Chapter 1 – Introduction to Composite Materials

1.1 Introduction
1.2 Classification
1.3 Recycling (reading only)
1.4 Mechanics Terminology (review only)

1.1 Introduction

1) **Four main categories of engineering materials**
2) **What's a composite?**
3) **Qualitive comparison of engineering materials**
4) **Quantitative comparison of engineering materials**
5) **Markets of composites**

1) Four main categories of engineering materials

- Metals
- Plastics (or Engineering Polymers)
- Ceramics
- Composites

2) What's a composite? (p. 2)

A composite is a material that consists of 2 or more distinct constituent materials.

- Constituent materials have to be *significantly* different such that the properties of the composite are noticeably different from those of its constituents.
- Typically, one constituent is stiff and provides strength; the other is softer, and is to embed or support the stiff constituent.
- The stiff constituent is known as the reinforcing phase while the soft supporting constituent is known as the matrix.
- Reinforcing phase may take different forms: particles, flakes, fibers, etc.

3) Qualitative comparison of engineering

Metals:

- Dominate structural applications;
- Have the longest design & processing history;
- Have good stiffness, strength , thermal stability, temperature resistance, and high thermal and electrical conductivity;
- Are heavy compared with plastics and composites;
- Require several machining operations to obtain the final product;

Plastics:

- Became very common in the 1990's;
- Are light-weight, easy to process, and resistant to corrosion;
- Are not suitable for high-temperature (> 100 °C) applications.

Ceramics:

- Provide great thermal stability and very high hardness;
- Are best suited for high-temperature and high-wear applications;
- Are resistant to most forms of chemical attacks;
- Are very brittle;
- Are difficult to machine.

Composites:

- Natural composites have been utilized for a long time;
- Industrial applications started in the 1960's with the introduction of polymer-based composites;
- Applications include, to list just a few, auto parts, sporting goods, aerospace parts, consumer goods, marine and oil industries;
- They enable part integration;
- They enable in-service monitoring by embedding sensors;
- They enable DFM (design for manufacture) and DFA (design for assembly);
- Properties can be tailor-made (by selecting constituent materials and lay-up sequence, by optimization, for example);
- They have better impact properties;
- They have better NVH (noise-vibration-harshness) characteristics;
- There is a lack of design database, handbooks and history
- Resistance to temperature, solvents and chemicals varies;
- They absorb moisture, compromising or affecting composite's properties and dimensional stability;
- Composites may not be repaired or recycled, depending on the matrix.

4) Quantitative comparison of engineering materials

- In general, the specific strength (strength-to-density ratio) of composites is, approximately, 3 to 5 times that of steel and aluminum;
- Specific stiffness (stiffness-to-density ratio) of composites is ~5 times that of steel, and ~2 times that of aluminum;
- For example, carbon-fiber-reinforced polymers (CFRP's) and Titanium alloys have similar modulus (~130 GPA) and strength (~1000 MPa) but their densities are ~1400 kg/m^3 and ~4400 kg/m^3 respectively.
- Relative cost is defined as:

$$C_R = \frac{\$\, per\ kg\ of\ material}{\$\, per\ kg\ of\ mild\ steel\ rod}$$

- For example, relative costs are, ~20 for CFRPs and ~70 for Ti-alloys.
- Sec. 5.4 of text has 4 examples of laminated design, showing that the savings in mass or weight over metals range 50% to 75%

5) Markets of composites (pp. 16-17)

- The text has stats 1990-1995;
- 2004's market was as follows (U.S. stats):
    - Transport, 32%
    - Construction, 20%

- Corrosion-resistant apps, 12%
    - Marine, 10%
    - Electrical, 10%
    - Consumer, 7%
    - Appliance, 5%
    - Aircraft, 1%
    - Others, 3%
- Total consumption (in 2004) was 4.0 billion lbs.
- R. MacNeil, U.S. Composites Market Outlook for 2005 and Beyond, *Composites Manufacturing*, Jan. 16-29, 2005.

1.2 Classification

The sections covers classification as well as manufacture of gibers, and applications of composites **(pp. 16—50)**.

Manufacture of composites can be subject or course in itself.

1. **Wood as a natural composite**
2. **Classification**
3. **Modern/Advanced/Man-made composites**
4. **Fibers used in advanced composites**
5. **Matrix materials**

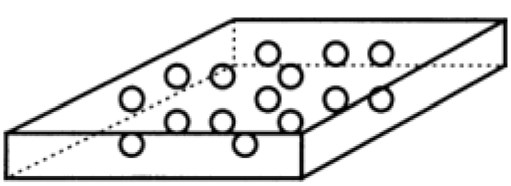
1. Wood

- Is the most commonly used natural composite;
- The 2 constituents are
  Fibers: long and stiff
  Cells: soft and to embed the fibers
- Modern/Advanced/Man-made composites imitate wood: strong reinforcing phase(s) embedded in softer supporting material(s).

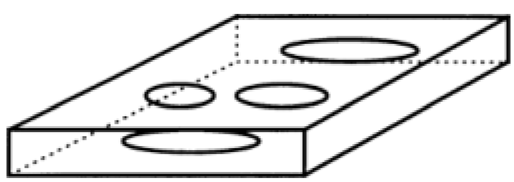
2. Classification based on the form of reinforcing phase (pp. 16-19 and Fig. 1.8)

a. <u>Particulate:</u> randomly dispersed particles in a soft matrix. (like cement)

b. <u>Flake:</u> randomly dispersed flakes or aligned flakes in a soft matrix (like glass)



Particulate composites



Flake composites

c. <u>Fiber-reinforced:</u>

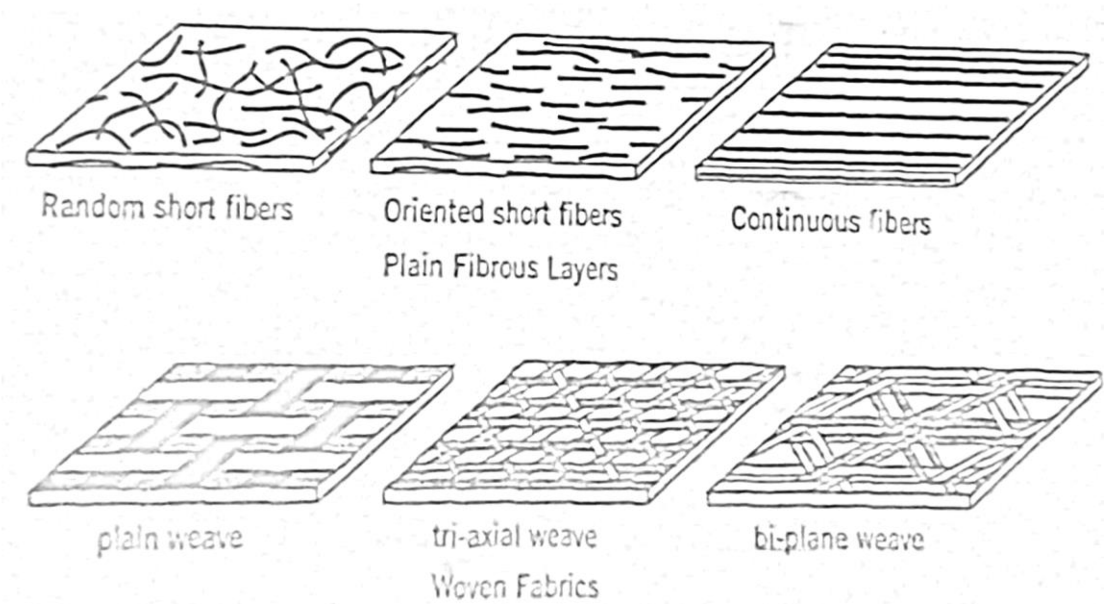
Diameter: $0.0001" \sim 0.005"$

Length: $L \leq 100D$, short fibers

Length: $L > 100D$, long fibers

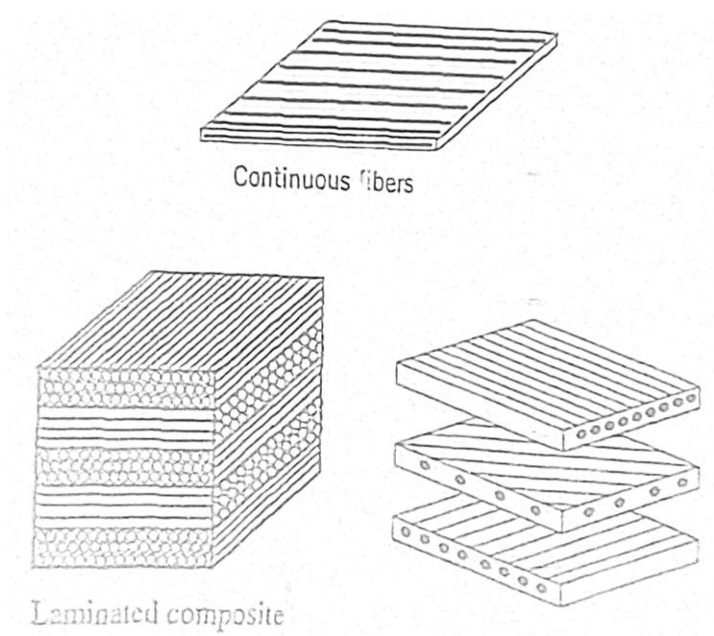
- Short fibers randomly oriented as in metals;

- Short fibers aligned in one direction;
- Long fibers aligned in one direction;
- Long fibers aligned in a few directions (weave).



d. Fiber-reinforced laminated:

- Thin layers of long and unidirectional fibers are the building blocks. A thin layer is known as a lamina or a ply;
- Layers of different materials, thicknesses and orientations are bonded together, as per the layer sequence. The result is the so-called laminate, or fiber-reinforced laminated composite;
- Fiber-reinforced laminated composites are commonly used in the design of the so-called high-performance components and structures.



3. Modern/Advanced/Man-made composites

- Imitating wood - strong fibers embedded in softer supporting materials
- Fibers:
  - Long (continuous) or short (discontinuous);
  - Aligned or random orientations
  - G – glass fibers;
  - C – carbon fibers including graphite fibers;
  - K – aramid fibers (Kevlar fibers; Kevlar is a trademark)
  - (…… etc)
- Matrix:
  - Polymers (→ PMCs polymer matrix composites);
  - Metals (→ MMCs);
  - Ceramics (→ CMCs);
  - (…… etc)

4. Fibers used in advanced composites

a. Glass fibers:

Most commonly used
High strength but low stiffness
Low cost
Insulating
Low CTE (coefficient of thermal expansion)
Poor abrasion resistance
(Manufacture of glass fibers: p. 22, Fig 1.9)
Types of glass fibers:
E – electrical (E-glass)
S – silica (S-glass)
C – corrosion
A – appearance
(…… etc) And their combinations
E-glass vs. S-glass

- They are more common than other glass fibers;
- E-glass is more common than S-glass;
- Compared with E-glass, S-glass has ~20 to 25% higher strength and stiffness, ~10% higher CTE, and is ~7 to 8 times as expensive;
- Table 1.6 (p. 21)

TABLE 1.6

Comparison of Properties of E-Glass and S-Glass

Property	Units	E-Glass	S-Glass
System of units: USCS			
Specific gravity	—	2.54	2.49
Young's modulus	Msi	10.5	12.4
Ultimate tensile strength	ksi	500	665
Coefficient of thermal expansion	µin./in./°F	2.8	3.1
System of units: SI			
Specific gravity	—	2.54	2.49
Young's modulus	GPa	72.40	85.50
Ultimate tensile strength	MPa	3447	4585
Coefficient of thermal expansion	µm/m/°C	5.04	5.58

b. Carbon and graphite fibers:

They are the so-called high-performance fibers (mainly for aerospace apps; lately used in auto-industry, civil infrastructures, offshore oil industry, etc.)

Carbon fibers are classified as, based on the precursors:

- PAN (poly-acrylo-nitrile, being most common)
- Pitch (bitumen)
- Rayon
- (……etc)

Manufacture of carbon fibers: p. 25, Fig. 1.11 (PAN-based carbon)

Table 1.8 (p. 25) compares PAN-based bs. Pitch-based carbon fibers.

CTE (coefficient of thermal expansion) is negative in longitudinal (axial) as well as radial directions.

Carbon fibers typically have a carbon content of 93-95%. If the carbon content gets to 99%, then carbon fibers become graphite fibers.

Graphite fibers vs. Carbon fibers;

- Graphite fibers have higher stiffness and strength, but higher cost as well.
- Processing temperature is 1900 $°C$ (3400 $°F$) compared with 1300 $°C$ (2400 $°F$) for carbon fibers.
- Graphite fibers are typically used in aircraft and aerospace applications.

c. Aramid fibers (Kevlar® fibers):

Kevlar® is the registered trademark of DuPont;

- Standard K-fibers include: K-29, K-49, K-129, and K-149 (or K29, K49, K129 and K149);
- Kevlar® 29 AP and Kevlar® 49 AP have higher performance than their respective standard counterparts;
- Kevlar® XP is the light weight version of the K-fibers, used for helmets and armors, for example.

Manufacture of Kevlar® fibers: ???

Table 1.9 (p. 26) compares K29 and K49

TABLE 1.9

Properties of Kevlar Fibers

Property	Units	Kevlar 29	Kevlar 49
System of units: USCS			
Specific gravity	—	1.44	1.48
Young's modulus	Msi	9	19
Ultimate tensile strength	ksi	525	525
Axial coefficient of thermal expansion	µin./in./°F	–1.111	–1.111
System of units: SI			
Specific gravity	—	1.44	1.48
Young's modulus	GPa	62.05	131.0
Ultimate tensile strength	MPa	3620	3620
Axial coefficient of thermal expansion	µm/m/°C	–2	–2

CTE is negative in longitudinal (axial) direction and positive in radial direction.

d. Other fibers:

Boron fibers:

- Strength and stiffness are at the same level as carbon;
- Fiber diameters (About 140 μm or 0.0055") are ~10 times those of carbon fibers;
- They are 300 times as expensive as E-glass;
- They can take higher buckling load.

5. Matrix materials

Functions of a matrix:

- Hold/embed fibers;
- Transmit forces between fibers;
- Protect fibers from the environment

a. Polymers

They are the common choice of a resin/matrix material.

When is a polymer a resin or a matrix?

- PMC's (Polymer-matrix-composites)
- Resin refers to polymer before and during processing
- Matrix refers to polymer after it is cured or solidified

Typical resins:

- Thermoset (e.g. epoxy, polyester, etc.);
- Thermoplastic (e.g. PEEK PPS, etc.).

Epoxy is the most commonly used resin/matrix.

b. Metals

Why metal matrix?

- MMCs (metal matrix composite)
- MMCs are insensitive to moisture
- MMCs have better resistance to wear and tear, to fatigue

Typical metals and fibers used in MCCs:

- Fibers: carbon, boron, SiC (silicon carbide)
- Metals: aluminum, magnesium, titanium
    - Boron-aluminum, for example

c. Ceramic

- CMCs (ceramic matrix composites)

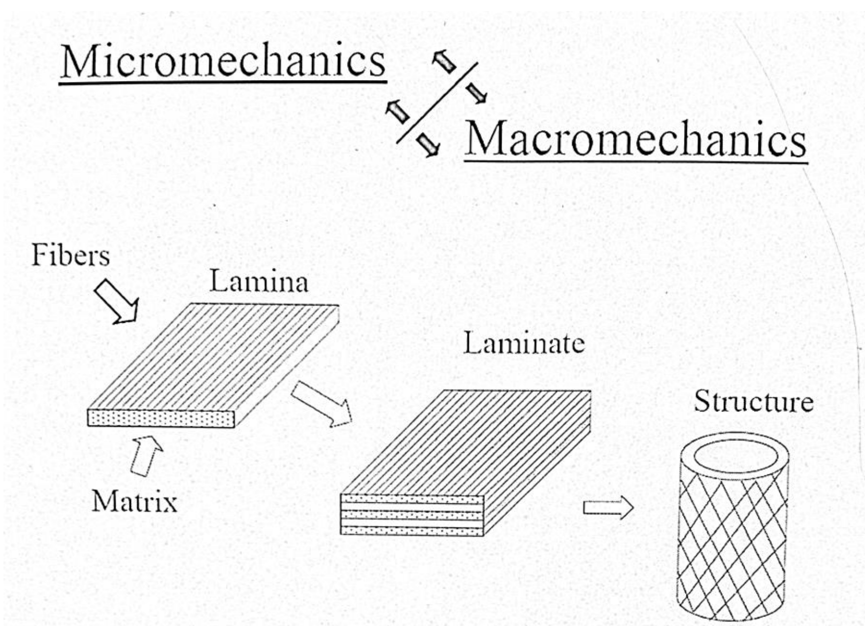
Typical CMCs:

- Carbon-ceramic
- SiC-ceramic

When ceramic is used as a material matrix to form CMCs fibers are to supplement ceramic, to reduce the brittleness of ceramics in particular.

Micromechanical Analysis (or Micromechanics – Chapter 3)

An analysis that starts with the properties (elastic moduli, strength, CTE, CME, etc.) of the constituent materials, and finds like properties of a unidirectional lamina.



Macromechanical Analysis (or Macromechanics)

A study of the stress-strain behavior of composites, using properties of unidirectional laminas found from micromechanics.

Level 1:
laminate level, Ch. 4

Level 2:
Structural level
beams (Ch. 6), plates (file on D2L), shells, etc.

Chapter 2 – Macro-Mechanical Analysis of a Lamina

2.1 Introduction

A lamina is typically in the order of $0.127\ mm$ or $0.005''$ thick and not isotropic

2.2 Review

1. Stress
2. Strain
- Normal
- Shear
3. Elastic moduli
4. Strain energy

2.3 Hooke's Law for Different Types of Materials

1. Homogeneous vs. heterogeneous materials
FGM – functionally graded materials (properties at different locations are different)

2. Anisotropic vs. isotropic materials
Isotropic materials are those whose properties are orientation independent.
For example, steel, aluminum.
They require 2 independent mechanical property constants. (Young's modulus, poisons ratio or Young's modulus, shear modulus)

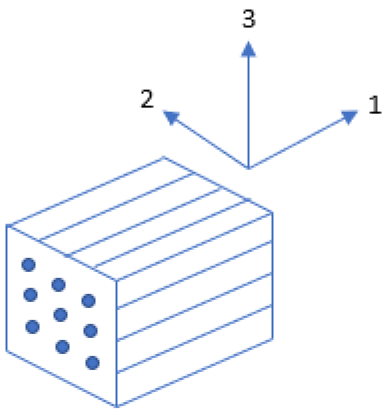
Anisotropic materials are those whose properties are orientation dependent.
For example, natural wood.
They require 21 independent constants, see (Eq. 2.25 – only looking at triangular matrix, it's symmetric)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}, \qquad (2.25)$$

3.Special Cases of Anisotropic Materials

3.1 Orthotropic Materials
They possess 3 mutually perpendicular principal planes, or 3 principal directions mutually perpendicular to each other.

For example, unidirectional continuous fiber-reinforced composite blocks are considered orthotropic.
Orthotropic materials require 9 independent constants.



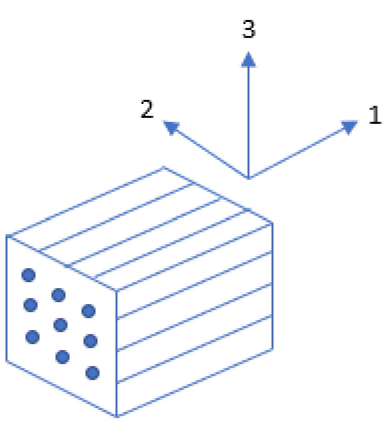
(3 axes and planes – first plane in direction of fiber)
1 – direction of fibers
2, 3 – perpendicular to fibers

E_1 E_2 E_3: Young's moduli
G_{12} G_{23} G_{13}: Shear moduli

Note:
$G_{ij} = G_{ji}$
But:
$v_{ij} \neq v_{ji}$ (but they are related)
Since:
$\frac{v_{ij}}{E_i} = \frac{v_{ji}}{E_j}$

3.2 Transversely isotropic materials
An orthotropic material is called transversely isotropic if one of its principal planes is a plane of isotropy that is, material properties are symmetric about one of its principal axes.

