2.8.1 Max. Stress Failure Theory

Given $\sigma_!, \sigma_2, and\ \tau_{12}$, failure of the lamina occurs when any one of the following is true,

$$\sigma_1 > (\sigma_1^T)_{ult} \quad \text{if } \sigma_1 \geq 0$$
$$\sigma_1 < -\left(\sigma_1^C\right)_{ult} \quad \text{if } \sigma_1 < 0$$
$$\sigma_2 > (\sigma_2^T)_{ult} \quad \text{if } \sigma_2 \geq 0$$
$$\sigma_2 < -\left(\sigma_2^C\right)_{ult} \quad \text{if } \sigma_2 < 0$$

In terms of SR (strength ratio), the theory reads

$$SR_1 = (\sigma_1^T)_{ult}/\sigma_1 \qquad \sigma_1 \geq 0$$
$$SR_1 = -\left(\sigma_1^C\right)_{ult}/\sigma_1 \qquad \sigma_1 < 0$$
$$SR_2 = (\sigma_2^T)_{ult}/\sigma_2 \qquad \sigma_2 \geq 0$$
$$SR_2 = -\left(\sigma_2^C\right)_{ult}/\sigma_2 \qquad \sigma_2 < 0$$
$$SR_6 = (\tau_{12})_{ult}/|\tau_{12}|$$

The minimum of all SR's is the SR of the lamina.
e.g. if SR_6 is the minimum, then the ST of the lamina is SR_6, and mode of failure is $6S$.

2.8.4 Max Strain Failure Theory

Given $\sigma_1, \sigma_2\ and\ \tau_{12}$, then

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

And failure occurs when any of the following is true,

$$\varepsilon_1 > (\varepsilon_1^T)_{ult} \qquad \varepsilon_1 \geq 0$$
$$\varepsilon_1 < -\left(\varepsilon_1^C\right)_{ult} \quad \varepsilon_1 < 0$$
$$\varepsilon_2 > (\varepsilon_2^T)_{ult} \qquad \varepsilon_2 \geq 0$$
$$\varepsilon_2 < -\left(\varepsilon_2^C\right)_{ult} \quad \varepsilon_2 < 0$$
$$|\gamma_{12}| > (\gamma_{12})_{ult}$$

In terms of SR,

$$SR_1 = (\varepsilon_1^T)_{ult}/\varepsilon_1 \qquad \varepsilon_1 \geq 0$$
$$SR_1 = -\left(\varepsilon_1^C\right)_{ult}/\varepsilon_1 \qquad \varepsilon_1 < 0$$
$$SR_2 = (\varepsilon_2^T)_{ult}/\varepsilon_2 \qquad \varepsilon_2 \geq 0$$
$$SR_2 = -\left(\varepsilon_2^C\right)_{ult}/\varepsilon_2 \qquad \varepsilon_2 < 0$$
$$SR_6 = (\gamma_{12})_{ult}/|\gamma_{12}|$$

Where

$$(\varepsilon_1^T)_{ult} = (\sigma_1^T)_{ult}/E_1$$
$$\left(\varepsilon_1^C\right)_{ult} = \left(\sigma_1^C\right)_{ult}/E_1$$
$$(\varepsilon_2^T)_{ult} = (\sigma_2^T)_{ult}/E_2$$
$$\left(\varepsilon_2^C\right)_{ult} = \left(\sigma_2^C\right)_{ult}/E_2$$
$$(\gamma_{12})_{ult} = (\tau_{12})_{ult}/G_{12}$$

2.8.5 Tsai-Hill Theory (Distortion Energy)

Given σ_1, σ_2, and τ_{12}, and they are increased proportionally to σ_1^f, σ_2^f and τ_{12}^f then failure occurs when

$$\left(\frac{\sigma_1^f}{F_1}\right)^2 - \left(\frac{\sigma_1^f}{F_2}\right)\left(\frac{\sigma_2^f}{F_2}\right) + \left(\frac{\sigma_2^f}{F_3}\right)^2 + \left(\frac{\tau_{12}^f}{F_4}\right)^2 \geq 1$$

To find SR,

$$\sigma_1^f = SR \cdot \sigma_1$$
$$\sigma_2^f = SR \cdot \sigma_2$$
$$\tau_{12}^f = SR \cdot \tau_{12}$$

Such that,

$$SR = \frac{1}{\sqrt{\left(\frac{\sigma_1^f}{F_1}\right)^2 - \left(\frac{\sigma_1^f}{F_2}\right)\left(\frac{\sigma_2^f}{F_2}\right) + \left(\frac{\sigma_2^f}{F_3}\right)^2 + \left(\frac{\tau_{12}^f}{F_4}\right)^2}}$$

