminates in which the layers take angles of 0° and 90° only.
→ $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$

For example, a symmetric cross-ply has, $[B] = [0]$
and $A_{16} = A_{26} = D_{16} = D_{26} = 0$;

3. Angle-Ply Laminates

They are laminates that consist of pairs of layers of the same constituents and mixture, and thickness, but oriented at $+\theta$ and $-\theta$.
→ $A_{16} = A_{26} = 0$

4. Anti-Symmetric Laminates

In 4.2, anti-symmetric laminates are defined as those in which fiber orientations of the top half of the laminate are opposite those of the bottom half.

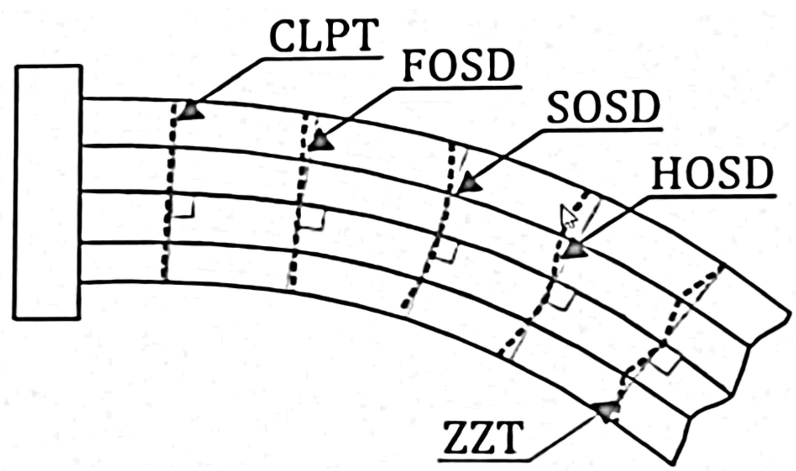
In terms of constituents and mixtures, and thicknesses, the top half and bottom half are mirror images of each other.

An anti-symmetric laminate always has even number of layers.
→ $A_{16} = A_{26} = D_{16} + D_{26} = 0$

5. Balanced Laminates

In 4.2, balanced laminates are defined as having pairs of "$+\theta$" and "$-\theta$" layers (θ cannot be $0°$ or $90°$).

Further, the pair of "$+\theta$" and "$-\theta$" layers must have the same constituents and mixture and thickness for a laminate to be balanced
→ $A_{16} = A_{26} = 0$



6. Quasi-Isotropic Laminates

Quasi-isotropic means behaving like an isotropic material, or independent of orientation. However, being quasi-isotropic does not mean being isotropic.

- Quasi-isotropic in terms of membrane action:
  (a) $[B] = [0]$; and
  (b) $A_{11} = A_{22}\,; A_{16} = A_{26} = 0\,; A_{66} = (A_{11} - A_{12})/2$
- Quasi-isotropic in terms of bending action:
  (a) $[B] = 0$; and
  (b) $D_{11} = D_{22}; D_{16} = D_{26} = 0; D_{66} = (D_{11} - D_{12})/2$

- Quasi-isotropic in terms of both membrane and bending actions:
  (a) [B]=[0]; and
  (b) $A_{11} = A_{22}\,; A_{16} = A_{26} = 0\,; A_{66} = (A_{11} - A_{12})/2$; and
  (c) $D_{11} = D_{22}; D_{16} = D_{26} = 0; D_{66} = (D_{11} - D_{12})/2$; or
  $[D] = \left(\frac{h^2}{12}\right) * [A]$

See Example 5.1 for a laminate that is quasi-isotropic in terms of membrane action.

How to make a quasi-isotropic laminate:

(1) Number of layers $N \geq 3$
(2) Orientations of two adjacent layers differ by $180°/N$.

For example, if $N = 3$, layups may be $[\ 60\ /\ 0\ /\ -80\ ]$ and $[\ 45\ /\ -15\ /\ -75\ ]$

5.3 Failure Criterion of a Laminate

1. Basic concepts and terminologies

Under the combined action of membrane and bending loads, layers will have different levels of stress, not to mention different constituents and mixtures, and giver orientations. That is, layers will have different SR's and different modes of failure.

Failure of a single layer does not lead to failure of the laminate. This is a huge advantage of lamiantes over isotropic materials.

<u>First-ply failure (FPF) and first-ply failure load:</u>
FPF refers to the phenomenon that one layer (or some layers) fails (or fail) before others.

FPF Load refers to the load level that causes FPF. This load equals the applied load times the SR of the laminate at FPF.

In general, the laminate will be able to continue to take on increased load, and more layers will fail, in a sequence (hence second-ply failure, …, and so on), unit the laminate fails, based on some pre-selected failure criteria.

<u>Ultimate-ply failure (UPF) and ultimate-ply failure load:</u>
UPF refers to when the load on the laminate is at such level that the laminate is considered failed, based on the pre-selected failure criterion. The load level that causes UPF is known as the **UPF Load.**

The process of layers in a laminate fail in some sequence as the load is increased is known as **progressive failure.**

<u>Last-ply failure (LPF) and last-ply failure load:</u>
If the progressive failure continues until the last ply (or plied) fails (or fail), the phenomenon is known as **LPF** and the corresponding load level is the **LPF Load.**

2. What determines "a laminate fails"

Termination criterion is used to determine if a laminate fails. A termination criterion can be,

- If fibers fail in tension (1T);

- If fibers fail (in tension or under compression 1T or 1C); or
- If a certain number of layers fail. Typical choice is 50% of layers, but it can be of a higher value, say 100%. Setting the value to 100% in fact gives rise to **LPF.**

3. What to do with a failed lamina?

- Originally occupied space by a failed lamina remains occupied by it; that is, the z coordinates of layers are unchanged during the progressive failure analysis.
- The failed lamina's stiffness and strengths will be discounted; the discount can be a total discount (e.g., $E_2 = 0$) or partial discount (e.g., $E_2 = 10\%$ of before-failure value).

It should be noted that answers to, (1) what termination criterion to use; and (2) how to discount a failed lamina, are not entirely technical, and far from definitive.