For example,
- Carbon/graphite and aramid fibers are transversely isotropic.
- Unidirectional continuous fiber-reinforced composites, when fibers are packed in a hexagonal way or very close to it, can be considered transversely isotropic.



Transversely isotropic materials require 5 independent constants.

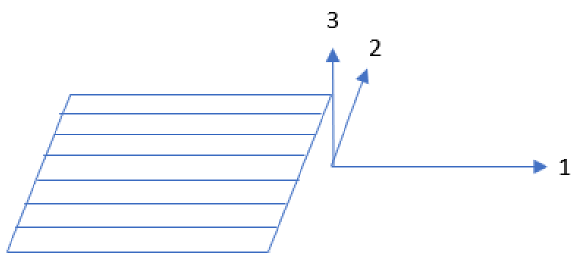
Axis 1 being the axis of symmetry
Axis 2, 3 being the plane of isotropy

E_1 $\quad E_2 = E_3$: Young's moduli
$G_{12} = G_{31}$ $\quad G_{23}$: Shear moduli
$v_{12} = v_{13}$ $\quad v_{23}$: Poisson's ratio

Also, E_2 G_{23} v_{23} are related.

3. Plane stress vs. Plane Strain
Thin unidirectional continuous fiber-reinforced composite layers become a plane stress case.

4 independent mechanical properties are required.



E_1 E_2 G_{12} v_{12}

Chapter 3 Micromechanical Analysis of a Lamina
7 Sections in total (pp. 203-314)
3.1 Introduction
3.2 Volume and Mass Fractions, Density, and Void Content
3.3 Evaluation of Elastic Moduli (pp. 215-270)
3.4 Ultimate Strengths (pp. 271-295)
3.5 CTE
3.6 CME
3.7 Accuracy

3.1 Introduction
1. Determination of the 4 elastic moduli (3.3), 5 ultimate strengths (3.4), and 4 transport properties (3.5, 3.6) by experimental means is costly; experimental results are also limited or restricted.

2. Transport properties include, CTE in 1- and 3- directions, and CME in 1- and 2- directions, respectively.

3. Unidirectional lamina is NOT homogenous. However, it's customary to assume homogeneity once micro-analysis is completed.

4. Lamina is the building block of composites. This is true from analysis as well as physical perspectives.

3.2 Volume and Mass Fractions, Density, and Void Content
- to quantify how much fibers and matrix there are;

- by volume fractions, or by mass (or weight) fractions;

- volume fractions and mass/weight fractions are related;

- void content can't be avoided in reality, but complicates analysis;

1. Volume Fractions
Consider the case of composite having one type of fibers, one type of matrix, and some voids.

$v_{c,f,m,v}$: Volumes of composite, fibers, matrix and voids respectively (eg. in^3).

Then volume fraction of fibers is:
$V_f = v_f/v_c$

Then volume fraction of matrix is:
$V_m = v_m/v_c$

Volume fraction of voids is:
$V_v = v_v/v_c$

$\therefore V_f + V_m + V_v = 1$

(Thus, $V_f + V_m = 1$ implies zero void content.)

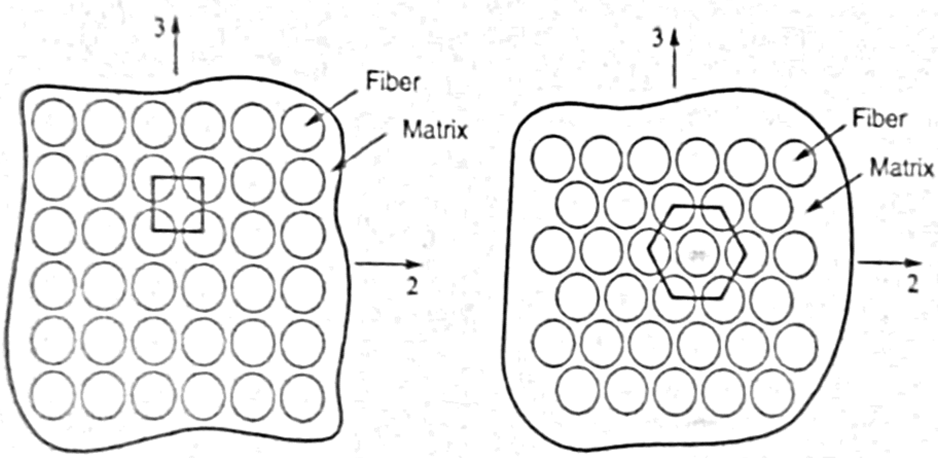
2. Limits on V_f
V_f has some upper limit, from the theoretical as well as the practical perspectives.

2.1 Theoretical Limit on v_F or v_{fmax} to be examined by RVE
RVE = Representative Volume Element

Used in determining v_{fmax} as well as elastic moduli and so on (see 3.3)

RVE is the smallest portion of material that is representative of the composite as a whole.

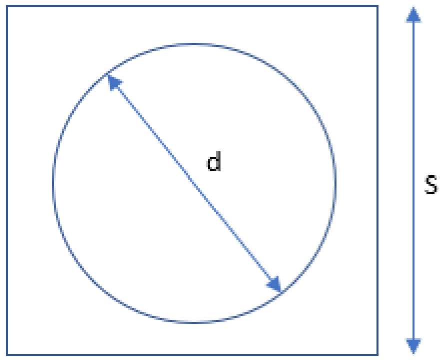
Here, "smallest" means smallest for analysis; it may or may not fit the strict mathematical definition.



(a) Square-packed array (b) Hexagonal-packed array

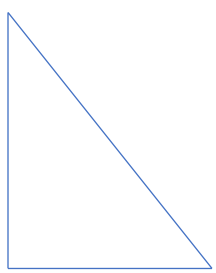
Cross section idealizations for micromechanics studies

For (a), smallest is 1/8 (half of 1/4
For (b), smallest is 1/12 (half of 1/6)



S
d
$l = length/depth\ of\ RVE$

$$v_c = S^2 l$$
$$v_f = \frac{\pi d^2}{4} l$$
$$v_f = \frac{v_f}{v_c} = \frac{\frac{\pi d^2}{4} l}{S^2 l} = \frac{\pi d^2}{4S^2}$$
$$\therefore v_{fmax} = v_f|_{s=d} = \frac{\pi}{4} = 0.785$$



For hexagonal packed:
$$v_f = \frac{\pi d^3}{2\sqrt{3}S^2}$$
$$v_{fmax} = 0.907$$

2.2 Practical Limit
For fiber-reinforced lamina with long and unidirectional fibers, $v_{fmax} = 60\%$
Elementary Materials Science, W.F. Hosford, ASM International, 2013 (Ch. 10, p. 117)

3. Density of the Composite
$\rho_c\ f\ m$: mass densitites (per unit volume) of composite, fibers, and matrix, respectively
$w_c\ f\ m$: masses of composite, fibers, and matrix, respectively

$\therefore w_c = \rho_c v_c \quad w_f = \rho_f v_f \quad w_m = \rho_m v_m$
And $\rho_c v_c = \rho_f v_f + \rho_m v_m$

$\therefore\ \rho_c = \rho_f V_f + \rho_m V_m$: works if there is void content
$V_f + V_m = 1$: works if there is ZERO void content

4. Mass Fractions
mass fraction of fibers is $W_f = w_f / w_c$
mass fraction of matrix is $W_m = w_m / w_c$
$\therefore W_f + W_m = 1$

5. Volume Fractions and Mass Fractions
These fractions are related
$\therefore w_c = \rho_c v_c \quad w_f = \rho_f v_f \quad w_m = \rho_m v_m$

$$\therefore W_f = \frac{w_f}{w_c} = \frac{\rho_f v_f}{\rho_c v_c} = \frac{\rho_f}{\rho_c} V_f$$

$$= \frac{\rho_f V_f}{\rho_f V_f + \rho_m V_m} \qquad (Eq. 1)$$

And:

$$W_m = \frac{\rho_m V_m}{\rho_f V_f + \rho_m V_m} \qquad (Eq. 2)$$

a) Knowing $V_m\ V_f\ \rho_m\ \rho_f$
Then $W_f\ W_m$ solved by (Eq. 1 and Eq. 2)

b) Knowing $W_m\ W_f\ \rho_m\ \rho_f$
Then $V_m\ V_f$ solved by (Eq. 1 or Eq. 2)

c) How to determine V_v or v_v?
mostly by experiments

1) Some specifications for determining void contents
For example:
ASTM D3171-06
Standard Test Method for Constituent Content of Composite Materials

ISO-14127:2008
Composites – Determination of resin, fiber and void content of composites reinforces with carbon fiber.

2) What the text has in terms of determining void content?
pp. 212-215
By means of theoretical density of composite ρ_{ct} and experimental density of composite ρ_{ce} leading to (Eq. 3.16)

$$V_v = \frac{v_v}{v_c} = \frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}}. \quad (3.16)$$

Example 3.2, which in essence is ASTM-D3171

d) Equations in 3.2 of the text (pp. 204-215)

- Some equations are valid only for the case of zero void content
- Condition under which an equation is valid isn't spelled out
- Equations (3.5a), (3.5b), (3.10)

$$W_f = \frac{\frac{\rho_f}{\rho_m}}{\frac{\rho_f}{\rho_m} V_f + V_m} V_f,$$

$$W_m = \frac{1}{\frac{\rho_f}{\rho_m}(1 - V_m) + V_m} V_m. \quad (3.5a, b)$$

$$\frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}. \quad (3.10)$$

e) Corrected volume fractions

In later sections of chapter 3, V_v is assumed to be zero.
When V_v is NOT zero, corrected volume fractions are to be used
$V_f' = \dfrac{v_f}{v_f + v_m}$; $V_m' = \dfrac{v_m}{v_f + v_m}$

3.3 Evaluation of Elastic Moduli

Recalling from 2.3
Orthotropic materials: 9 independent constants
E_1 E_2 E_3: Young's moduli in the 1, 2, and 3 directions, respectively.
G_{12} G_{23} G_{13}: Shear moduli on the 1-2, 2-3, and 3-1 planes respectively.
$v_{12} v_{23}$ v_{31}: Poisson's ratio
1st subscript: strain in the loading direction
2nd subscript: lateral strain

Note:
$G_{ij} = G_{ji}$
But:
$v_{ij} \neq v_{ji}$ (but they are related)
Since:
$\frac{v_{ij}}{E_i} = \frac{v_{ji}}{E_j}$ (but $i \neq j$)

Lamina: 4 independent constants
$E_1\ E_2\ G_{12}\ v_{12}$
v_{12}: major Poisson's ratio
v_{21}: minor Poisson's ratio

Major assumptions
1. Fibers are isotropic
(this assumption is okay with G-fibers; C- and K- fibers are transversely isotropic)

2. Matrix is isotropic

3. Void content is zero
$V_f + V_m = 1$
or use V_f' and V_m'

4. Perfect bonding exists between fibers and matrix
Need to know (i.e., need to determine beforehand)
fibers: $E_f\ v_f\ G_f\ V_f$ (or V_f')
matrix: $E_m\ v_m\ G_m\ V_m$ (or V_m')

(only two are independent)

4 Subsections
3.3.1 Strength of Materials Approach (Mechanics of Materials)
3.3.2 Semi-Empirical Approach
3.3.3 Theory of Elasticity Approach
3.3.4 Transversely Isotropic Fibers (C- and K- fibers are transversely isotropic)

Example 1

A unidirectional fiber-reinforced composite consists of one type of fibers and one type of matrix. The weight fraction of matrix is 0.45. The specific gravity of the fibers and matrix is 2.5 and 1.3, respectively.

a) Find the specific gravity of the composite. Assume zero void content.

b) Find the specific gravity of the composite if $V_v = 5\%$

Solution

a) Given:

$\rho_f = 2.5$

$\rho_m = 1.3$

$W_m = 45\% = 0.45$

$W_f + W_m = 1$

$\therefore W_f = 0.55$

$$(Eq. 1): W_f = \frac{\rho_f V_f}{\rho_f V_f + \rho_m V_m}$$

$\therefore 0.715V_m = 1.125V_f$

$$(Eq. 2): W_m = \frac{\rho_m V_m}{\rho_f V_m + \rho_m V_m}$$

$\therefore 0.715V_m = 1.125V_f$

But we also know that:

$(Eq. 3): V_f + V_m = 1$

Solving:

$V_f = 0.3886$

$V_m = 0.6114$

Thus:

$\rho_c = \rho_f V_m + \rho_m V_m = 1.766$

b) $W_f = 0.55$

$0.715V_m = 1.125V_f$ (still holds true)

$V_f + V_m = 1 - V_v = 0.95$

Solving:

$V_f = 0.3692$

$V_m = 0.5808$

$\rho_c = 1.678$

Example 2 (similar to midterm question – last year)
A unidirectional fiber-reinforced composite consists of two types of fibers (fiber 1 and fiber 2) and one type of matrix. The total volume is v_c. Void content is assumed zero. The volume fractions of the fibers and the matrix are V_{f1} and V_{f2} and V_m respectively. The weight densities (per unit volume) of the fibers and matrix are ρ_{f1} ρ_{f2} and ρ_m respectively. Gravitational acceleration is g.
a) Express the total mass of the composite in terms of ρ_{f1}, ρ_{f1}, ρ_m, V_{f1}, V_{f2}, V_m, v_c, and g; and
b) Express the mass fraction of fiber 2, in terms of ρ_{f1}, ρ_{f2}, ρ_m, V_{f1}, V_{f2}, and V_m

Solution
a) $w_c = w_{f1} + w_{f2} + w_m$

$$w_c = \frac{\rho_{f1} v_{f1} + \rho_{f2} v_{f2} + \rho_m v_m}{g} \cdot \frac{v_c}{v_c}$$

$$w_c = \frac{\rho_{f1} V_{f1} + \rho_{f2} V_{f2} + \rho_m V_m}{g} \cdot v_c$$

b) $w_c = w_{f1} + w_{f2} + w_m$
Then:

$$w_{f2} = \frac{\rho_{f2} v_{f2}}{g} v_c$$

$$\therefore W_{f2} = \frac{w_{f2}}{w_c} = \frac{\rho_{f2} V_{f2}}{\rho_{f1} V_{f1} + \rho_{f2} V_{f2} + \rho_m V_m}$$

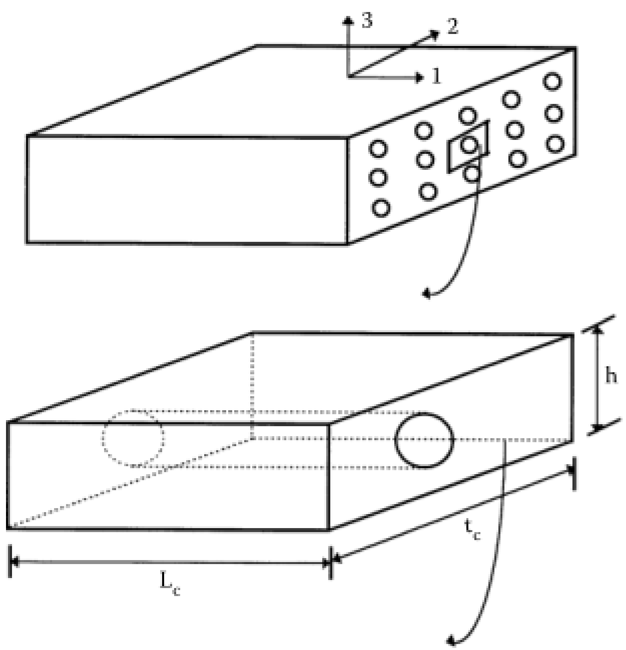
3.3.1 Strengths of Materials Approach

(*) Simple expression for E_1 E_2 v_{12} and G_{12}

(*) Being simple & being accurate

$$\begin{cases} RVE \rightarrow Model? \\ Mathematical\ manipulation? \end{cases}$$

(*) E_1 E_2 v_{12} G_{12} in terms of:
E_f v_f G_f V_f
E_m v_m G_m V_m



RVE for 3.3.1:
Rectangular packing of fibers
d: diameter of fibers
h, t_C: diameter of rectangle
$L_c = l_c$; length of composite

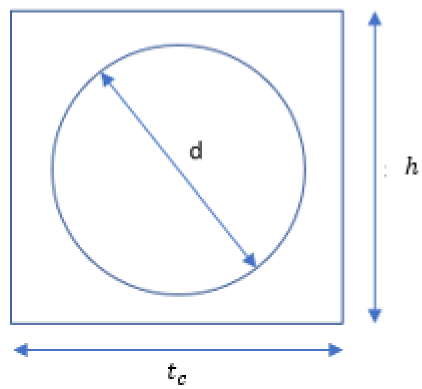
Square packing → Rectangular packing:
Principle is to preserve V_f
$\therefore h = t_c = s$
s being spacing, see 3.2

Hexagonal packing → Rectangular packing:
Principle remains to preserve V_f
By section 3.2:
$$V_f = \frac{\pi}{2\sqrt{3}} \frac{d^2}{S^2}$$

Hexagonal packing:

$$V_f = \frac{\pi}{2\sqrt{3}} \frac{d^2}{S^2}$$

Rectangular packing:



(Let l_c be the depth)

$$v_f = \frac{\pi}{4} d^2 \, l_c$$
$$v_c = h \, t_c \, l_c$$
$$V_f = \frac{v_f}{v_c} = \frac{\frac{\pi}{4} d^2 \, l_c}{h \, t_c \, l_c} = \frac{\pi d^2}{4 \, h \, t_c}$$

Equating rectangular packing to hexagonal packing:

$$\therefore \frac{\pi d^2}{4 \, h \, t_c} = \frac{\pi}{2\sqrt{3}} \frac{d^2}{S^2}$$

$h \, t_c = \sqrt{3} \, S^2$ (and further, $t_c = \alpha h$)

3.3.1.1 Longitudinal Young's Modulus E_1

RVE: Fig 3.3 of text, top and middle diagrams in particular.

Determining E_1:

l_c: length of RVE

A_c: cross-section of RVE

$\therefore v_c = A_c l_c$

$\therefore v_f = V_f v_c$

$= (V_f A_c) l_c$

$V_f A_c$: cross-section of fibers

$v_m = V_m v_c$

$= (V_m A_c) l_c$

$V_m A_c$: cross-section of matrix

Under F:

Axial loading, statically indeterminate

$$\delta = \frac{PL}{EA}$$
$$k = \frac{EA}{L}$$
$$k_f = \frac{E_f V_f A_c}{l_c}$$
$$k_m = \frac{E_m V_m A_c}{l_c}$$
$$k_c = \frac{E_1 A_c}{l_c}$$

$\therefore k_c = k_f + k_m$
$\therefore \frac{E_1 A_c}{l_c} = \frac{E_f V_f A_c}{l_c} + \frac{E_m V_m A_c}{l_c}$
$\therefore E_1 = E_f V_f + E_m V_m$ (Rule of mixture)

Rule of Mixture
It refers to the method of estimating composite's property by volume-weighted average of like properties of the constituents.