Original Tsai-Hill (Eq. 2.150):

$$F_1 = F_2 = (\sigma_1^T)_{ult}$$
$$F_3 = (\sigma_2^T)_{ult}$$
$$F_4 = (\tau_{12})_{ult}$$

Modified Tsai-Hill (Eq. 2.151):

$$F_1 = (\sigma_1^T)_{ult} \quad \sigma_1 \geq 0$$
$$F_1 = \left(\sigma_1^C\right)_{ult} \quad \sigma_1 < 0$$
$$F_2 = (\sigma_1^T)_{ult} \quad \sigma_2 \geq 0$$
$$F_2 = \left(\sigma_1^C\right)_{ult} \quad \sigma_2 < 0$$
$$F_3 = (\sigma_2^T)_{ult} \quad \sigma_2 \geq 0$$
$$F_3 = \left(\sigma_2^C\right)_{ult} \quad \sigma_2 < 0$$
$$F_4 = (\tau_{12})_{ult}$$

Modified Tsai-Hill takes into account:

a) The different strengths in tension and under compression
b) The interaction between σ_1 and σ_2
   $\therefore \left(\frac{\sigma_1}{F_2}\right)\left(\frac{\sigma_2}{F_2}\right)$
   and choices for F_2

2.8.6 Tsai-Wu Failure Theory (Total Strain Energy)

Define:

$$H_1 = \frac{1}{(\sigma_1^T)_{ult}} - \frac{1}{(\sigma_1^C)_{ult}}$$
$$H_2 = \frac{1}{(\sigma_2^T)_{ult}} - \frac{1}{(\sigma_2^C)_{ult}}$$
$$H_{11} = \frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$

$$H_{22} = \frac{1}{(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}$$

$$H_{66} = \frac{1}{[(\tau_{12})_{ult}]^2}$$

And Tsai-Hill:

$$H_{12} = -\left(\frac{1}{2}\right)\frac{1}{[(\sigma_1^T)_{ult}]^2}$$

Hoffman:

$$H_{12} = -\left(\frac{1}{2}\right)\frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$

von Mises-Hencky:

$$H_{12} = -\left(\frac{1}{2}\right)\frac{1}{\sqrt{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}}$$

Then given σ_1, σ_2, and τ_{12}, and assuming they are to be increased proportionally to σ_1^f, σ_2^f, and τ_{12}^f, failure occurs when:

$$H_1\sigma_1^f + H_2\sigma_2^f + H_{11}\left(\sigma_1^f\right)^2 + H_{22}\left(\sigma_2^f\right)^2 + 2H_{12}\left(\sigma_1^f\sigma_2^f\right) + H_{66}\left(\sigma_6^f\right)^2 \geq 1$$

In terms of SR:

$$a(SR)^2 + 2b(SR) - 1 = 0$$

Where:

$$a = H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + 2H_{12}\sigma_1\sigma_2 + H_{66}\sigma_6^2$$
$$b = \left(\frac{1}{2}\right)(H_1\sigma_1 + H_1\sigma_2)$$

SR is the root (one of the roots) of the quadratic equation.
Compared with Tsai-Hill theory, Tsai-Wu theory considers:

a) The so-called 1st order effects;
b) The interaction between σ_1 and σ_2 with more sophistication.

Example 3

A unidirectional graphite/epoxy lamina ($\theta = 50°$) is subject to $\sigma_x = \sigma$, $\sigma_y = -\sigma$, and $\tau_{xy} = 0$ (where σ is in Pa). Find the allowable σ, using (1) the max. stress theory; (2) the max strain theory; (3) the Tsai-Hill theory; and (4) the Tsai-Wu theory. Also indicate the mode of failure where available. Set $SR = 2$.

Given, for the lamina,

$E_1 = 181\ GPa$

$E_2 = 10.3\ GPa$

$v_{12} = 0.28$

$G_{12} = 7.2\ GPa$

$(\sigma_1^T)_{ult} = 1500\ MPa$

$(\sigma_1^C)_{ult} = 500\ MPa$

$(\sigma_2^T)_{ult} = 40\ MPa$

$(\sigma_2^C)_{ult} = 245\ MPa$

$(\tau_{12})_{ult} = 70\ MPa$

Solution:

From Example 2, local stresses and strains are,

$$\begin{Bmatrix} -0.1736\sigma \\ 0.1736\sigma \\ -0.9848\sigma \end{Bmatrix} \text{ and } \begin{Bmatrix} -0.001228\sigma \\ 0.01713\sigma \\ -0.1368\sigma \end{Bmatrix} (10^{-9})$$

(1) Maximum stress theory

$$SR_1 = -\frac{(\sigma_1^C)_{ult}}{\sigma_1} = -\frac{500(10^6)}{(-0.1736\sigma)} = 2$$

So, $\sigma = 1440\ MPa$

$$SR_2 = \frac{(\sigma_2^T)_{ult}}{\sigma_2} = -\frac{40(10^6)}{(0.1736\sigma)} = 2$$

So, $\sigma = 115.2\ MPa$

$$SR_3 = \frac{(\tau_{12})_{ult}}{|\tau_{12}|} = -\frac{70(10^6)}{(0.9848\sigma)} = 2$$

So, $\sigma = 35.54\ MPa$

Therefore, $\sigma_{all} = 35.54\ MPa$ and the lamina's mode of failure is $6S$.

(2) Maximum strain theory

$$(\varepsilon_1^C)_{ult} = \frac{(\sigma_1^C)_{ult}}{E_1} = 0.00276$$

$$SR_1 = -\frac{(\varepsilon_1^C)_{ult}}{\varepsilon_1} = -\frac{0.00276}{-0.001228\sigma(10^{-9})} = 2$$

So, $\sigma = 1124\ MPa$

$$(\varepsilon_2^T)_{ult} = \frac{(\sigma_2^T)_{ult}}{E_2} = 0.00388$$

$$SR_2 = \frac{(\varepsilon_2^T)_{ult}}{\varepsilon_2} = -\frac{0.00388}{0.01713\sigma(10^{-9})} = 2$$

So, $\sigma = 113.3\ MPa$

$$(\gamma_{12})_{ult} = \frac{(\tau_{12})_{ult}}{|G_{12}|} = 0.00972$$

$$SR_6 = \frac{(\gamma_{12})_{ult}}{|\gamma_{12}|} = -\frac{0.00972}{0.1368\sigma(10^{-9})} = 2$$

So, $\sigma = 35.53\ MPa$

Therefore, $\sigma_{all} = 35.53\ MPa$ and the mode of failure of the lamina is $6S$.

(3) Tsai-Hill theory
Modified Tsai-Hill:

$$2 = \frac{1}{\sqrt{2.16935(10^{-16})\sigma^2}}$$

$\sigma_{all} = 33.95\ MPa$

(4) Tsai-Wu theory
Tsai-Hill form:
$a = 2.01058(10^{-16})\sigma^2$
$b = 1.93198(10^{-9})\sigma$

The quadratic equation is:
$a(2^2) + 2b(2) - 1 = 0$

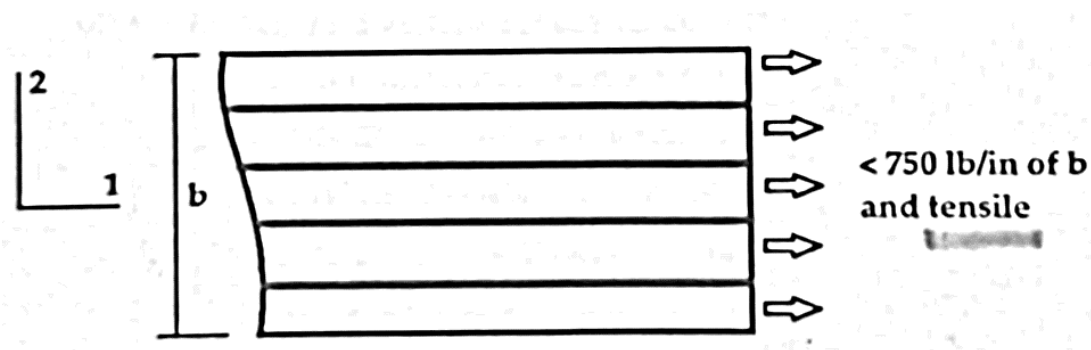
And $\sigma_{all} = 30.78\ MPa$

Chapter 4: Macromechanical Analysis of Laminates

4.1 Introduction

Why laminate?