E_1:
RVE: Fig 3.3, middle diagram
Statically indeterminate problem
Calculation: Example 3.3
Comparing with experimental results: Fig 3.6
E_1 is by rule-of-mixture
E_1 is accurate

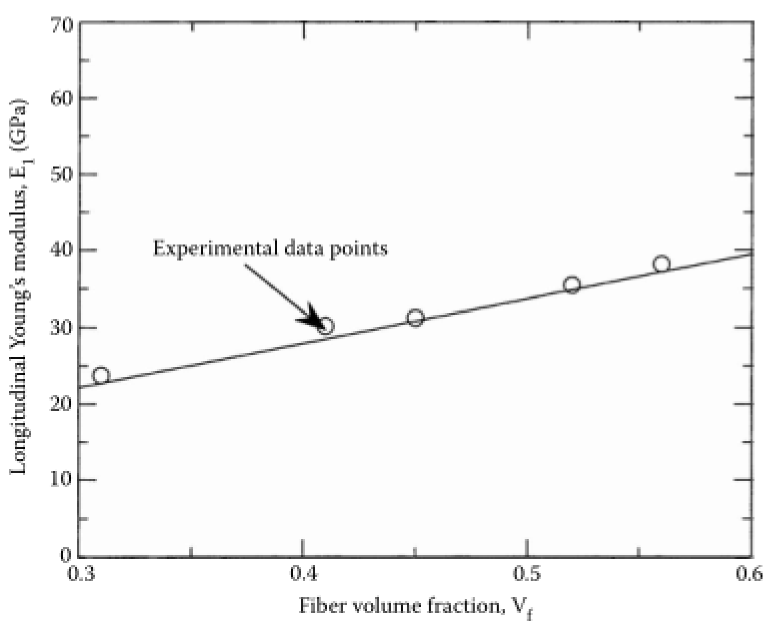
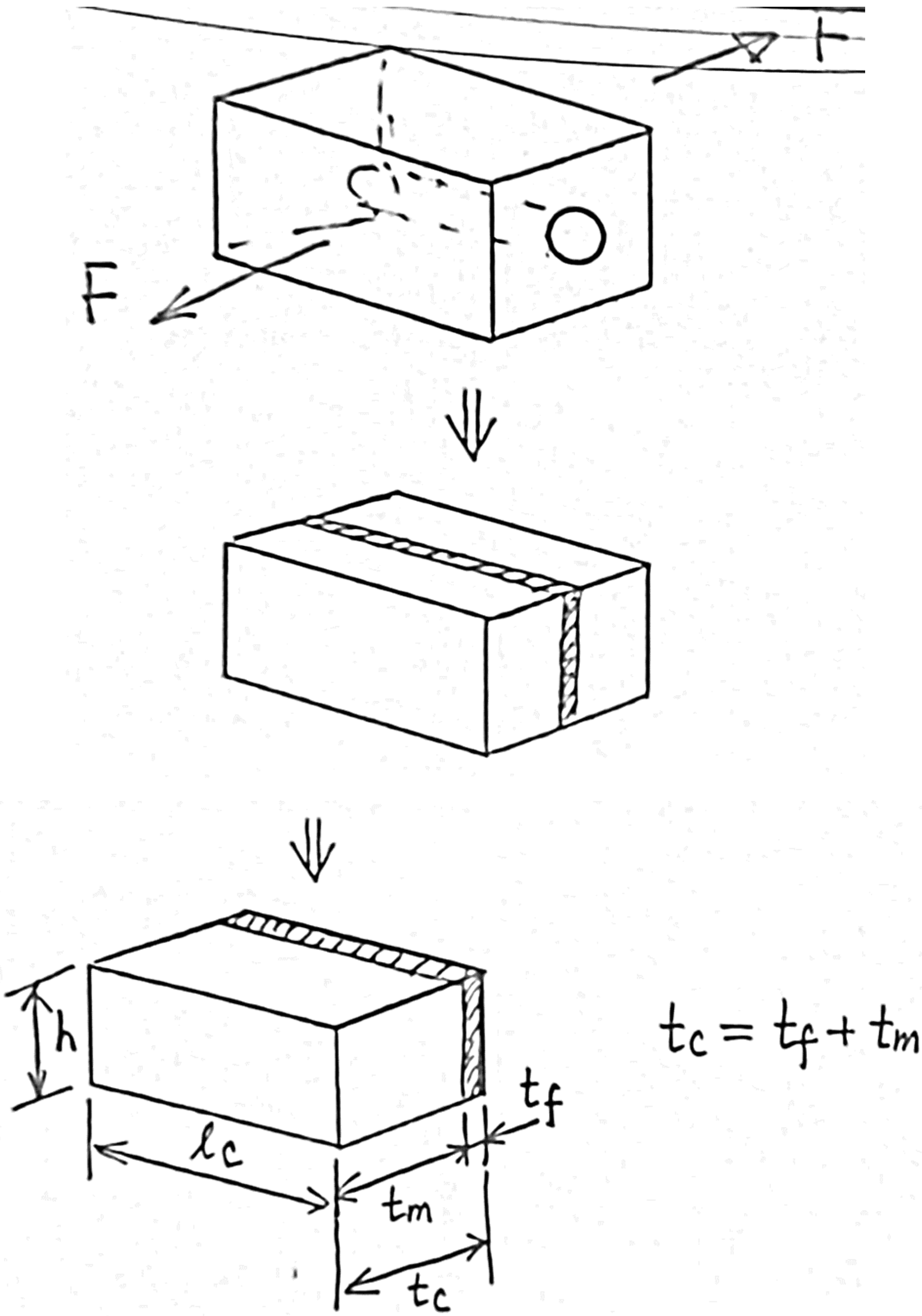


Figure 3.6

3.3.1.2 Transverse Young's Modulus, E_2

This time, RVE is loaded in the 2-direction.



E_2:
Axial loading

$$k = \frac{EA}{L}$$

$$k_f = \frac{E_f(h\, l_c)}{t_f}$$

$$k_m = \frac{E_m(h\, l_c)}{t_m}$$

$$k_c = \frac{E_2(h\, l_c)}{t_c}$$

$$\therefore \frac{1}{k_c} = \frac{1}{k_f} + \frac{1}{k_m}$$

$$\therefore \frac{t_c}{E_2(h\, l_c)} = \frac{t_f}{E_f(h\, l_c)} + \frac{t_m}{E_m(h\, l_c)}$$

$$\therefore \frac{1}{E_2} = \frac{t_f}{E_f\, t_C} + \frac{t_m}{E_m\, t_c}$$

$\therefore \frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$ (Inverse rule of mixture)

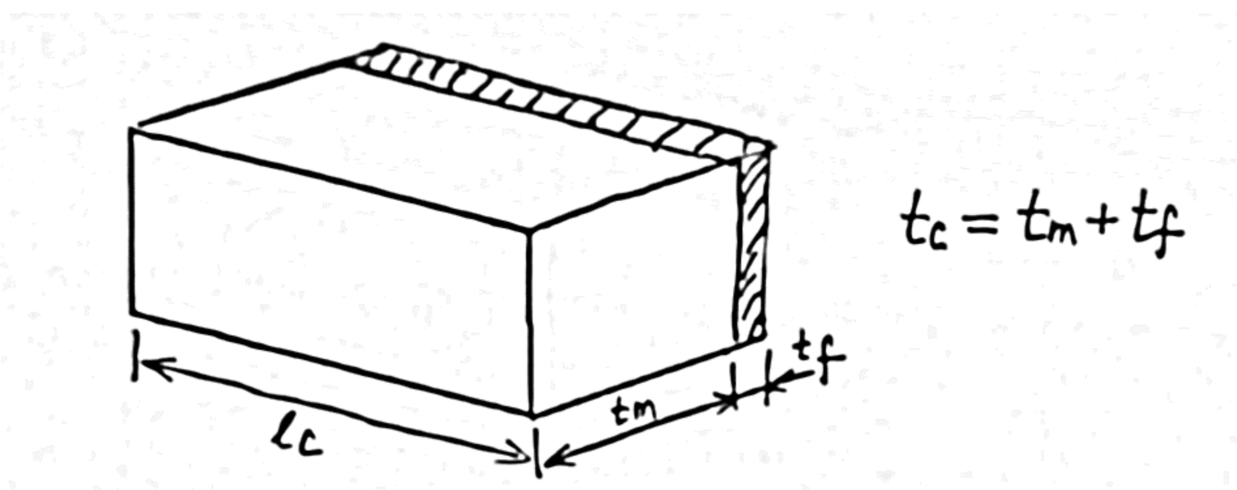
E_2:
RVE: Figure 3.7
Calculation: Example 3.4
Comparing with experimental results: Fig 3.10 inverse ROM results in lower-bound solution.
Solutions may be as low as 40%~50% of where values should be, depending on the difference between E_f and E_m (or E_f/E_m) and V_f

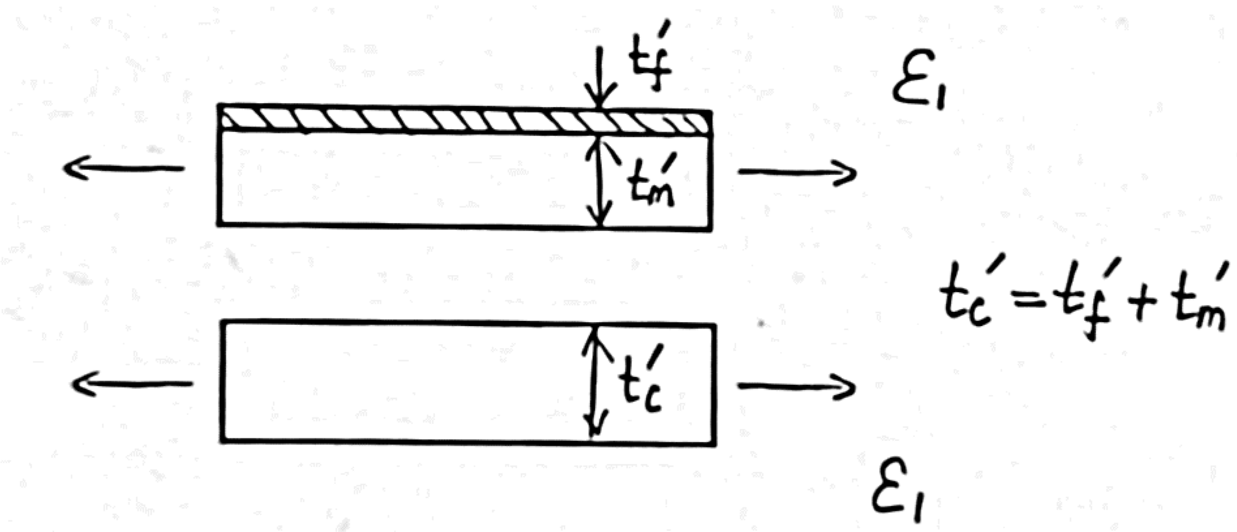
Typically, the higher the ratio E_f/E_m, and/or the higher the V_f value, the higher the deduction.

Inverse rule of mixture is simple, but not accurate, because it's not true axial attention.

Midterm: October 31st (Thursday) during class time.

3.3.1.3 Major Poisson's Ratio v_{12}





Poisson's ratio:

$$v = -\frac{lateral\ strain}{longitudinal\ strain} = -\frac{lateralstrain}{\varepsilon_1}$$

$$Normal\ strain\ \varepsilon = \frac{\Delta l}{l_o} = \frac{l' - l_o}{l_o}$$

$\therefore$ *Fibers*: *lateral strain*

$$= \frac{t_f' - t_f}{t_f}$$

$$v_f = -\frac{t_f' - t_f}{t_f\ \varepsilon}$$

$$t_f' = t_f - v_f t_f \varepsilon_1\ (\boldsymbol{Eq.1})$$

Matrix:

$$t_m' = t_m - v_m t_m \varepsilon_1\ (\boldsymbol{Eq.2})$$

Composite:

$$t_c' = t_c - v_{12} t_c \varepsilon_1\ (\boldsymbol{Eq.3})$$

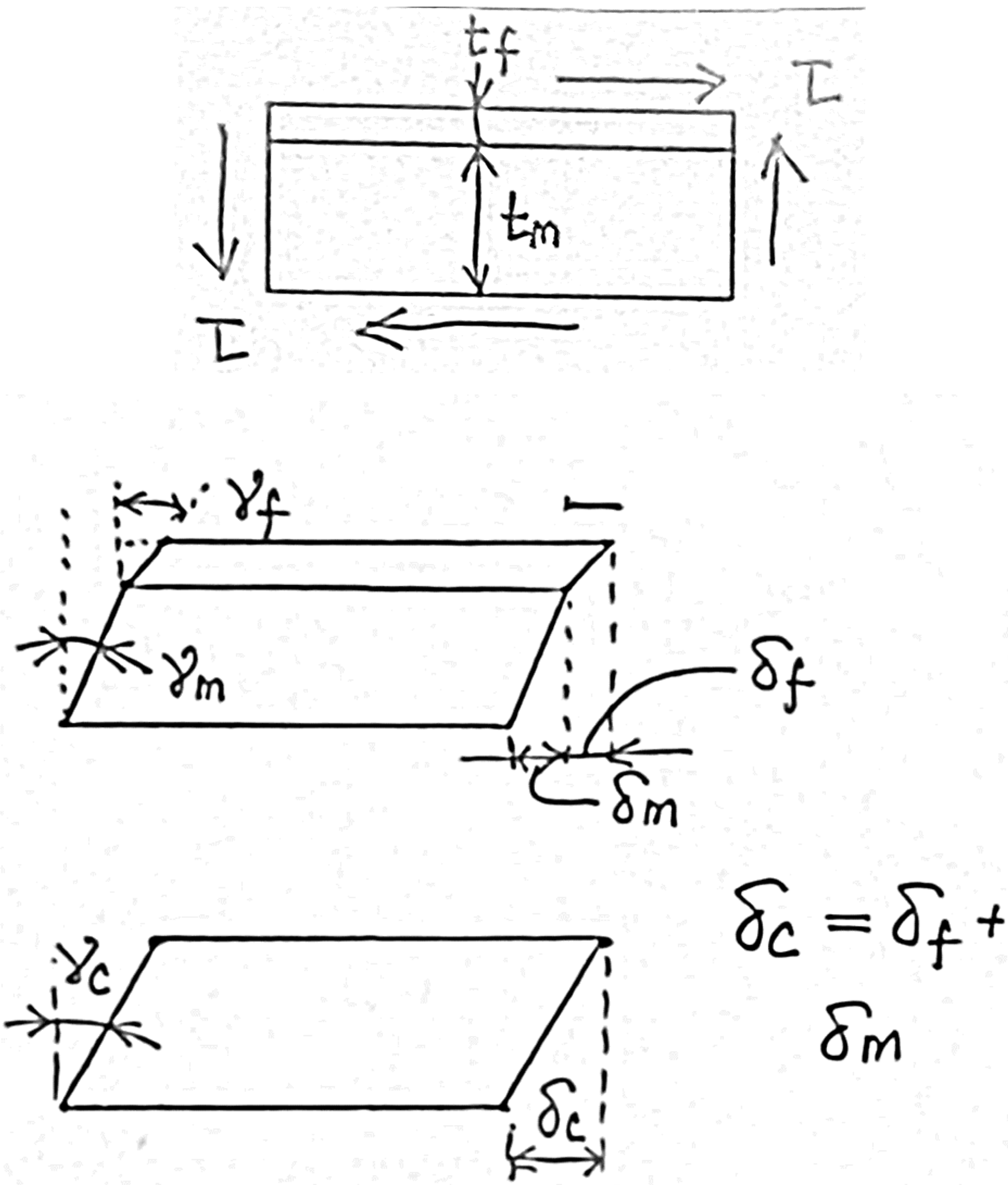
$$(\boldsymbol{Eq.1}) + (\boldsymbol{Eq.2}): t_f' + t_m' = t_f + t_m - (v_f t_f)\varepsilon_1 = (v_m t_m)\varepsilon_1\ (\boldsymbol{Eq.4})$$

$$(\boldsymbol{Eq.3}) + (\boldsymbol{Eq.4}):\ \therefore -v_{12} t_c \varepsilon_1 = -v_f t_f \varepsilon_1 - v_m t_m \varepsilon_1$$

$$\therefore v_{12} = v_f V_f + v_m V_m\ (rule\ of\ mixture)$$

v_{12}:
RVE: Fig. 3.11
Calculation: Example 3.5
v_{12} by rule-of-mixture is accurate

3.3.1.4 In-plane Shear Modulus G_{12}



G_{12}:

$Matrix\colon v_m = \tan^{-1}\frac{\delta_m}{t_m} \approx \frac{\delta_m}{t_m}$

$(\tan^{-1} x \approx x \ \ for\ very\ small\ angles)$

But,

$$\gamma_m = \frac{\tau}{G_m}$$

$$\therefore \frac{\delta_m}{t_m} = \frac{\tau}{G_m}$$
$$\therefore \delta_m = \frac{\tau\, t_m}{G_m}$$

$$Fibers: \delta_f = \frac{\tau\, t_f}{G_f}$$

$$Composite\ \delta_c = \frac{\tau\, t_c}{G_{12}}$$

And:
$$\frac{\tau\, t_c}{G_{12}} = \frac{\tau\, t_m}{G_m} + \frac{\tau\, t_f}{G_f}$$
$$\therefore \frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}\ (inverse\ rule\ of\ mixture)$$

G_{12}:
RVE: Fig. 3.12
Calculation: Example 3.6
Comparing with experimental results: Fig. 3.13
G_{12} is by inverse rule-of-mixture expression which gives lower bound solution.
Discussions pertaining E_2 are mostly applicable with the exceptions of smaller deduction (25~40%), and the ratio G_f/G_m instead of E_f/E_m

4 sub-sections
3.3.1 Strength of Materials Approach (Mechanics of Materials)
3.3.2 Sem-Empirical Approach
3.3.3 Theory of Elasticity Approach
3.3.4 Transversely Isotropic Fibers (c- and k- fibers are transversely isotropic)

3.3.2 Semi-Empirical Models
A. Some historical notes

- Mechanics of Materials approach yields simple expressions, rule of mixture (ROM) or inverse ROM; General observation is, ROM-based expressions give good accuracy, but the inverse ROM-based ones are far from satisfactory.
- A series of formal approaches (As they are known as) took place after realizing shortcomings of the inverse ROM. Such approaches were based on theory of elasticity, in one form or the others. The differences lie in the methodologies of solving the PDEs involved.
- Basic assumptions behind the formal approaches are, noting that some of the basic assumptions have been used in 3.3.1
    - Fibers and matrix are homogeneous and isotropic;
    - The resulting composite is homogeneous and orthotropic;
    - Void content is zero;

    - There is perfect bonding between constituents;
    - Constituents and resulting composite are linearly elastic;
    - Composite is initially stress-free;
    - Fibers are regularly spaced and aligned.
- The formal approaches include the following methods, to list just a few
    - Classical or exact method (See 3.3.3)
    - Variation methods or energy methods that are either analytical or numerical. The former gives rise to bounds on elastic moduli; and the latter typically leads to finite difference method and finite element method.
    - Mori-Tanaka models (or inclusion models). The key aspect is to assume that the average strain of the inclusion (i.e., the fibers) is related to the average strain of the matrix by a to-be-determined fourth-order tensor.
    - Self-consistent models (suitable for composites having particulate or short fibers as reinforcing phase) …..
- To meet the desire of engineers to have simple yet accurate formulas, efforts were taken, in the 1960' to 1970', to interpolate existing theoretical as well as experimental results.
    - Experimental data to best-fit ROM- or inverse ROM-based formulas with modifying factors (wasn't successful);
    - Experimental data to best-fit re-arranged formulas from self-consistent models, and simplified by introducing factors (was successful);
    - Halpin-Tsai formulas/equations are the best-known outcome of such effort; self-consistent models with factors.

Example: Two fiber-reinforced laminas of unidirectional continuous fibers consistent of pitch-based graphite fibers and epoxy, and Kevlar 49 and epoxy respectively. The laminas have the same corrected volume fractions: $V_f' = 58\%$ and $V_m' = 42\%$. The Young's moduli of the fibers can be found in Tables 1.8, 1.9, and 1.10 of the text. Poisson's ratios are, 0.22 for graphite, 0.35 for Kevlar 49, and 0.32 for epoxy, respectively.
For each of the laminas, determine the four elastic moduli by:
(a) The mechanics of materials approach;
(b) The semi-empirical approach; and
(c) The theory of elasticity approach.

From the Tables: $E_{fG} = 55\ Mpsi$, $E_{fK} = 19\ Mpsi$, and $E_m = 0.55\ Mpsi$
Shear moduli of the fibers and epoxy are evaluated: $G_{fG} = 22.5\ Mpsi$, $G_{fK} = 7.04\ Mpsi$m $G_m = 0.208\ Mpsi$.

(a) The mechanics of materials approach:

	Graphite/Epoxy	Kevlar/Epoxy
$E_1, Mpsi$	32.13	11.25
E_f/E_m	100	34.5
$E_2, Mpsi$	1.29	1.26
v_{12}	0.262	0.337
G_f/G_m	108	33.8
$G_{12}, Mpsi$	0.489	0.420

(b)

Experimental as well as analytical work leading to the H-T formulas was conducted at US Wright Patterson Air Force Base (Dayton, Ohio) during the 1970s.

Halpin and Tsai then published an internal technical report [Environmental Factors in Composite Materials Design, J.C. Halpin and S.W. Tsai, AFML TR-423] summarizing the work, hence Halpin-Tsai formulas or equations.

In the text, Halphin and Haphin are incorrect spelling.

C. State-of-the-art in terms of predicting elastic moduli

Predicting E_2, G_{12}, and G_{23} (G_{23} is required when dealing with transversely isotropic fibers) remains a challenge, especially when new fibers and matrix are considered.

Chamis model, also a semi-empirical model, seems robust.

Bridging model, which is an analytical model, proves to be reliable, albeit not straightforward.

FE models, however detailed, have not proven themselves more accurate than analytical models or semi-empirical models.

A composite's elastic modulus P_c is found by:

$$P_c = \frac{P_m\left(1 + \zeta\eta V_f\right)}{1 - \eta V_f}$$

Where:
P_c can be either E_1, $E2$, or G_{12} for example;
V_f is the volume fraction of fibers;
ζ is a factor that is used to describe the influence of geometry of the fibers;
ζ depends on V_f, and has different values for different moduli of the composite;
$\eta = \dfrac{P_f/P_m - 1}{P_f/P_m + \zeta}$
P_m, P_f are the corresponding moduli of the matrix, and the fibers, respectively.

Fibers should be interpreted as reinforcing phases, within the context of the H-T formulas.

Factor ζ and Recommendation

Unidirectional continuous fibers

E_1	ROM
E_2	$\zeta = 2 + 40V_f^{10}$
G_{12}	$\zeta = 1 + 40V_f^{10}$
v_{12}	ROM

Particulate

E_1	$\zeta = 2 + 40V_f^{10}$
E_2	$\zeta = 2 + 40V_f^{10}$
G_{12}	$\zeta = 1 + 40V_f^{10}$
v_{12}	ROM

Example: Two fiber-reinforced laminas of unidirectional continuous fibers consistent of pitch-based graphite fibers and epoxy, and Kevlar 49 and epoxy respectively. The laminas have the same corrected volume fractions: $V_f' = 58\%$ and $V_m' = 42\%$. The Young's moduli of the fibers can be found in Tables 1.8, 1.9, and 1.10 of the text. Poisson's ratios are, 0.22 for graphite, 0.35 for Kevlar 49, and 0.32 for epoxy, respectively.
For each of the laminas, determine the four elastic moduli by:
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(b) The semi-empirical approach; and
(c) The theory of elasticity approach.

From the Tables: $E_{fG} = 55\ Mpsi$, $E_{fK} = 19\ Mpsi$, and $E_m = 0.55\ Mpsi$
Shear moduli of the fibers and epoxy are evaluated: $G_{fG} = 22.5\ Mpsi$, $G_{fK} = 7.04\ Mpsi$m $G_m = 0.208\ Mpsi$.

(b) H-T Formulas (for Kevlar/Epoxy only)
$E_1 = 11.25$ (from ROM approach)
$v_{12} = 0.337$ (from ROM approach)

E_2:
$\zeta = 2 + 40V_f^{10}$
$= 2.172$

$$\eta = \frac{P_f/P_m - 1}{P_f/P_m + \zeta}$$

$$\eta = \frac{(34.5) - 1}{(34.5) + 2.172} = 0.9135$$

$$P_c = \frac{P_m(1 + \zeta\eta V_f)}{1 - \eta V_f}$$

$$E_2 = \frac{E_m(1 + \zeta\eta V_f)}{1 - \eta V_f} = 2.516\ Mpsi$$

(should be higher than number obtained from mechanics of materials approach.)

G_{12}:
$\zeta = 1 + 40V_f^{10}$
$= 1.172$

$$\eta = \frac{(33.8) - 1}{(33.8) + 1.172} = 0.9379$$

$G_{12} = 0.7469\ Mpsi$

	Graphite/Epoxy	Kevlar/Epoxy
$E_1, Mpsi$	32.13	11.25
E_f/E_m	100	34.5
$E_2, Mpsi$	2.79	2.52
v_{12}	0.262	0.337
G_f/G_m	108	33.8
$G_{12}, Mpsi$	0.803	0.747

	Graphite/Epoxy	Kevlar/Epoxy
E_f/E_m	100	34.5
$E_{2(IROM)}/E_{2(H-T)}$	46%	50%
G_f/G_m	108	33.8
$G_{12(IROM)}/G_{12(H-T)}$	61%	56%

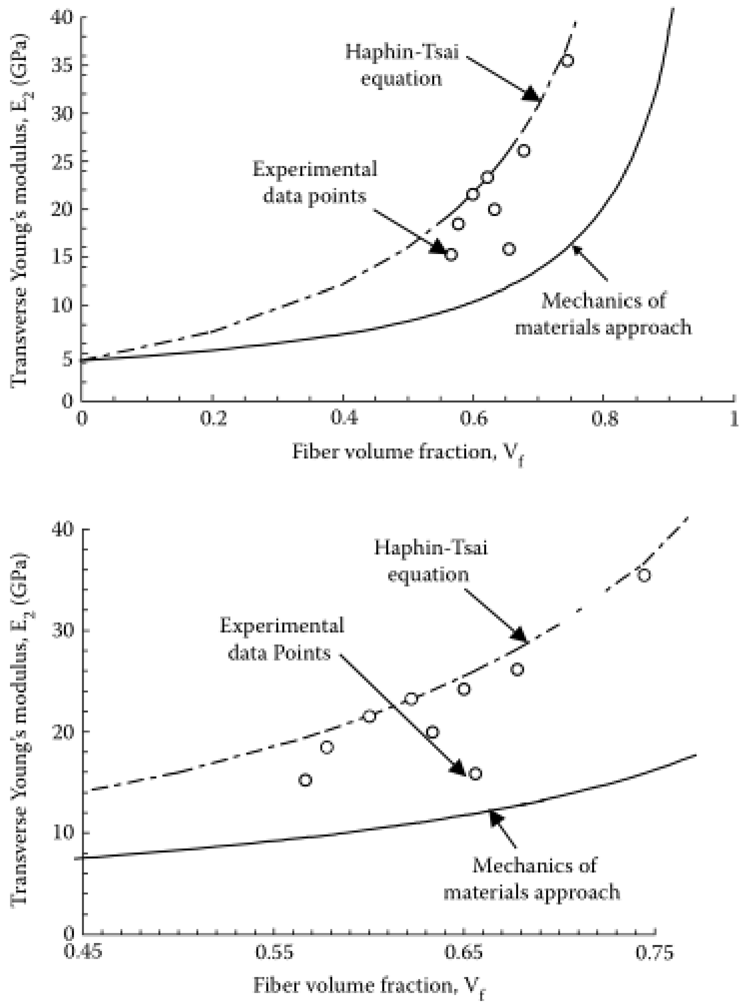


FIGURE 3.15
Theoretical values of transverse Young's modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

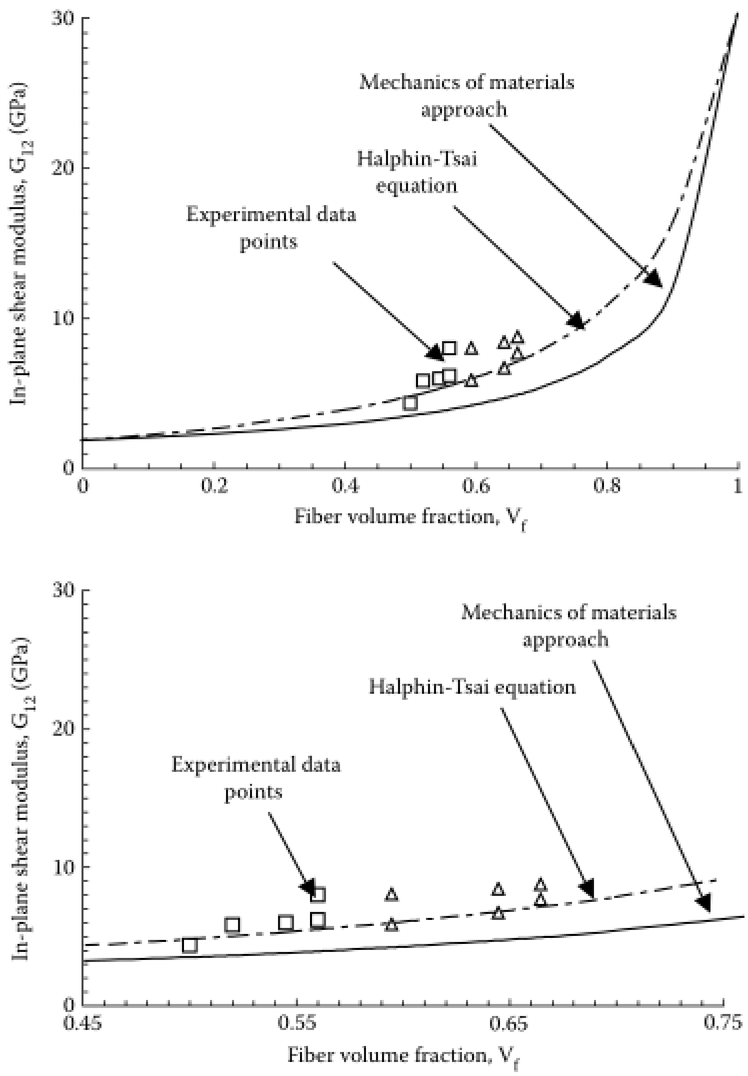


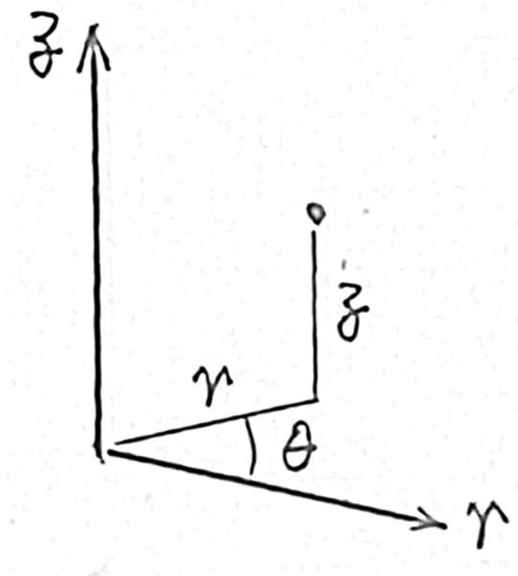
FIGURE 3.17
Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

Theory of Elasticity Using the Cylindrical Coordinates

The following elasticity theory is for homogeneous, isotropic and linearly elastic materials. Therefore, it is applicable to isotropic fiber as a cylinder, and matrix as a cylinder.

As a reminder, composites reinforced with unidirectional continuous fibers are not isotropic. Carbon or graphite fibers, and aramid fibers are transversely isotropic.

1. Cylindrical Coordinates (r, $\boldsymbol{\theta}, \boldsymbol{z}$)



Typically, z represents direction 1; r and θ form the 2-3 plane.

2. Unknowns

The 15 unknowns are:

Displacements $u_r \; u_\theta \; u_z$
Stresses $\sigma_r \; \sigma_\theta \; \sigma_z \; \tau_{r\theta} \; \tau_{\theta z} \; \tau_{zr}$
Strains $\varepsilon_r \; \varepsilon_\theta \; \varepsilon_z \; \gamma_{r\theta} \; \gamma_{\theta z} \; \gamma_{zr}$

3. Relations Governing the 15 Unknowns

There are 3 sets of relations.

(A) The strain displacement relations:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$
$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$
$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$
$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

Where $u_r \; u_\theta$ and u_z are displacements in the r, θ, and z directions, respectively.

(B) The stress-strain relations (or the Hooke's law, or the consecutive relations)

$$\{\sigma\} = [\sigma_r \; \sigma_\theta \; \sigma_z \; \tau_{r\theta} \; \tau_{\theta z} \; \tau_{zr}]^T$$
$$\{\varepsilon\} = [\varepsilon_r \; \varepsilon_\theta \; \varepsilon_z \; \gamma_{r\theta} \; \gamma_{\theta z} \; \gamma_{zr}]^T$$

$$\{\sigma\} = [C]\{\varepsilon\}$$

Where $[C\}$ is the 6x6 matrix:

$$[C] = \begin{bmatrix} C_1 & C_2 & C_2 & 0 & 0 & 0 \\ C_2 & C_1 & C_2 & 0 & 0 & 0 \\ C_2 & C_2 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}$$

Where C_1 and C_2 are constants in terms of E and v

(C) The equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{zr}}{\partial z} + \bar{R} = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{\partial \tau_{\theta z}}{\partial z} + \bar{\theta} = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r} + \bar{Z} = 0$$

Where terms with over-bar are body forces in the r, θ, and z directions, respectively. Body forces are forces per unit volume.

For example, if the cylinder is heavy and z takes the vertical direction, then $\bar{Z} = -\rho g$with ρ being the mass density per unit volume and g being the gravitational acceleration.

Centrifugal force is another example of body force: $\bar{R} = \rho r \omega^2$. Again, ρ is the mass density per unit volume.

Note that sets (A) and (C) are differential. Set (B) is linear.

4. Axisymmetric Problems

It means symmetry about the z-axis, as a result, (a) the unknowns are now functions of r and z only; and (b) $u_\theta = 0$.

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_\theta = \frac{u_r}{r} + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}$$
$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \cancel{\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}} = 0$$

$$\gamma_{\theta z} = \cancel{\frac{1}{r}\frac{\partial u_z}{\partial \theta}} \cancel{+} \cancel{\frac{\partial u_\theta}{\partial z}} = 0$$

$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

As a result, $\gamma_{r\theta}$ and $\gamma_{\theta z} = 0$. Because of homogenous, isotropic, and linearly elastic material assumption, $\tau_{r\theta} = \tau_{\theta z} = 0$. Vectors $\{\sigma\}$ and $\{\varepsilon\}$ are reduced to 4x1; [C] is a 4x4 matrix.

5. Additional simplifications

5.1 All body forces are zero.

- Second equilibrium equation is satisfied, automatically:

$$\cancel{\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}} \cancel{+} \cancel{\frac{\partial \tau_{r\theta}}{\partial r}} \cancel{+} \cancel{\frac{2\tau_{r\theta}}{r}} \cancel{+} \cancel{\frac{\partial \tau_{\theta z}}{\partial z}} \cancel{+ \bar{\theta}} = 0$$

5.2 $\tau_{zr} = 0$, or no shear deformation on any $z - r$ plane.

- The third equilibrium equation becomes:

$$\frac{\partial \sigma_z}{\partial z} \cancel{+} \cancel{\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}} \cancel{+} \cancel{\frac{\partial \tau_{zr}}{\partial r}} \cancel{+} \cancel{\frac{\tau_{zr}}{r}} \cancel{+ \bar{Z}} = 0$$

That is:

$$\frac{\partial \sigma_z}{\partial z} = 0$$

Which means σ_z is either a constant, or a function of r only. The option $\sigma_z = constant$ is chosen.

- The first equilibrium equation simplified to:

$$\frac{\partial \sigma_r}{\partial r} \cancel{+} \cancel{\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}} + \frac{\sigma_r - \sigma_\theta}{r} \cancel{+} \cancel{\frac{\partial \tau_{zr}}{\partial z}} \cancel{+ \bar{R}} = 0$$

Or:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (Eq. 1)$$

- $\gamma_{zr} = 0$ due to homogeneous, isotropic, and linearly elastic material assumption. **

6. PDE to ODE

Making use of strain-stress relations (the inverse of stress-strain relations) such that equation $(Eq. 1)$ is in terms of strains ε_r, ε_θ, and ε_z.

From strain-displacement relations,

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{u_r}{r} + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} = 0$$
$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} = 0$$
$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0$$

Making use of the following in particular

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_\theta = \frac{u_r}{r}$$
$$\varepsilon_z = constant$$

$$\sigma_z = constant$$

Then the final form of $(Eq. 1)$ is in terms of displacements.

By now, there is only one displacement, u_r, that is involved. Also, u_r is no longer a function of z.

Therefore, the subscript r in u_r can be dropped (i.e., u now denotes the radial displacement), and the partial derivatives become ordinary derivatives. The final form of $(Eq\ 1)$ is:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \quad (Eq. 2)$$

The solution to $(Eq. 2)$ is:

$$u = Ar + \frac{B}{r} \quad (Eq. 3)$$

Where A and B are constants to be determined by boundary conditions.

$(Eq. 2)$ and $(Eq. 3)$ are numbered as $(Eq. 3.73)$ and $(Eq. 3.78)$, respectively, in the text.

To determine constants A and B, two types of boundary conditions need to be considered.

1st : displacement-type of essential B.C.'s
2nd : stress or force-type or natural B.C.'s

With the first type of B.C.'s, (Eq. 3.78) or (Eq. 3) can be applied directly.

With the second type of B.C.'s,
$u \rightarrow \varepsilon_r\ \varepsilon_\theta\ \varepsilon_z$ (strain-displacement relation)
$\varepsilon_s' \rightarrow \sigma_r\ \sigma_z$ (stress-strain relation)

$u = Ar + B/r$
$\varepsilon_r = \partial u/\partial r = du/dr = A - B/r^2$
$\varepsilon_\theta = u/r = A + B/r^2$
$\varepsilon_z = \varepsilon_1$

If material is homogeneous, isotropic and linearly elastic:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} = [C] \begin{Bmatrix} A - B/r^2 \\ A + B/r^2 \\ \varepsilon_1 \end{Bmatrix}$$

$[C]$ is a constant matrix, in terms of E and v. Details are given in (Eq. 3.70) or (Eq. 3.71)

$$\therefore\ \sigma_r = (C_{11} + C_{12})A + \frac{C_{12} - C_{11}}{r^2} B + C_{12}\varepsilon_1 \quad (Eq. 4)$$

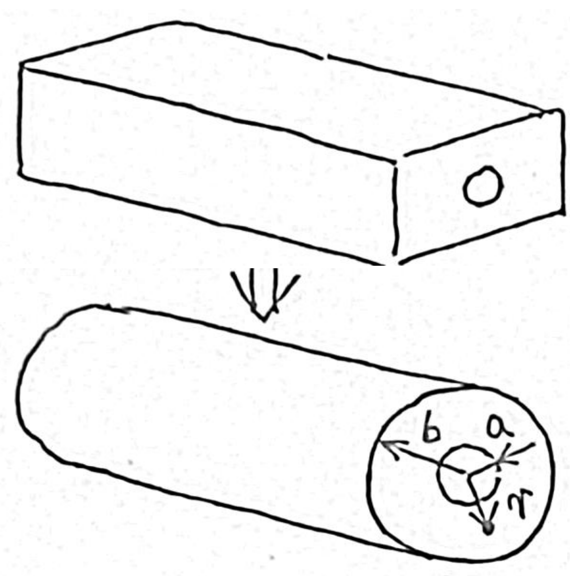
$$\sigma_z = 2C_{12}A + C_{11}\varepsilon_1 \quad (Eq. 5)$$

C_{11}, C_{12} are given by (Eq. 3.72)

3.3.3 Elasticity Approach

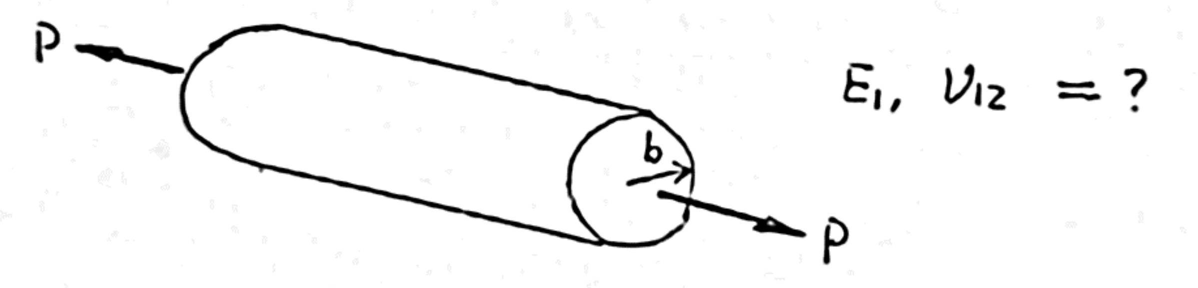
3.3.3.1 Longitudinal Young's Modulus E_1

1) RVE: two concentric cylinders (this RVE is also used with v_{12}) such RVE is known as CCA or CAM
CCA: composite cylinder assembly
CAM: cylindrical assembly model



Fibers: $0 \leq r \leq a$
Matrix: $a \leq r \leq b$
$\therefore V_f = \frac{a^2}{b^2} \quad (Equation\ 3.63)$
or $a/b = \sqrt{V_f}$

2) A homogenous cylinder representing composite



Assuming that ε_1 is the longitudinal/axial strain developed, after application of P.

$$\therefore\ \sigma_1 = \frac{P}{\pi b^2} \quad (Eq. 3.64)$$
meanwhile, $\sigma_1 = E_1 \varepsilon_1 \quad (Eq. 3.65)$
$$\therefore E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{P}{\pi b^2 \varepsilon_1} \quad (Eq. 3.66)$$
objective is to write P in terms of ε_1 such that E_1 is independent of P.