1. A single lamina (or layer, ply)
- 0.005" or 0.125 mm thick → not suitable as an engineering component;
- 750-lb per inch of width along fiber direction → not high enough for engineering application;



2. Unidirectional laminate (which has many layers, but fibers take the same direction)
- Transverse direction: rather weak;
- Fiber direction: compressive strength is low;
- Laminate be loaded along fiber direction by tensile load, which limits its applicaions

3. Optimal solutions

Having layers stacked with different
- Angles
- Thickness
- Position (top, …, middle, …, bottom)
- Constituents

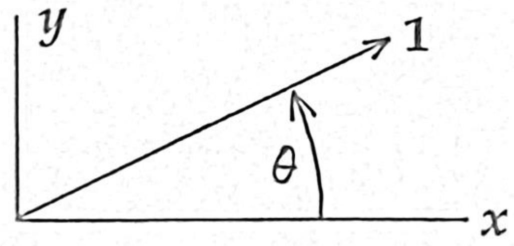
4.2 Laminate Code

Laminate code, also known as **Layup Sequence**, or **Stacking Sequence**, is a set of notation or convention to describe how layers (or piles, laminas) are stacking on top of one another.

The notation or convention is yet to be standardized.

1. Reference Axis

CCW angles (from x) are considered positive.



2. Layer Numbering

Top to bottom	more common
Bottom to top	easier for manual layup

3. Layers with Identical Constituents and Uniform Thickness

A) Long hand notation

$[\ \theta_1 /\ \theta_2 /\ \theta_3 /\ \ldots\ / \theta_N]$ where the θ's are in degrees.

For example, $[0\ /\ 90\ /\ 45\ /\ 90\ /\ 0\ ]$

The commonly used angles are, starting with the most preferred to the least $0°, 90°$
$\pm 45°$
$\pm 30°,\quad \pm 60°$
$\pm 15°,\quad \pm 75°$

B) Repeated orientation
$[0\ /\ 0\ /\ 90\ /\ 90\ ] \rightarrow [0_2 /\ 90_2]$
$[0\ /\ 90\ /\ 0\ /\ 90] \rightarrow [(0\ /\ 90)^2]$
$[0\ /\ 0\ \ /\ 90\ /\ 45\ /\ 90\ /\ 90\ /\ 45\ /\ 90\ /\ 0\ /\ 0] \rightarrow [0_2\ /\ (90\ /\ 45\ /\ 90)_2\ /\ 0_2]$

C) Balanced laminate
For every occurrence of "$+\theta$" (other than $0°$ and $90°$), a "$-\theta$" is placed, either adjacent to the "$+\theta$" layer, or separated by some layers.

$[\ 45\ /\ 45\ /\ 0\ /\ 45\ /\ -45] \rightarrow [45_2\ /\ 0\ /\ -45_2]$
$[45\ /\ -45\ /\ 0\ /\ 45/\ -45] \rightarrow [\pm 45\ /\ 0\ \ /\ \pm 45]$
$[45\ /\ -45\ /\ 0\ /\ -45\ /\ 45]$
$[45\ /\ -45\ /\ -45\ /\ 45\ /\ 45\ /\ -45\ ] \rightarrow [\pm \mp \pm 45]$

D) Symmetry
Symmetry means fiber orientations of the top half of the laminate are mirror image of the bottom half.
A symmetric laminate can have even or odd number of layers.

D.1) Even number of layers.
$[0\ /\ 0\ /\ 90\ /\ 45\ /\ 90\ /\ 90\ /\ 45\ /\ 90\ /\ 0\ /\ 0\ ] \rightarrow [0\ /\ 0\ /\ 90\ /\ 45\ /\ 90\ ]_s \rightarrow [0_2\ /\ 90\ /\ 45\ /\ 90]_s$

Notes:

- Only the top half of the sequence is notated
- s (subscript) is to indicate only a symmetric half of the entire sequence is given

D.2) Odd number of layers
$[0 \;/\; 90 \;/\; 0] \rightarrow [0 \;/\; \overline{90}]$

Note: the overbar indicates the layer about whose mid-plane the laminate is symmetric.
$[0 \;/\; 90 \;/\; 45 \;/\; 90 \;/\; 0] \rightarrow \left[0 \;/\; 90 \;/\; \overline{45}\right]$
$[0 \;/\; 0 \;/\; 90 \;/\; 45 \;/\; 90 \;/\; 90 \;/\; 45 \;/\; 90 \;/\; 0 \;/\; 0] \rightarrow [0_2 \;/\; 90 \;/\; 45 \;/\; 90]_s \rightarrow \left[0_2 \;/ (\; 90 \;/\; \overline{45}\;)\right]_s$
$[45 \;/\; -45 \;/\; 0 \;/\; -45 \;/\; 45] \rightarrow [\pm 45 \;/\; \bar{0}\;]$

Anti-symmetry means that fiber orientations of the top half of the laminate are opposite those of the bottom half.

In the context of laminate code, 90° is considered “opposite” of 0°, and vice verse.

An anti-symmetric laminate always has even number of layers.