3) The "fibers" cylinder
(Eq. 3) becomes:
$$u_f = A^f r + \frac{B^f}{r} \quad (0 \leq r \leq a)$$
B^f must be zero
$\therefore u^f = A^f \cdot r \quad (0 \leq r \leq a) \quad (Eq. 3.81)$

$(Eq. 4)$ and $(Eq. 5)$ become:
$$\left\{ \begin{matrix} \sigma_r^f = \left(C_{11}^f + C_{12}^f\right) A^f + C_{12}^f \varepsilon_1 \\ \sigma_z^f = 2C_{12}^f + C_{11}^f \varepsilon_1 \end{matrix} \right\} (Eq. 3.84)$$
Where $0 \leq r \leq a$

4) The "matrix" cylinder
$(Eq. 3) \sim (Eq. 5)$ become:
$$u^m = A^m r + \frac{B^m}{r}$$
$$\left\{ \begin{matrix} \sigma_r^m = (C_{11}^m + C_{12}^m) A^m + \frac{C_{12}^m - C_{11}^m}{r} B^m + C_{12}^m \varepsilon_1 \\ \sigma_z^m = 2C_{12}^m A^m + C_{11}^m \varepsilon_1 \end{matrix} \right\} (Eq. 3.86)$$
Where $0 \leq r \leq a$

So far, unknown constants are:
A^f A^m B^m and ε_1
And ε_1 is related to P.

5) Boundary conditions
5.1) At interface between fibers and matrix cylinders where $r = a$
$u^f = u^m \quad (Eq. 3.88) \sim (Eq. 3.89)$
$\sigma_r^f = \sigma_r^m \quad (Eq. 3.90\,) \sim (Eq. 3.91)$

5.2) At the outer surface of the matrix cylinder
$where\, r = b$
$\sigma_r^m = 0 \quad (Eq. 3.92) \sim (Eq. 3.93)$

The above three boundary conditions are involved with $A^f\, A^m\, B^m$ and ε_1;
A^f A^m B^m are then solved in terms of ε_1;

5.3) On any cross-section of CCA or CAM, static equilibrium requires:
$$\int\int_A \sigma_z \, dA = P$$
or $\int\int_{A_f} \sigma_z^f \, dA + \int\int_{A_m} \sigma_z^m \, dA = P \quad (Eq. 3.94) \sim (Eq. 3.97)$

Which results in a relation between P and ε_1.

6) Grand finale
back to (Eq. 3.66)
$$E_1 = \frac{P}{\pi b^2 \varepsilon_1}$$

After lengthy simplification:
$$E_1 = -\frac{2E_m E_f V_f \left(\nu_f - \nu_m\right)^2 \left(1 - V_f\right)}{E_f\left(2\nu_m^2 V_f - \nu_m + V_f \nu_m - V_f - 1\right) + E_m\left(-1 - 2V_f \nu_f^2 + \nu_f - V_f \nu_f + 2\nu_f^2 + V_f\right)} \quad (3.98)$$

3.3.3.2 Major Poisson's Ratio v_{12}
$\left(\frac{u^m}{r}\right)_{r=b}$ is the lateral strain
and:
$$\left(\frac{u^m}{r}\right)_{r=b} = \frac{A^m b + \frac{B^m}{b}}{b} = A^m + \frac{B^m}{b^2}$$
$$\therefore v_{12} = -\frac{A^m + \frac{B^m}{b^2}}{\varepsilon_1}$$
After lengthy simplification:

$v_{12} = v_f V_f + v_m V_m$

$$v_{12} = + \frac{V_f V_m (\nu_f - \nu_m)(2E_f \nu_m^2 + \nu_m E_f - E_f + E_m - E_m \nu_f - 2E_m \nu_f^2)}{(2\nu_m^2 V_f - \nu_m + \nu_m V_f - 1 - V_f)E_f + (2\nu_f^2 - V_f \nu_f - 2V_f \nu_f^2 + V_f + \nu_f - 1)E_m} \quad (3.103)$$

See example 3.10 for numerical application

3.3.3.3 Transverse Young's Modulus E_2

- CCA model gives lower and upper bounders of E_2
- 3-phase model gives an exact solution for G_{23} which will lead to $E_2 = 2(1 + v_{23})G_{23}$
- Example 3.11 for detailed steps

a) (missed note)

b) CCA model was used, together with energy method (which is different from the classical method of solving PDEs/ODEs)

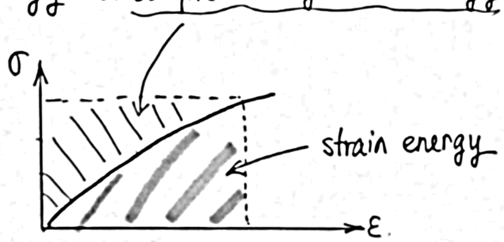
c) Upper bound: maximum potential energy principle

actual strain energy (which is completely unknown due to complexity of problem)	$\leq$	strain energy by trial functions meeting certain conditions

d) Lower bound: minimum complementary energy principle

actual complementary strain energy (unknown)	$\leq$	complementary energy by trial functions meeting certain conditions

e. Strain energy vs. complementary strain energy



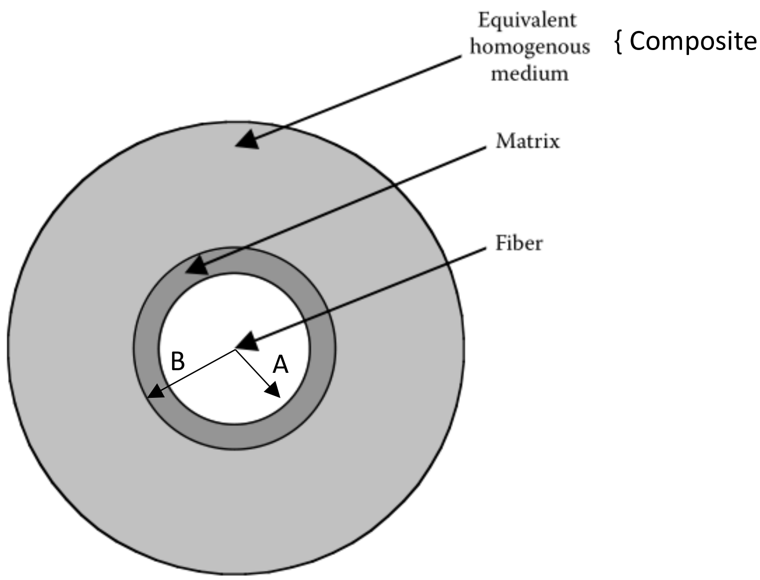
Strain energy: in terms of strains and elastic moduli
Complementary strain energy: in terms of stresses and compliances (compliances are the inverse of elastic moduli)

f) the Principle of Minimum Potential Energy:
Of all displacement fields satisfying the prescribed displacement boundary conditions, the field which satisfied stress equilibrium minimizes the stored elastic energy of the system.

g) the Principle of Minimum Complementary Energy:
Among those stress distributions that satisfy the stress equilibrium condition at each point, and that are in equilibrium with the external loads acting on the body, the true stress distribution minimizes the strain energy.

B) 3-phase model (Fig 3.20)
Exact solution for G_{23}: work by R.M. Christensen and K.H. Lo, "Solutions for Effective Shear Properties in Three Phase Sphere and Cylinder Models", J. Mech. And Phys. of Solids, 1979.

Then: $E_2 = 2(1 + v_{23})G_{23}$



C) Example 3.11

3.3.3.4 Axial Shear Modulus G12 (or In-Plane Shear Modulus)
1) RVE:
CCA or CAM, see Fig. 3.19
But it's no longer an axisymmetric case

2) Coordinates
Rectangular coordinates: x_1, x_2, x_3
Displacements: u_1 – axial displacement
u_2, u_3 — displacement along 2- and 3- axis.
Cylindrical coordinates: r, θ, z

3) Solution method:
Simi-inverse method
A certain form of displacement solution is assumed, typically with parameters to be determined so that equilibrium and/or boundary conditions can be met.

4) Assumed displacements

$$u_1 = -\frac{\gamma_{12}^0}{2} x_2 + F\left(x_2, x_3\right)$$

$$u_2 = \frac{\gamma_{12}^0}{2} x_1$$

$$u_3 = 0 \ , \qquad\qquad (3.113a, b, c)$$

Where, F(x_2, x_3) is a unknown function to be determined;
γ_{12}^0 is imposed shear strain, similar to the ε_1 that is assumed in 3.3.3.1 and 3.3.3.2 in order to evaluate E_1 and v_{12}. Here, γ_{12}^0 is imposed so as to evaluate G_{12}.

5) Roadmap to G_{12}
- Condition $F(x_2, x_3)$ must meet: (Eq. 3.117)
- Transformation F(x_2, x_3) to $F(r, \theta)$: (Eq. 3.118 ~ Eq. 3.126)
- Solution of $F(r, \theta)$: (Eq. 126), $F(r, \theta) = \left(Ar + \frac{B}{r}\right) cos\theta$
- $u_1(r, \theta)$, $\tau_{1r}(r, \theta)$
- u_1, τ_{1r} for fiber cylinder (A_1, γ_{12}^0)
- u_1, τ_{1r} for matrix cylinder $(A_2, B_2, \gamma_{12}^0)$
- 4 boundary conditions
$r = a$, 2 conditions
$r = b$, 2 conditions
A_1, A_2, B_2 in terms of γ_{12}^0
$\therefore\ \tau_{12}^m$ in terms of γ_{12}^0

$$G_{12} = \frac{\tau_{12}^m|_{r=b,\ \theta=0}}{\gamma_{12}^0}$$

Final expression:

$$G_{12} = G_m \left[\frac{G_f\left(1+V_f\right)+G_m\left(1-V_f\right)}{G_f\left(1-V_f\right)+G_m\left(1+V_f\right)}\right]. \qquad\qquad (3.160)$$

6) Example 3.12 for numerical applications of (3.160).

Example: Two fiber-reinforced laminas of unidirectional continuous fibers consistent of pitch-based graphite fibers and epoxy, and Kevlar 49 and epoxy respectively. The laminas have the same corrected volume fractions: $V_f' = 58\%$ and $V_m' = 42\%$. The Young's moduli of the fibers can be found in Tables 1.8, 1.9, and 1.10 of the text. Poisson's ratios are, 0.22 for graphite, 0.35 for Kevlar 49, and 0.32 for epoxy, respectively.
For each of the laminas, determine the four elastic moduli by:
(a) The mechanics of materials approach;
(b) The semi-empirical approach; and
(c) The theory of elasticity approach.

(a) The mechanics of materials approach

	Graphite/Epoxy	Kevlar/Epoxy
$E_1, Mpsi$	32.13	11.25
E_f/E_m	100	34.5
$E_2, Mpsi$	1.29	1.26
v_{12}	0.262	0.337
G_f/G_m	108	33.8
$G_{12}, Mpsi$	0.489	Revised: 0.477

(b) The semi-empirical approach
- from previous notes

(c) The theory of elasticity approach
- from MATLAB

<u>Summary of results:</u>
Graphite/Epoxy

	$Mech\ of\ Mat'ls$	$H-T$	$Elasticity$
$E_1, Mpsi$	32.13	32.13	32.13
E_f/E_m	100	100	100
$E_2, Mpsi$	1.29	2.79	2.01
v_{12}	0.262	0.262	0.255
G_f/G_m	108	108	108
$G_{12}, Mpsi$	0.489	0.803	0.759

Kevlar Epoxy

	$Mech\ of\ Mat'ls$	$H-T$	$Elasticity$
$E_1, Mpsi$	11.25	11.25	11.25
E_f/E_m	34.5	34.5	34.5
$E_2, Mpsi$	1.26	2.52	1.88
v_{12}	0.337	0.337	0.340
G_f/G_m	33.8	33.8	33.8
$G_{12}, Mpsi$	0.477	0.747	0.711

Summary re: Elasticity approach

1) CCA or CAM, and 3-phase model
2) "Exact" solutions: E_1 v_{12} G_{12}, $G_{23} \rightarrow E_2$
3) Values typically fall between those by mech of materials , and by H-T.

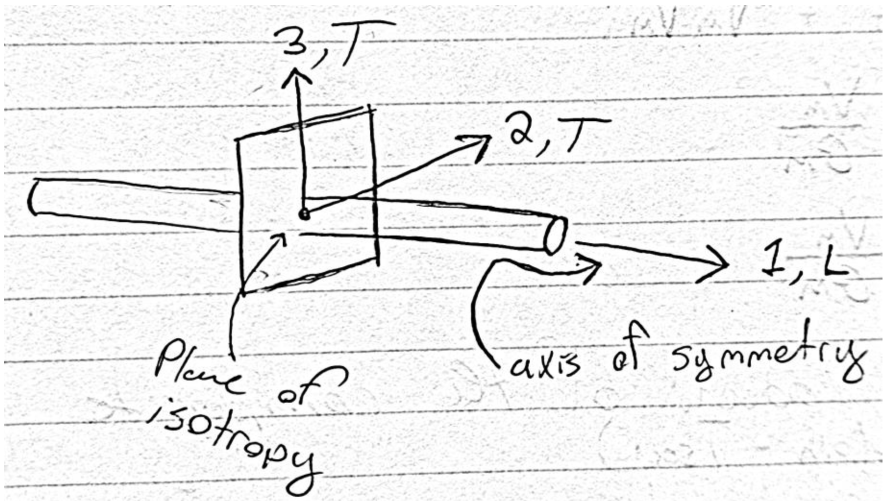
Fig 3.21 comparing E_2 by 3 approaches

Fig 3.22 comparing G_{12} by 3 approaches

4) Derivations seem lengthy, so do some final expressions
5) It introduced artificial voids.

3.3.4 Transversely Isotropic Fibers

- Glass fibers are isotropic
- Carbon/graphite and aramid fibers are transversely isotropic



1) Elastic modulus for the fibers
E_{fL}: Young's Modulus in the Longitudinal direction
E_{fT}: Young's Modulus in the Transverse direction
v_{fL}: Major Poisson's Ratio (that is v_{fLT})
v_{fT}: Minor Poisson's Ratio (or v_{fTL})
G_{fT}: Shear modulus in the L-T plane

2) Elastic moduli of the composite (Mechanics of Materials Approach)

$$E_1 = E_{fL}V_f + E_mV_m$$

$$v_{12} = v_{fL}V_f + v_mV_m$$

$$\frac{1}{E_2} = \frac{V_f}{E_{fT}} + \frac{V_m}{G_m}$$

$$\frac{1}{G_{12}} = \frac{V_f}{G_{f1}} + \frac{V_m}{G_m}$$

3) Elastic moduli of the composite (Halpin-Tsai Method)
E_2: $E_f \leftarrow E_{fT}$
G_{12}: $G_f \leftarrow G_{fT}$

4) Elastic moduli of the composite (Elasticity approach)
E_1: $E_f \leftarrow E_{fL}$; $v_f \leftarrow v_{fL}$
v_{12}: $v_f \leftarrow v_{fL}$; $E_f \leftarrow E_{fL}$
G_{12}: $G_f \leftarrow G_{fT}$
E_2: $v_f \leftarrow v_{fT}$
$G_f \leftarrow G_{fTT}$
$E_f \leftarrow E_{fT}$

G_{fTT} fibers shear modulus in the T-T plane

$$G_{fTT} = \frac{E_{fT}}{2(1 + v_{fT})}$$

Example: Find E_1, E_2, G_{12}, and v_{12} by the three approaches.

Graphite fibers:
$V_f = 0.6$
$E_{fL} = 345\ GPa$
$E_{fT} = 9.66\ GPa$
$G_{fT} = 2.07\ GPa$
$v_{fL} = 0.2$

Epoxy fibers:
$V_m = 0.4$
$E_m = 3.45\ GPa$
$G_m = 1.28\ GPa$
$v_m = 0.35$

Solution:
$v_{fT} = 0.0056$
$G_{fTT} = 4.80\ GPa$

Mechanics of Materials method:
$E_1 = 208.38\ GPa$
$E_2 = 5.6163\ GPa$
$G_{12} = 1.6602\ GPa$
$v_{12} = 0.26000$

Halpin-Tsai method:
$E_2 = 6.4988\ GPa$
$\rho = 2.2419$
$n = 0.35701$

$G_{12} = 1.7070\ GPa$
$\rho = 1.2419$
$n = 0.21587$

Elasticity method:
$E_1 = 208.42\ GPa$
$v_{12} = 0.25103$
$G_{12} = 1.7019\ GPa$

$A = -38.7795$
$B = 22.3345$
$C = 69.2433$

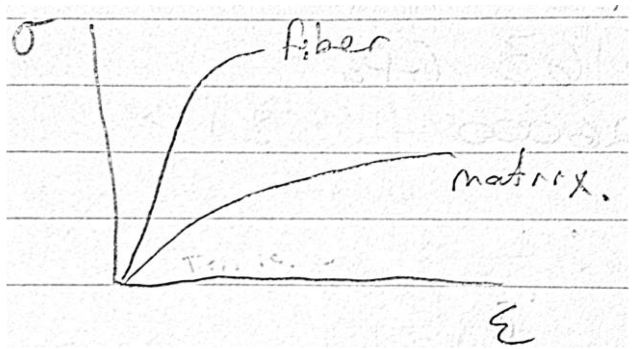
$G_{23} = 2.59971\ GPa$
$v_{23} = 0.275602$

$E_2 = 6.6324\ GPa$

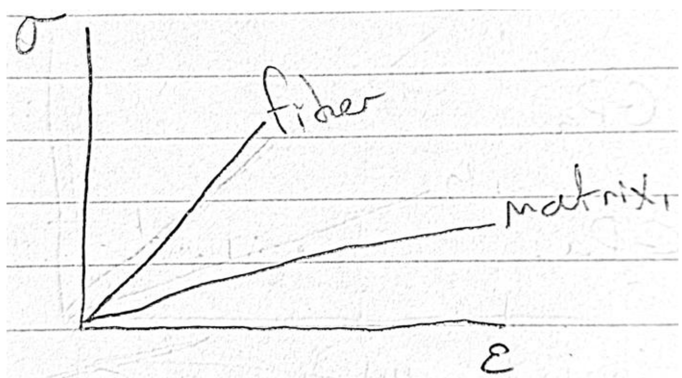
3.4 Ultimate Strength of a Unidirectional Lamina

It's observed that,
1) Fibers behave like ductile materials, but matrix behaves like brittle material

2) Both are not linearly elastic



Within the section, it's assumed that $\sigma - \varepsilon$ plots for both fiber and matrix are linear up to failure (by breakage or by fracture)



What we need before determining ultimate strengths.
Fibers: V_f or V_f'
$(\sigma_f)_{ult}$: strength of fibers in tension and compression
$(\tau_f)_{ult}$: ultimate shear strength of fibers

Matrix: V_m or V_m'
$(\sigma_m)_{ult}$ or $(\sigma_m^T)_{ult}$: ultimate strength of matrix in tension
$(\sigma_m^C)_{ult}$: ultimate strength of matrix under compression
$(\tau_m)_{ult}$: ultimate shear strength of matrix

Composite: E_1, E_2, v_{12}, G_{12}
What we determine:
$(\sigma_1^T)_{ult}$: Ultimate longitudinal tensile strength
$(\sigma_1^C)_{ult}$: Ultimate longitudinal compressive strength
$(\tau_{12})_{ult}$: Ultimate shear strength

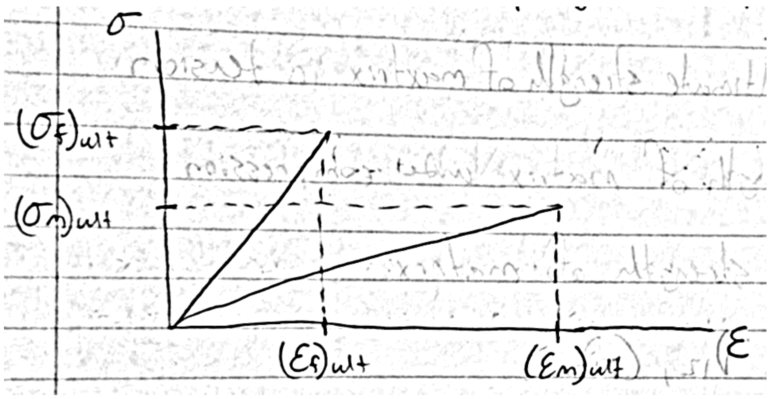
5 sub-sections to determine such ultimate strength

3.4.1 Longitudinal Tensile Strength $\left(\sigma_1^T\right)_{ult}$

There are 2 scenarios:
Fibers fail first; or matrix fails first

<u>Fibers-fail-first</u>:
Typically takes place for:

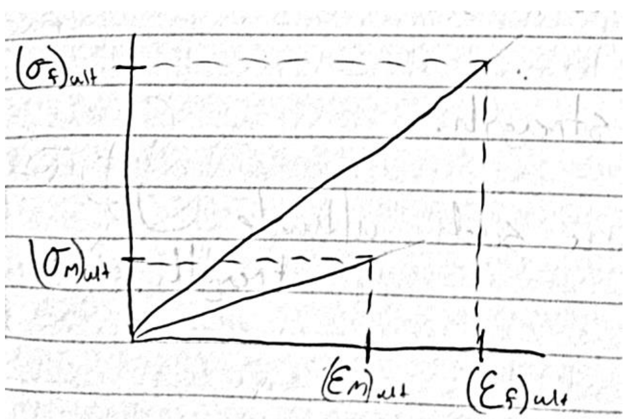
- MMCs
- Thermoplastic polymer composites



$\left(\varepsilon_f\right)_{ult} < (\varepsilon_m)_{ult}$ indicates that fibers fail first.

<u>Matrix-fail-first</u> (not in the textbook):
Typically takes place if:

- V_f is low
- PMC



$(\varepsilon_m)_{ult} < \left(\varepsilon_f\right)_{ult}$ indicates that matrix fail first.

$\because$ fibers and matrix taking the load together
$\therefore \sigma_1$ follows ROM, similar to ROM on E_1
$\therefore \sigma_1 = V_f\sigma_f + V_m\sigma_m$
$= V_f E_f \varepsilon + V_m E_m \varepsilon \quad (Eq. A)$

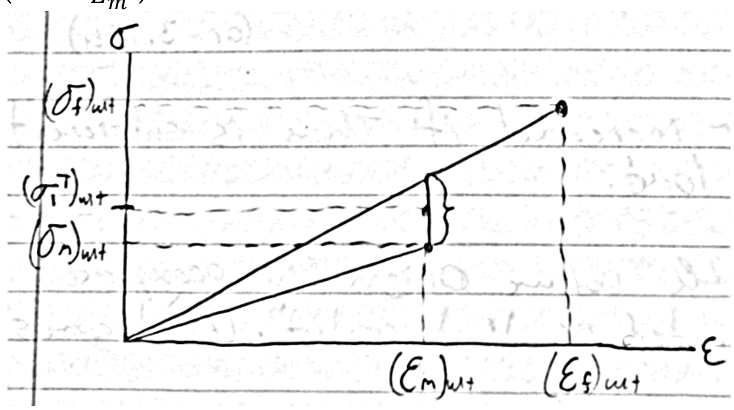
$\because$ matrix-fails-first $(\varepsilon_m)_{ult} < (\varepsilon_f)_{ult}$

$\therefore \varepsilon = (\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m}$

$\therefore \sigma_1 = (\sigma_m)_{ult}\left(V_m + \frac{V_f E_f}{E_m}\right) \quad (Eq. A1)$

The full load transfers to the fibers, but due to low V_f, fibers see a large jump in stress and fails immediately.

$\therefore (\sigma_1^T)_{ult} = (\sigma_m)_{ult}\left(V_m + \frac{V_f E_f}{E_m}\right) \quad (Eq. B)$



Fibers-fail-first

Fibers and matrix taking the load together

$\therefore \sigma_1 = V_f E_f \varepsilon + V_m E_m \varepsilon \quad (Eq. A)$

Load reaches the level that will break the fibers;

$\because (\varepsilon_f)_{ult} < (\varepsilon_m)_{ult}$

$\therefore \varepsilon = (\varepsilon_f)_{ult} = \frac{(\sigma_f)_{ult}}{E_f}$

$\therefore \sigma_1 = (\sigma_f)_{ult}\left(V_f + \frac{V_m E_m}{E_f}\right) \quad (Eq. A2)$

Load transfers to the matrix, causing increase in stress in the matrix, and fracture in matrix, leading to failure of composite.

$\therefore (\sigma_1^T)_{ult} = (\sigma_f)_{ult}\left(V_f + \frac{V_m E_m}{E_f}\right) \quad (Eq. C)$

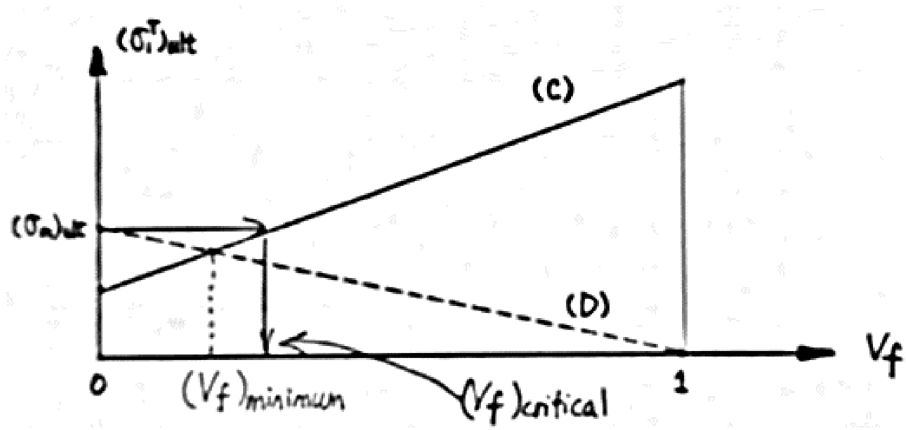
Loads can be further increased if there is sufficient matrix to take the load.

Once fibers break, the volume originally occupied by the fibers is regarded as void content. (Eq. A) becomes:

$\sigma_1 = V_m \sigma_m$

$(\sigma_1^T) = V_m(\sigma_m)_{ult} \quad (Eq. D)$

Now the question is which $(\sigma_1^T)_{ult}$ to use, (Eq. C) or (Eq. D)?



$$\left(V_f\right)_{minimum} = \frac{(\sigma_m)_{ult} - E_m \cdot \frac{\sigma_f}{E_f}}{\left(\sigma_f\right)_{ult}\left(1 - \frac{E_m}{E_f}\right) + (\sigma_m)_{ult}}$$

If $V_f < \left(V_f\right)_{minimum}$ use $(Eq.D)$
If $V_f \geq \left(V_f\right)_{minimum}$ use $(Eq.C)$

However it is practically impossible if $(\sigma_1^T)_{ult}$ by $(Eq.C)$ or $(Eq.D)$ is less than $(\sigma_m)_{ult}$

$$\left(V_f\right)_{critical} = \frac{(\sigma_m)_{ult} - E_m \cdot \frac{\left(\sigma_f\right)_{ult}}{E_f}}{\left(\sigma_f\right)_{ult} - \left(\sigma_f\right)_{ult}\frac{E_m}{E_f}}$$

If $V_f < \left(V_f\right)_{critical}$ $(\sigma_1^T)_{ult} = (\sigma_m)_{ult}$
If $V_f \geq \left(V_f\right)_{critical}$ use $(Eq.C)$

3.4.2 Longitudinal Compressive Strength $\left(\sigma_1^C\right)_{ult}$

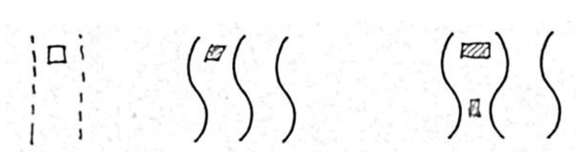
1. Modes of failure (3 or 4)

1) Tensile Failure:

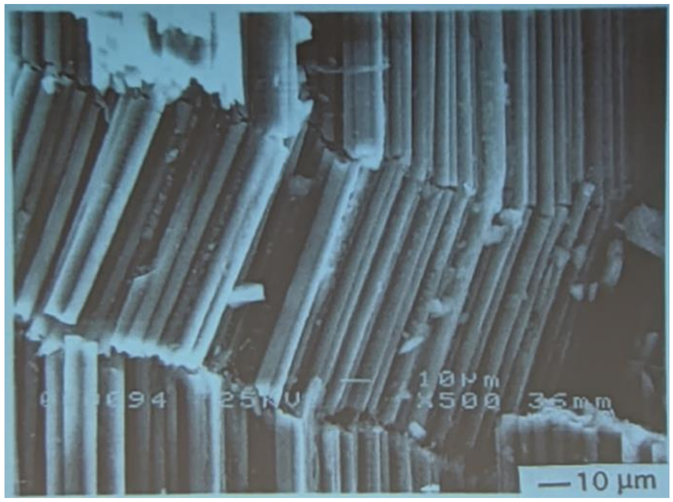
- Excessive tensile strain in the matrix
  - Matrix fractures or fiber-matrix bonding fractures
- Common with thermoset

2) Microbuckling: refers to fibers buckling “inside” the matrix

- Common with Kevlar
- Two possible microbucklng modes
  - In-phase or shear mode
  - Out-of-phase or extensional mode



3) Kinging/Shearing: refers to direct shear failure of fibers, or kinking of fibers if not sheared.



2. Tensile failure

$$(\sigma_1^c)_{ult} = \frac{E_1(\varepsilon_2^T)_{ult}}{\nu_{12}}. \qquad (3.169)$$

3. Microbuckling

Extensional mode:

$$S_1^c = 2\left[V_f + (1-V_f)\frac{E_m}{E_f}\right]\sqrt{\frac{V_f E_m E_f}{3(1-V_f)}}, \qquad (3.173a)$$

Shear mode:

$$S_2^c = \frac{G_m}{1-V_f}. \qquad (3.173b)$$

4. Shearing/kinking:

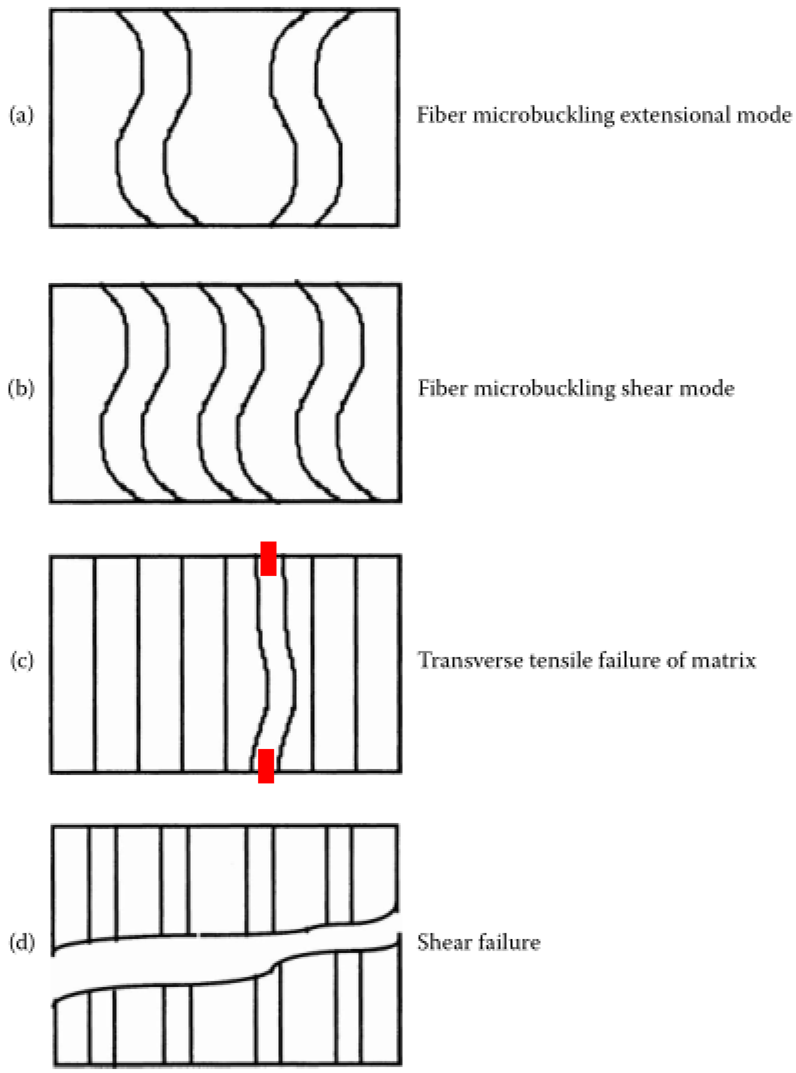
$$(\sigma_1^c)_{ult} = 2\left[(\tau_f)_{ult}V_f + (\tau_m)_{ult}V_m\right]. \qquad (3.175)$$

5. Conclusion:

4 values of $\left(\sigma_1^C\right)_{ult}$ by $(Eq. 3.169)$, $(Eq. 3.173a)$, $(Eq. 3.173b)$, and $(Eq. 3.175)$

Choose the smallest value.

Example 3.14 for detail.



3.4.3 Transverse Tensile Strength $(\sigma_2^T)_{ult}$

$$(\sigma_2^T)_{ult} = E_2(\varepsilon_2^T)_{ult}, \tag{3.182}$$

Example 3.15 for detail.

3.4.4 Transverse Compressive Strength $(\sigma_2^C)_{ult}$

$$(\sigma_2^C)_{ult} = E_2(\varepsilon_2^C)_{ult}, \tag{3.183}$$

Example 3.17 for detail.

3.4.5 In-Plane Shear Strength $(\tau_{12})_{ult}$

$$(\tau_{12})_{ult} = G_{12}(\gamma_{12})_{ult}$$

$$= G_{12}\left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_{mult}. \qquad (3.191)$$

Example 3.17 for detail.

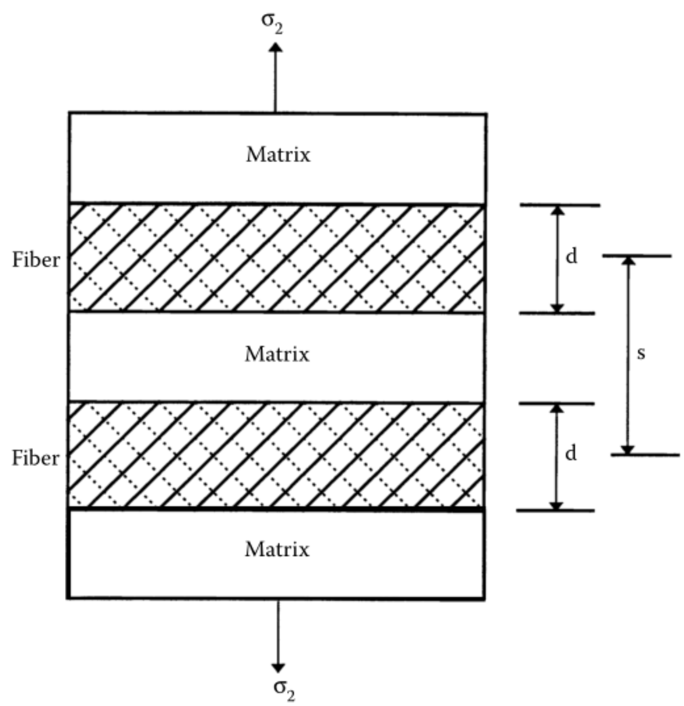
A few notes regarding equation in subsections 3.4.2 ~ 3.4.4:

1) S_1^C and S_2^C are by buckling analysis (or eigenvalue analysis)
2) All other equations are mostly results of doing simple mechanics of materials analyses; the RVEs are similar or identical to those used in 3.3.1
3) Where $(\varepsilon_2^T)_{ult}$ is needed, it should be the lesser of

- Empirical formula, $(Eq. 3.170)$
- Mechanics of materials approach $(Eq. 3.171)$

4) ROM is applied a few times.

$(\varepsilon_2^T)_{ult}$: mechanics of materials formula:



Transverse-direction deformations
Fiber: $\delta_f = \varepsilon_f \cdot d$
Matrix: $\delta_m = \varepsilon_m \cdot (s - d)$
Composite: $\delta_c = \varepsilon_c \cdot s$

But, $\delta_c = \delta_f + \delta_m$

$\therefore\ \varepsilon_c \cdot s = \varepsilon_f \cdot d + \varepsilon_m \cdot (s-d)$

$\varepsilon_c = \frac{d}{s}\varepsilon_f + \left(1-\frac{d}{s}\right)\varepsilon_m$

$E_f\varepsilon_f = E_m\varepsilon_m\ (same\ stress\ in\ fibers\ and\ in\ matrix)$

$\therefore \varepsilon_c = \varepsilon_m\left[\frac{d}{s}\frac{E_m}{E_f} + \left(1-\frac{d}{s}\right)\right]$

If ε_m reaches $(\varepsilon_m^T)_{ult}$, ε_c reaches its ultimate value $(\varepsilon_2^T)_{ult}$; that is:

$(\varepsilon_2^T)_{ult} = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1-\frac{d}{s}\right)\right](\varepsilon_m^T)_{ult}$

Where,

$(\varepsilon_m)_{ult}$ = ultimate tensile strain of the matrix

d = diameter of the fibers

s = center-to-center spacing between fibers

$\frac{d}{s}$ depends on packing and V_f

And, $(\varepsilon_2^T)_{ult}$ the empirical formula

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult}(1 - V_f^{1/3}), \quad (3.170)$$

State-of-the-art in terms of predicting or evaluating ultimate strengths

a) With the availability of new (and newer) materials, there are composites with combinations of brittle fibers and brittle matrix, brittle fibers and ductile matrix, in addition to ductile fibers plus brittle matrix as discussed in class.
b) In terms of the combination of ductile fibers plus brittle matrix, the trend seems to be moving away from assuming linear $\sigma - \varepsilon$ up to failure for matrix; Various approaches are seen to deal with the non-linearity, and to various degree of success.
c) It is known that predicting elastic moduli remains a challenge. It is an even bigger challenge for predicting ultimate strengths.