$[0 \;/\; 90 \;/\; 0 \;/\; 90] \rightarrow [(0 \;/\; 90)_2]$
$[45 \;/\; -45 \;/\; 45 \;/\; -45] \rightarrow [\pm 45_2]$
$[45 \;/\; 30 \;/\; 35 \;/\; -45 \;/\; -30 \;/\; -45] \rightarrow \pm[(45 \;/\; \overline{30})]$
$[45 \;/\; -45 \;/\; -45 \;/\; 45 \;/\; 45 \;/\; -45] \rightarrow [\pm \mp \pm 45]$

4. Layers with Identical Constituents but Non-uniform Thickness
Two options:

- Long-hand notation, with thickness as subscript
- Spelling out the detail, in English

For example,

The laminate code is $[(0 \;/\; 90)_2 \;/\; \bar{0}]$ where the 0° layers have a thickness of $0.2\ mm$ each, and the 90° layers have a thickness of $0.25\ mm$ each.

Where:
$[(0 \;/\; 90)_2 \;/\; \bar{0}] \rightarrow [0 \;/\; 90 \;/\; 0 \;/\; 90 \;/\; 0 \;/\; 0 \;/ 90 \;/\; 0 \;/\; 90]$

$0.2\ mm \cdot 5 = 1\ mm$
$0.25\ mm \cdot 4 = 1\ mm$

Then total thickness is $2\ mm$

5. Hybrid Laminates
These are laminates whose layers are of different constituent materials.

$\left[0^{\mathrm{K}} \;/\; 0^{K} \;/\; 45^{C} \;\;/\; -45^{C} \;/ 90^{\mathrm{G}} \;/\; -45^{C} \;/\; 45^{C} \;/\; 0\text{^}\mathrm{K} \;/\; 0\text{^}\mathrm{K}\right] \rightarrow \left[0_2^{K} \;/\; \pm 45^{C} \;/\; \overline{90}^{G} \;\;\right]$
$[0^{\mathrm{K}} \;/\; 0^{\mathrm{K}} \;\;/\; 45^{C} \;/\; -45^{G} \;/\; 90^{G} \;/\; -45^{G} \;/\; 45^{C} \;/\; 0^{\mathrm{K}} \;/\; 0^{\mathrm{K}}\;] \rightarrow \left[0_2^{K} \;\;/\; 45^{C} \;/\; -45^{G} \;/\; \overline{90^{G}}\right]$

6. Brain Teaser

Given the following 22-later sequence, write the shortest possible code.
$[45 \;/\; -45 \;/\; 0 \;/\; 0 \;/\; 45 \;/\; -45 \;/\; 0 \;/\; 0 \;/\; 90 \;/\; 0 \;/\; 0 \;/\; 0 \;/\; 0 \;\;/\; 90 \;/\; 0 \;/\; 0 \;/\; -45 \;/\; 45 \;/\; 0 \;/\; 0 \;/\; -45 \;/\; 45\;]$

Solution:
$[\;(\pm 45 \;/\; 0_2)_2 \;/\; 90 \;/\; 0_2]_s$ `

7. More Terminologies

A) Unidirectional laminates: laminates in which all layers have the same θ.
For example, $[0_6]$ or $[0]_6$ and $[45]_{10}$

B) Cross-ply laminates: laminates in which the layers take angles of 0° and 90° only.
For example, $[0\ /\ \overline{90}]$, $[0\ /\ 90\ /\ 0\ /\ 90] \rightarrow [(0\ /\ 90)_2\ ]$

C) Angle-ply laminates: laminates that consist of pairs of layers of same material (that is, same fiver and matrix, and same mixture) and thickness, and oriented at $+\theta$ and $-\theta$.
For example,
$[45\ /\ -45\ /\ 45\ /\ -45]$
$[45\ /\ -45\ /\ -45\ /\ 45\ /\ 45\ /\ -45]$
$[45\ /\ 30\ /\ 45\ /\ -45\ /\ -30\ \ -45]$

4.3 Stress-Strain Relations for a Laminate
4.3.1: $\sigma - \varepsilon$ relation for a one-dimensional isotropic beam
4.3.2 $\sim$ 4.3.4: Classical laminated plates theory (CLPT)
Key features:

- Membrane stretching is considered
- Bending is considered
- The two actions are not kinematically coupled, but kinetically coupled
- Transverse shear is not considered

Other laminated plate theories:
Membrane & bending actions are coupled
Transverse shear is considered

$$\begin{Bmatrix} 1st\ order\ theory \\ 2nd\ order\ theory \\ higher\ order\ theory \\ zigzag\ theory \\ \ldots \end{Bmatrix}$$