Chapter 2: Macromechanical Analysis of a Lamina

2.1

2.2

2.3 – done before chapter 3 (previously covered)

2.4

2.5

2.6

2.7 – stiffness matrix, compliance matrix, and their applications

2.8 – failure theories of a lamina

2.9 – hydrothermal situation (not covered)

Basics

Contacted notation, $[C]$ and $[S]$

Stress vector

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Strain vector

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Stiffness matrix $[C]$:

$$\{\sigma\} = [C]\{\varepsilon\}$$

Compliance matrix $[S]$:

$$\{\varepsilon\} = [S]\{\sigma\}$$

And:

$$[S] = [C]^{-1}$$
$$[C] = [S]^{-1}$$

Shear strains are the so-called engineering shear strains instead of torsional shear strains

Example 1:

A fiber-reinforced lamina of unidirectional continuous fibers consists of the high-strength graphite fiber and epoxy. The lamina has $V_f = 0.45$ and zero void content. Find $(\sigma_1^T)_{ult}$ of the lamina, given the following:

$E_f = 280\ GPa$
$\left(\sigma_f\right)_{ult} = 5700\ MPa$
$E_m = 3.45\ GPa$
$(\sigma_m)_{ult} = 60\ Mpa$

$$\left(\varepsilon_f\right)_{ult} = \frac{\left(\sigma_f\right)_{ult}}{E_f} = 0.020$$
$$(\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m} = 0.017$$

Therefore, matrix fails first, and Eq. (B) is used to determine $(\sigma_1^T)_{ult}$:

$$(\sigma_1^T)_{ult} = (\sigma_m)_{ult}\left(V_m + V_f \frac{E_f}{E_m}\right) = 2{,}224\ MPa$$

Example 2:

A fiber-reinforced lamina of unidirectional continuous fibers consists of the high-modulus graphite fiber and epoxy. The lamina has $V_f = 0.45$ and zero void content. Find $(\sigma_1^T)_{ult}$ of the lamina, given the following:

$E_f = 530\ GPa$
$\left(\sigma_f\right)_{ult} = 1900\ MPa$
$E_m = 3.45\ GPa$
$(\sigma_m)_{ult} = 60\ Mpa$

$$\left(\varepsilon_f\right)_{ult} = \frac{\left(\sigma_f\right)_{ult}}{E_f} = 0.0036$$
$$(\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m} = 0.017$$

Therefore, fiber fails first, and we need to decide what equation to use.
$(Eq. 3.165)$: $V_{minimum} = 0.0244 = 2.44\%$
$(Eq. 3.166)$: $V_{critical} = 0.0252 = 2.52\%$

So, use Eq. (C) or $(Eq. 3.164)$

$$(\sigma_1^T)_{ult} = \left(\sigma_f\right)_{ult}\left(V_f + V_m \frac{E_m}{E_f}\right) = 861.8\ MPa$$

Example 3:

For the lamina in Example 2, evaluate $(\sigma_1^T)_{ult}$, $(\sigma_1^C)_{ult}$, $(\sigma_2^T)_{ult}$, $(\sigma_2^C)_{ult}$ and $(\tau_{12})_{ult}$, given the following:

$E_f = 530\ GPa$
$v_f = 0.23$
$G_f = 215\ GPa$
$(\sigma_f)_{ult} = 1900\ MPa$
$(\tau_f)_{ult} = 36\ MPa$

$E_m = 3.45\ GPa$
$v_m = 0.30$
$G_m = 1.33\ GPa$
$(\sigma_m)_{ult} = 72\ MPa$
$(\tau_m)_{ult} = 34\ MPa$

$E_1 = 240.4\ GPa$
$E_2 = 11.66\ GPa$
$v_{12} = 0.2685$
$G_{12} = 3.472\ GPa$
Hexagonal packing
Solution:
$(\sigma_1^T)_{ult} = 861.8\ MPa$
$(\sigma_1^C)_{ult} = 69.80\ MPa$
$(\sigma_2^T)_{ult} = 56.87\ MPa$

$(\sigma_2^C)_{ult} = 103.5\ MPa$
$(\tau_{12})_{ult} = 26.63\ MPa$

Textbook changes (errors in equations):

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^2),$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4). \qquad (2.104a\text{–}f)$$

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$$= \frac{1}{2}\tan^{-1}\left(-\frac{2\times10^6 + 3\times10^6}{2(4\times10^6)}\right)$$

$$= 16.00^0$$

8. The principal strains are given by[4]

Bottom line should be -16.00°

from Equation (3.2) as

$$w_f + w_m = w_c \cdot$$

From the definition of the density of a single material,

$$w_c = r_c v_c,$$
$$w_f = r_f v_f, \text{ and} \tag{3.3a–c}$$
$$w_m = r_m v_m.$$

Bottom line should be $w_m = \rho_m v_m$

FIGURE 3.23
Longitudinal and transverse direction in a transversely isotropic fiber.

$$E_1 = E_{fL} V_f + E_m V_m,$$

$$\frac{1}{E_2} = \frac{V_f}{E_{fT}} + \frac{V_m}{E_m},$$

$$\nu_{12} = \nu_{fT} V_f + \nu_m V_m,$$

Bottom line should be $v_{12} = v_{fL} V_f + v_m V_m$

$$+\frac{1}{2}\begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix}(10^9)[(0.0075)^2 - (0.0025)^2]$$

$$[B] = \begin{bmatrix} -3.129\times10^6 & 9.855\times10^5 & -1.972\times10^6 \\ 9.855\times10^5 & 1.158\times10^6 & -1.972\times10^6 \\ -1.072\times10^6 & -1.072\times10^6 & 9.855\times10^5 \end{bmatrix} Pa\text{-}m^2 .$$

Highlighted areas should be zero.

Midterm Review

Chapter 1
Definition of composite materials:
Reinforcing phase: purpose, shapes, types of fibers
matrix: purpose, materials choices for matrix

Manufacture of Fibers

Applications

Chapter 2
2.3: Independent mechanical properties vs. Types of materials
(and why we use those constants as well)
e.g. orthotropic materials, 9 transversely isotropic materials (what do we need to determine 9 transversely isotropic materials, what is the plane of symmetry), resulting 5…

Chapter 3
3.2: V_f, V_m, W_f, W_m, void content
a few fibers + a few matrices + voids
$V_f', V_m'; V_{fmax}, RVE$
When an equation in the text is only valid for zero void content

3.3: Isotropic fibers + isotropic matrix
transversely isotropic fibers + isotropic matrix
mech. of mat'ls
Halpin-Tsai
elasticity (E_1, E_2, v_{12} won't appear on midterm, too long – but G_{12} could)

2.4 Hooke's Law for a 2D Unidirectional Lamina

1. Plane stress

$\sigma_3 = \tau_{23} = \tau_{31} = 0$

$\gamma_{23} = \gamma_{31} = 0$

However, $\varepsilon \neq 0$ (See Eq. 2.76)

2. [C] and [S] for plane stress situation

$[C]_{6x6} \rightarrow [Q]_{3x3}$ Reduced stiffness/compliance matrix

$[S]_{6x6} \rightarrow [S]_{3x3}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad (Eqn. 2.78 *)$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \quad (Eqn. 2.77 *)$$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q_{22} & 0 \\ sym. & & Q_{66} \end{bmatrix} \quad (Eqn. 2.78)$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ sym. & & Q_{66} \end{bmatrix} \quad (Eqn. 2.77)$$

3. Q_{ij} $(2.93\ a{\sim}d)$– in terms of $E_1\ E_2\ G_{12}\ and\ v_{12}$

$S_{ij}(2.92\ a{\sim}d)$ – in terms of $E_1\ E_2\ G_{12}\ and\ v_{12}$

$i, j = 1, 2, 6$

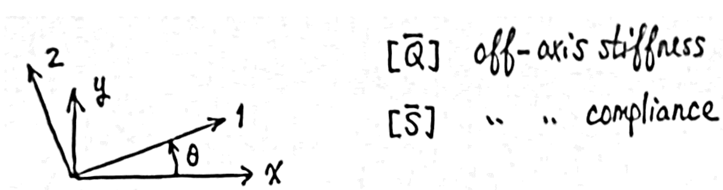
2.5 Hooke's Law for a 2D Unidirectional Angle Lamina (off-axis stiffness and compliance)

In 2.4, we have:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad or \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

[Q], [S]: used with 1-2-3 coordinates, or local coordinates.

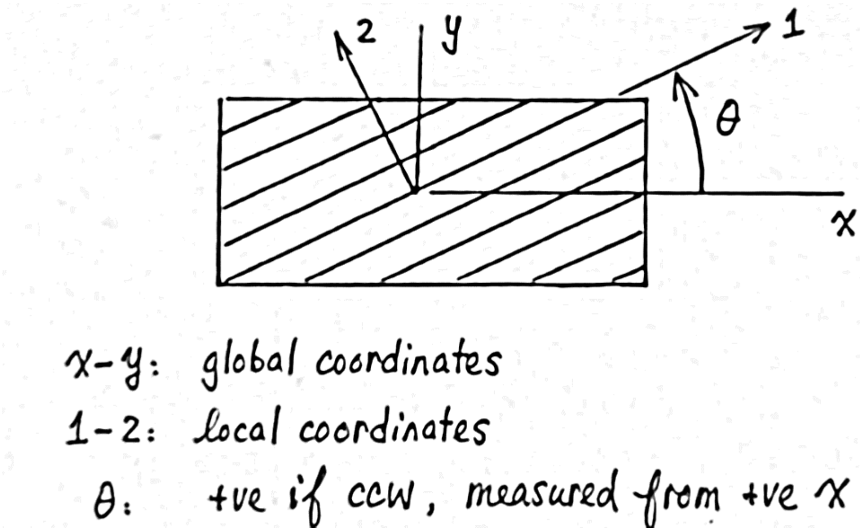
For application, more than 1 lamina will be used; and the laminas are typically placed at various angles, hence angle lamina.



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad or \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\bar{S}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$[Q] \rightarrow [\bar{Q}]? \quad [S] \rightarrow [\bar{S}]?$$

Step 1: Transformation Matrix [T]



Global stresses: $\sigma_x, \sigma_y, \tau_{xy}$
Local stresses: $\sigma_1, \sigma_2, \sigma_6 = \tau_{12}$

Define $c = cos\theta,\ \ s = sin\theta$
Then:

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}$$

$[T]$ is orthogonal, i.e., $[T]^{-1} = [T(-\theta)]$

$$\therefore \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$or \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

Where:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T^{-1}][Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

Step 2: [T] is applicable to tensorial strains, i.e.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\varepsilon_6 \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

<u>Step 3:</u> Reuter's matrix

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\varepsilon_6 \end{Bmatrix}$$

$$= [R][T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

$$= [R][T][R^{-1}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

<u>Step 4:</u> Subs into (A)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Where:
$[\bar{Q}] = [T]^{-1}[Q][R][T][R]^{-1}$

And it can be shown that:
$[\bar{S}] = [R][T]^{-1}[R]^{-1}[S][T]$
$[\bar{S}] = [\bar{Q}]^{-1}$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q_{22} & 0 \\ sym. & & Q_{66} \end{bmatrix}$$

$[\bar{Q}]$ = full matrix symmetric ($Eqn. 2.104\ a{\sim}f$)

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ sym. & & S_{66} \end{bmatrix}$$

$[\bar{S}]$ = full matrix symmetric ($Eqn. 2.104\ a{\sim}f$)

2.6 Engineering Constants of an Angle Lamina

Elastic moduli in the x and y directions:

$$E_x = 1/\overline{S_{11}}$$
$$E_y = 1/\overline{S_{22}}$$

Shear modulus in the $x - y$ plane:

$$G_{xy} = 1/\overline{S_{66}}$$

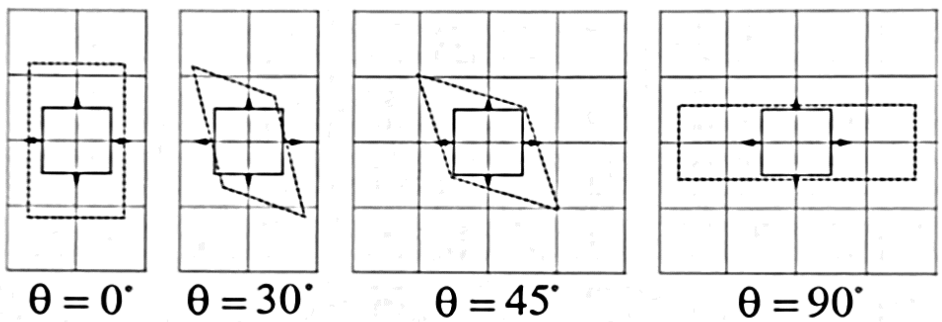
Poisson's ratios:

$$v_{xy} = -\overline{S_{12}}/\overline{S_{11}}$$
$$v_{yx} = -\overline{S_{12}}/\overline{S_{22}}$$

Shear coupling factors:
Unlike isotropic materials, an angle lamina may develop shear strains when subject to normal stresses; or when being stretched/compressed, shear stress may be developed.

$m_x = -\overline{S_{16}} \cdot E_1$ (non-dimensional quantity)
relating ε_x to τ_{xy}, or σ_x to γ_{xy}

$m_y = -\overline{S_{26}} \cdot E_1$ (non-dimensional quantity)
relating ε_x to τ_{xy}, or σ_y to γ_{xy}



No shear strain (deformation) if angle is 0° or 90°

2.8 Strength Failure Theories of an Angle Lamina
7 sub-sections

Overview:
2.8.7 - Comparing theories, and with experimental data

2.8.2 – Strength ratio
2.8.3 – Failure envelopes

2.8.1 – 4 theories (Note: τ_{12} and σ_6 are interchangeable, so are γ_{12} and ε_6)
2.8.4
2.8.5
2.8.6

Overview on Strength Theories

A) Purpose of strength theories

Similar to isotropic materials such as metals, strength theories are to allow for determination of when failure occurs if a component is in 2- or 3- dimensional state of stress.

B) Strength theories available to isotropic materials

Ductile materials

- Max. shear stress theory
- Distortion energy theory (von Mises theory)

Brittle materials

- Max. normal stress theory
- Coulomb-Mohr theory
- Modified Coulomb-Mohr theory

C) Challenges when dealing with unidirectional laminas

- They are direction/orientation dependent
- Tensile and compressive strengths are different in both the longitudinal and transverse directions; e.g., $(\sigma_1^T)_{ult} > \left(\sigma_1^C\right)_{ult}$, but $(\sigma_2^T)_{ult} < \left(\sigma_2^C\right)_{ult}$
- They retain part of ductile behavior; at the same time , they retain part of brittle behavior

D) List of strength theories

Unidirectional Laminas	Isotropic Materials
Max. stress	Max. normal stress
Max. strain	Max. normal strain
Tsai-Hill	Distortion energy
Tsai-Wu (Quadratic)	Total strain energy

2.8.7 Comparison of Experimental Results with Failure Theories

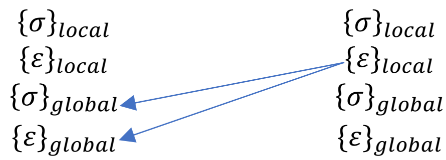
1) Max. stress & max strain theories don't compare well with experimental results.

2) Tsai-Hill and Tsai-Wu theories don't compare well with experimental results.

3) Tsai-Hill or Tsai-Wu theory, however, doesn't indicate the specific mode of failure, which max. stress and max. strain theories do.

Failure Mode	Shorthand Notation
1: Tensile failure in longitudinal direction (or fiber direction)	1T
2: Compressive failure in longitudinal direction	1C
3: Tensile failure in transverse failure	2T
4: Compressive failure in the transverse direction	2C
5: in-plane shear	6S

Transformation between local (1-2-3) axes and global (x-y-z) axes:



Example 1:

A unidirectional graphite/epoxy lamina ($\theta = 50°$) is subject to $\sigma_x = 0$, $\sigma_y = -3\ MPa$, $\tau_{xy} = 4\ MPa$. Find the local stresses and local strains. Given, for the lamina, $E_1 = 181\ GPa$, $E_2 = 10.3\ GPa$, $v_{12} = 0.28$, and $G_{12} = 7.2\ GPa$.

Solution:

Global stresses → $(via\ [T])$ Local stresses → $(via\ [S])$ Local strains

$$[T] = \begin{bmatrix} 0.4132 & 0.5868 & 0.9848 \\ 0.5868 & 0.4131 & -0.9848 \\ -0.4924 & 0.4924 & -0.1736 \end{bmatrix}$$

$$[S] = \begin{bmatrix} 0.5525 & -0.1547 & 0 \\ & 9.709 & 0 \\ sym. & & 13.89 \end{bmatrix} (10^{-9}) \left(\frac{1}{Pa}\right)$$

Local stress $= [T]$ $*$global stress $= \begin{Bmatrix} 2.179 \\ -5.179 \\ -2.172 \end{Bmatrix} (MPa)$

Local strain $= [S]$ $*$global stress $= \begin{Bmatrix} -0.0200 \\ -0.505 \\ -0.302 \end{Bmatrix} (10^{-3})$

Example 2:

A unidirectional graphite/epoxy lamina ($\theta = 50°$) is subject to $\sigma_x = \sigma_1$, $\sigma_y = -\sigma$, and $\tau_{xy} = 0$ (where σ is in Pa). Find the local stresses and local strains in terms of σ. Given, for the lamina, $E_1 = 181\ GPa$, $E_2 = 10.3\ GPa$, $v_{12} = 0.28$, $G_{12} = 7.2\ GPa$.

Solution:

Global stresses → $(via\ [T])$ Local stresses → $(via\ [S])$ Local strains

Local stress $= [T]$ $*$global stress

$$[T] \begin{Bmatrix} \sigma \\ -\sigma \\ 0 \end{Bmatrix} = \sigma \begin{Bmatrix} -0.1736 \\ 0.1736 \\ -0.9848 \end{Bmatrix}$$

Local strain $= [S]$ $*$local stress

$$\sigma[S] \begin{Bmatrix} -0.1736 \\ 0.1736 \\ -0.9848 \end{Bmatrix} = \sigma \begin{Bmatrix} -0.001228 \\ 0.01713 \\ -0.1368 \end{Bmatrix} (10^{-9})$$

2.8.1 Max. Stress Failure Theory

Given $\sigma_!, \sigma_2, and\ \tau_{12}$, failure of the lamina occurs when any one of the following is true,

$$\sigma_1 > (\sigma_1^T)_{ult} \quad \text{if } \sigma_1 \geq 0$$
$$\sigma_1 < -(\sigma_1^C)_{ult} \quad \text{if } \sigma_1 < 0$$
$$\sigma_2 > (\sigma_2^T)_{ult} \quad \text{if } \sigma_2 \geq 0$$
$$\sigma_2 < -(\sigma_2^C)_{ult} \quad \text{if } \sigma_2 < 0$$

In terms of SR (strength ratio), the theory reads

$$SR_1 = (\sigma_1^T)_{ult}/\sigma_1 \qquad \sigma_1 \geq 0$$
$$SR_1 = -(\sigma_1^C)_{ult}/\sigma_1 \qquad \sigma_1 < 0$$
$$SR_2 = (\sigma_2^T)_{ult}/\sigma_2 \qquad \sigma_2 \geq 0$$
$$SR_2 = -(\sigma_2^C)_{ult}/\sigma_2 \qquad \sigma_2 < 0$$
$$SR_6 = (\tau_{12})_{ult}/|\tau_{12}|$$

The minimum of all SR's is the SR of the lamina.
e.g. if SR_6 is the minimum, then the ST of the lamina is SR_6, and mode of failure is $6S$.

2.8.4 Max Strain Failure Theory

Given $\sigma_1, \sigma_2\ and\ \tau_{12}$, then

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

And failure occurs when any of the following is true,

$$\varepsilon_1 > (\varepsilon_1^T)_{ult} \qquad \varepsilon_1 \geq 0$$
$$\varepsilon_1 < -(\varepsilon_1^C)_{ult} \qquad \varepsilon_1 < 0$$
$$\varepsilon_2 > (\varepsilon_2^T)_{ult} \qquad \varepsilon_2 \geq 0$$
$$\varepsilon_2 < -(\varepsilon_2^C)_{ult} \qquad \varepsilon_2 < 0$$
$$|\gamma_{12}| > (\gamma_{12})_{ult}$$

In terms of SR,

$$SR_1 = (\varepsilon_1^T)_{ult}/\varepsilon_1 \qquad \varepsilon_1 \geq 0$$
$$SR_1 = -(\varepsilon_1^C)_{ult}/\varepsilon_1 \qquad \varepsilon_1 < 0$$
$$SR_2 = (\varepsilon_2^T)_{ult}/\varepsilon_2 \qquad \varepsilon_2 \geq 0$$
$$SR_2 = -(\varepsilon_2^C)_{ult}/\varepsilon_2 \qquad \varepsilon_2 < 0$$
$$SR_6 = (\gamma_{12})_{ult}/|\gamma_{12}|$$

Where

$$(\varepsilon_1^T)_{ult} = (\sigma_1^T)_{ult}/E_1$$
$$(\varepsilon_1^C)_{ult} = (\sigma_1^C)_{ult}/E_1$$
$$(\varepsilon_2^T)_{ult} = (\sigma_2^T)_{ult}/E_2$$
$$(\varepsilon_2^C)_{ult} = (\sigma_2^C)_{ult}/E_2$$
$$(\gamma_{12})_{ult} = (\tau_{12})_{ult}/G_{12}$$

2.8.5 Tsai-Hill Theory (Distortion Energy)

Given σ_1, σ_2, and τ_{12}, and they are increased proportionally to σ_1^f, σ_2^f and τ_{12}^f then failure occurs when

$$\left(\frac{\sigma_1^f}{F_1}\right)^2 - \left(\frac{\sigma_1^f}{F_2}\right)\left(\frac{\sigma_2^f}{F_2}\right) + \left(\frac{\sigma_2^f}{F_3}\right)^2 + \left(\frac{\tau_{12}^f}{F_4}\right)^2 \geq 1$$

To find SR,

$$\sigma_1^f = SR \cdot \sigma_1$$
$$\sigma_2^f = SR \cdot \sigma_2$$
$$\tau_{12}^f = SR \cdot \tau_{12}$$

Such that,

$$SR = \frac{1}{\sqrt{\left(\frac{\sigma_1^f}{F_1}\right)^2 - \left(\frac{\sigma_1^f}{F_2}\right)\left(\frac{\sigma_2^f}{F_2}\right) + \left(\frac{\sigma_2^f}{F_3}\right)^2 + \left(\frac{\tau_{12}^f}{F_4}\right)^2}}$$

Original Tsai-Hill (Eq. 2.150):

$$F_1 = F_2 = (\sigma_1^T)_{ult}$$
$$F_3 = (\sigma_2^T)_{ult}$$
$$F_4 = (\tau_{12})_{ult}$$

Modified Tsai-Hill (Eq. 2.151):

$$F_1 = (\sigma_1^T)_{ult} \quad \sigma_1 \geq 0$$
$$F_1 = \left(\sigma_1^C\right)_{ult} \quad \sigma_1 < 0$$
$$F_2 = (\sigma_1^T)_{ult} \quad \sigma_2 \geq 0$$
$$F_2 = \left(\sigma_1^C\right)_{ult} \quad \sigma_2 < 0$$
$$F_3 = (\sigma_2^T)_{ult} \quad \sigma_2 \geq 0$$
$$F_3 = \left(\sigma_2^C\right)_{ult} \quad \sigma_2 < 0$$
$$F_4 = (\tau_{12})_{ult}$$

Modified Tsai-Hill takes into account:

a) The different strengths in tension and under compression
b) The interaction between σ_1 and σ_2
   $\therefore \left(\frac{\sigma_1}{F_2}\right)\left(\frac{\sigma_2}{F_2}\right)$
   and choices for F_2

2.8.6 Tsai-Wu Failure Theory (Total Strain Energy)

Define:

$$H_1 = \frac{1}{(\sigma_1^T)_{ult}} - \frac{1}{(\sigma_1^C)_{ult}}$$
$$H_2 = \frac{1}{(\sigma_2^T)_{ult}} - \frac{1}{(\sigma_2^C)_{ult}}$$
$$H_{11} = \frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$

$$H_{22} = \frac{1}{(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}$$

$$H_{66} = \frac{1}{[(\tau_{12})_{ult}]^2}$$

And Tsai-Hill:

$$H_{12} = -\left(\frac{1}{2}\right)\frac{1}{[(\sigma_1^T)_{ult}]^2}$$

Hoffman:

$$H_{12} = -\left(\frac{1}{2}\right)\frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$

von Mises-Hencky:

$$H_{12} = -\left(\frac{1}{2}\right)\frac{1}{\sqrt{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}}$$

Then given σ_1, σ_2, and τ_{12}, and assuming they are to be increased proportionally to σ_1^f, σ_2^f, and τ_{12}^f, failure occurs when:

$$H_1\sigma_1^f + H_2\sigma_2^f + H_{11}\left(\sigma_1^f\right)^2 + H_{22}\left(\sigma_2^f\right)^2 + 2H_{12}\left(\sigma_1^f\sigma_2^f\right) + H_{66}\left(\sigma_6^f\right)^2 \geq 1$$

In terms of SR:

$$a(SR)^2 + 2b(SR) - 1 = 0$$

Where:

$$a = H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + 2H_{12}\sigma_1\sigma_2 + H_{66}\sigma_6^2$$
$$b = \left(\frac{1}{2}\right)(H_1\sigma_1 + H_1\sigma_2)$$

SR is the root (one of the roots) of the quadratic equation.
Compared with Tsai-Hill theory, Tsai-Wu theory considers:

a) The so-called 1st order effects;
b) The interaction between σ_1 and σ_2 with more sophistication.

Example 3

A unidirectional graphite/epoxy lamina ($\theta = 50°$) is subject to $\sigma_x = \sigma$, $\sigma_y = -\sigma$, and $\tau_{xy} = 0$ (where σ is in Pa). Find the allowable σ, using (1) the max. stress theory; (2) the max strain theory; (3) the Tsai-Hill theory; and (4) the Tsai-Wu theory. Also indicate the mode of failure where available. Set $SR = 2$.

Given, for the lamina,

$E_1 = 181\ GPa$

$E_2 = 10.3\ GPa$

$v_{12} = 0.28$

$G_{12} = 7.2\ GPa$

$(\sigma_1^T)_{ult} = 1500\ MPa$

$(\sigma_1^C)_{ult} = 500\ MPa$

$(\sigma_2^T)_{ult} = 40\ MPa$

$(\sigma_2^C)_{ult} = 245\ MPa$

$(\tau_{12})_{ult} = 70\ MPa$

Solution:

From Example 2, local stresses and strains are,

$$\begin{Bmatrix} -0.1736\sigma \\ 0.1736\sigma \\ -0.9848\sigma \end{Bmatrix} \text{ and } \begin{Bmatrix} -0.001228\sigma \\ 0.01713\sigma \\ -0.1368\sigma \end{Bmatrix} (10^{-9})$$

(1) Maximum stress theory

$$SR_1 = -\frac{(\sigma_1^C)_{ult}}{\sigma_1} = -\frac{500(10^6)}{(-0.1736\sigma)} = 2$$

So, $\sigma = 1440\ MPa$

$$SR_2 = \frac{(\sigma_2^T)_{ult}}{\sigma_2} = -\frac{40(10^6)}{(0.1736\sigma)} = 2$$

So, $\sigma = 115.2\ MPa$

$$SR_3 = \frac{(\tau_{12})_{ult}}{|\tau_{12}|} = -\frac{70(10^6)}{(0.9848\sigma)} = 2$$

So, $\sigma = 35.54\ MPa$

Therefore, $\sigma_{all} = 35.54\ MPa$ and the lamina's mode of failure is $6S$.

(2) Maximum strain theory

$$(\varepsilon_1^C)_{ult} = \frac{(\sigma_1^C)_{ult}}{E_1} = 0.00276$$

$$SR_1 = -\frac{(\varepsilon_1^C)_{ult}}{\varepsilon_1} = -\frac{0.00276}{-0.001228\sigma(10^{-9})} = 2$$

So, $\sigma = 1124\ MPa$

$$(\varepsilon_2^T)_{ult} = \frac{(\sigma_2^T)_{ult}}{E_2} = 0.00388$$

$$SR_2 = \frac{(\varepsilon_2^T)_{ult}}{\varepsilon_2} = -\frac{0.00388}{0.01713\sigma(10^{-9})} = 2$$

So, $\sigma = 113.3\ MPa$

$$(\gamma_{12})_{ult} = \frac{(\tau_{12})_{ult}}{|G_{12}|} = 0.00972$$

$$SR_6 = \frac{(\gamma_{12})_{ult}}{|\gamma_{12}|} = -\frac{0.00972}{0.1368\sigma(10^{-9})} = 2$$

So, $\sigma = 35.53\ MPa$

Therefore, $\sigma_{all} = 35.53\ MPa$ and the mode of failure of the lamina is $6S$.

(3) Tsai-Hill theory
Modified Tsai-Hill:

$$2 = \frac{1}{\sqrt{2.16935(10^{-16})\sigma^2}}$$

$\sigma_{all} = 33.95\ MPa$

(4) Tsai-Wu theory
Tsai-Hill form:
$a = 2.01058(10^{-16})\sigma^2$
$b = 1.93198(10^{-9})\sigma$

The quadratic equation is:
$a(2^2) + 2b(2) - 1 = 0$

And $\sigma_{all} = 30.78\ MPa$

Chapter 4: Macromechanical Analysis of Laminates

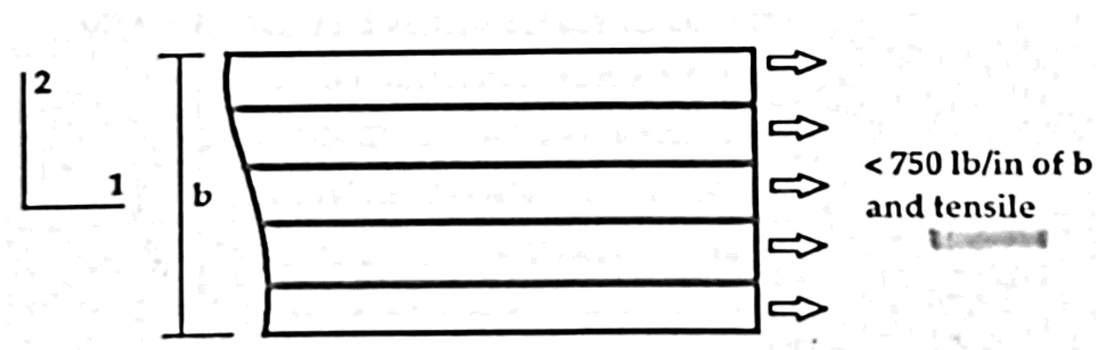
4.1 Introduction
4.2 Laminate Code
4.3 Stress-Strain Relations for a Laminate
(or CLPT – Classical Laminated Plates Theory, and [ABD])
4.4 In-Plane and Flexural Modulus of a Laminate
(or application of [ABD])
4.5 Hygrothermal Effects in a Laminate

4.1 Introduction

Why laminate?

1. A single lamina (or layer, ply)

- 0.005” or 0.125 mm thick → not suitable as an engineering component;
- 750-lb per inch of width along fiber direction → not high enough for engineering application;



2. Unidirectional laminate (which has many layers, but fibers take the same direction)

- Transverse direction: rather weak;
- Fiber direction: compressive strength is low;
- Laminate be loaded along fiber direction by tensile load, which limits its applicaions

3. Optimal solutions

Having layers stacked with different

- Angles
- Thickness
- Position (top, …, middle, …, bottom)
- Constituents

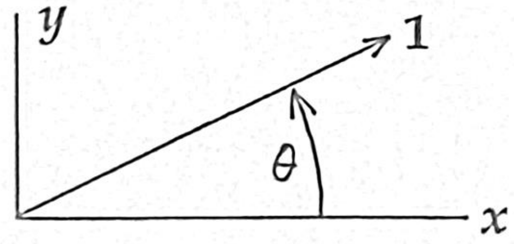
4.2 Laminate Code

Laminate code, also known as **Layup Sequence**, or **Stacking Sequence**, is a set of notation or convention to describe how layers (or piles, laminas) are stacking on top of one another.

The notation or convention is yet to be standardized.

1. Reference Axis

CCW angles (from x) are considered positive.



2. Layer Numbering

Top to bottom	more common
Bottom to top	easier for manual layup

3. Layers with Identical Constituents and Uniform Thickness

A) Long hand notation

$[\,\theta_1 /\, \theta_2 /\, \theta_3 / \ldots /\theta_N]$ where the θ's are in degrees.

For example, $[0\ /\ 90\ /\ 45\ /\ 90\ /\ 0\,]$

The commonly used angles are, starting with the most preferred to the least $0°, 90°$
$\pm 45°$
$\pm 30°, \quad \pm 60°$
$\pm 15°, \quad \pm 75°$

B) Repeated orientation
$[0\ /\ 0\ /\ 90\ /\ 90\,] \rightarrow [0_2 /\ 90_2]$
$[0\ /\ 90\ /\ 0\ /\ 90] \rightarrow [(0\ /\ 90)^2]$
$[0\ /\ 0\ \ /\ 90\ /\ 45\ /\ 90\ /\ 90\ /\ 45\ /\ 90\ /\ 0\ /\ 0] \rightarrow [0_2\ /\ (90\ /\ 45\ /\ 90)_2\ /\ 0_2]$

C) Balanced laminate
For every occurrence of $" + \theta"$ (other than $0°$ and $90°$), a $" - \theta"$ is placed, either adjacent to the $" + \theta"$ layer, or separated by some layers.

$[\,45\ /\ 45\ /\ 0\ /\ 45\ /\ -45] \rightarrow [45_2\ /\ 0\ /\ -45_2]$
$[45\ /\ -45\ /\ 0\ /\ 45/\ -45] \rightarrow [\pm 45\ /\ 0\ \ /\ \pm 45]$
$[45\ /\ -45\ /\ 0\ /\ -45\ /\ 45]$
$[45\ /\ -45\ /\ -45\ /\ 45\ /\ 45\ /\ -45\,] \rightarrow [\pm \mp \pm 45]$

D) Symmetry
Symmetry means fiber orientations of the top half of the laminate are mirror image of the bottom half. A symmetric laminate can have even or odd number of layers.

D.1) Even number of layers.
$[0\ /\ 0\ /\ 90\ /\ 45\ /\ 90\ /\ 90\ /\ 45\ /\ 90\ /\ 0\ /\ 0\,] \rightarrow [0\ /\ 0\ /\ 90\ /\ 45\ /\ 90\,]_s \rightarrow [0_2\ /\ 90\ /\ 45\ /\ 90]_s$

Notes:

- Only the top half of the sequence is notated
- s (subscript) is to indicate only a symmetric half of the entire sequence is given

D.2) Odd number of layers
$[0\ /\ 90\ /\ 0] \rightarrow [0\ /\ \overline{90}]$

Note: the overbar indicates the layer about whose mid-plane the laminate is symmetric.
$[0\ /\ 90\ /\ 45\ /\ 90\ /\ 0] \rightarrow \left[0\ /\ 90\ /\ \overline{45}\right]$
$[0\ /\ 0\ /\ 90\ /\ 45\ /\ 90\ /\ 90\ /\ 45\ /\ 90\ /\ 0\ /\ 0] \rightarrow [0_2\ /\ 90\ /\ 45\ /\ 90]_s \rightarrow \left[0_2\ /(\ 90\ /\ \overline{45}\ )\right]_s$
$[45\ /\ -45\ /\ 0\ /\ -45\ /\ 45] \rightarrow [\pm 45\ /\ \bar{0}\ ]$

Anti-symmetry means that fiber orientations of the top half of the laminate are opposite those of the bottom half.

In the context of laminate code, 90° is considered "opposite" of 0°, and vice verse.

An anti-symmetric laminate always has even number of layers.

$[0\ /\ 90\ /\ 0\ /\ 90] \rightarrow [(0\ /\ 90)_2]$
$[45\ /\ -45\ /\ 45\ /\ -45] \rightarrow [\pm 45_2]$
$[45\ /\ 30\ /\ 35\ /\ -45\ /\ -30\ /\ -45] \rightarrow \pm[(45\ /\ \overline{30})]$
$[45\ /\ -45\ /\ -45\ /\ 45\ /\ 45\ /\ -45] \rightarrow [\pm \mp \pm 45]$

4. Layers with Identical Constituents but Non-uniform Thickness
Two options:

- Long-hand notation, with thickness as subscript
- Spelling out the detail, in English

For example,

The laminate code is $[(0\ /\ 90)_2\ /\ \bar{0}]$ where the 0° layers have a thickness of $0.2\ mm$ each, and the 90° layers have a thickness of $0.25\ mm$ each.

Where:
$[(0\ /\ 90)_2\ /\ \bar{0}] \rightarrow [0\ /\ 90\ /\ 0\ /\ 90\ /\ 0\ /\ 0\ /90\ /\ 0\ /\ 90]$

$0.2\ mm \cdot 5 = 1\ mm$
$0.25\ mm \cdot 4 = 1\ mm$

Then total thickness is $2\ mm$

5. Hybrid Laminates
These are laminates whose layers are of different constituent materials.

$\left[0^{\mathrm{K}}\ /\ 0^{K}\ /\ 45^{C}\ \ /\ -45^{C}\ /90^{\mathrm{G}}\ /\ -45^{C}\ /\ 45^{C}\ /\ 0\text{^K}\ /\ 0\text{^K}\right] \rightarrow \left[0_2^{K}\ /\ \pm 45^{C}\ /\ \overline{90}^{G}\ \right]$
$[0^{\mathrm{K}}\ /\ 0^{\mathrm{K}}\ \ /\ 45^{C}\ /\ -45^{G}\ /\ 90^{G}\ /\ -45^{G}\ /\ 45^{C}\ /\ 0^{\mathrm{K}}\ /\ 0^{\mathrm{K}}\ ] \rightarrow \left[0_2^{K}\ \ /\ 45^{C}\ /\ -45^{G}\ /\ \overline{90^{G}}\right]$

6. Brain Teaser

Given the following 22-later sequence, write the shortest possible code.
$[45\ /\ -45\ /\ 0\ /\ 0\ /\ 45\ /\ -45\ /\ 0\ /\ 0\ /\ 90\ /\ 0\ /\ 0\ /\ 0\ /\ 0\ \ /\ 90\ /\ 0\ /\ 0\ /\ -45\ /\ 45\ /\ 0\ /\ 0\ /\ -45\ /\ 45\ ]$

Solution:
$[\ (\pm 45\ /\ 0_2)_2\ /\ 90\ /\ 0_2]_s$ `

7. More Terminologies

A) Unidirectional laminates: laminates in which all layers have the same θ.
For example, $[0_6]$ or $[0]_6$ and $[45]_{10}$

B) Cross-ply laminates: laminates in which the layers take angles of 0° and 90° only.
For example, $[0\ /\ \overline{90}]$, $[0\ /\ 90\ /\ 0\ /\ 90] \rightarrow [(0\ /\ 90)_2\ ]$

C) Angle-ply laminates: laminates that consist of pairs of layers of same material (that is, same fiver and matrix, and same mixture) and thickness, and oriented at $+\theta$ and $-\theta$.
For example,
$[45\ /\ -45\ /\ 45\ /\ -45]$
$[45\ /\ -45\ /\ -45\ /\ 45\ /\ 45\ /\ -45]$
$[45\ /\ 30\ /\ 45\ /\ -45\ /\ -30\ \ -45]$

4.3 Stress-Strain Relations for a Laminate
4.3.1: $\sigma - \varepsilon$ relation for a one-dimensional isotropic beam
4.3.2 $\sim$ 4.3.4: Classical laminated plates theory (CLPT)
Key features:

- Membrane stretching is considered
- Bending is considered
- The two actions are not kinematically coupled, but kinetically coupled
- Transverse shear is not considered

Other laminated plate theories:
Membrane & bending actions are coupled
Transverse shear is considered

$$\left\{\begin{matrix} 1st\ order\ theory \\ 2nd\ order\ theory \\ higher\ order\ theory \\ zigzag\ theory \\ \ldots \end{matrix}\right\}$$

2) Coordinate setup

$x - y - z$: global coordinates

$x - y$ plan coincides with the mid-plane of the laminate;

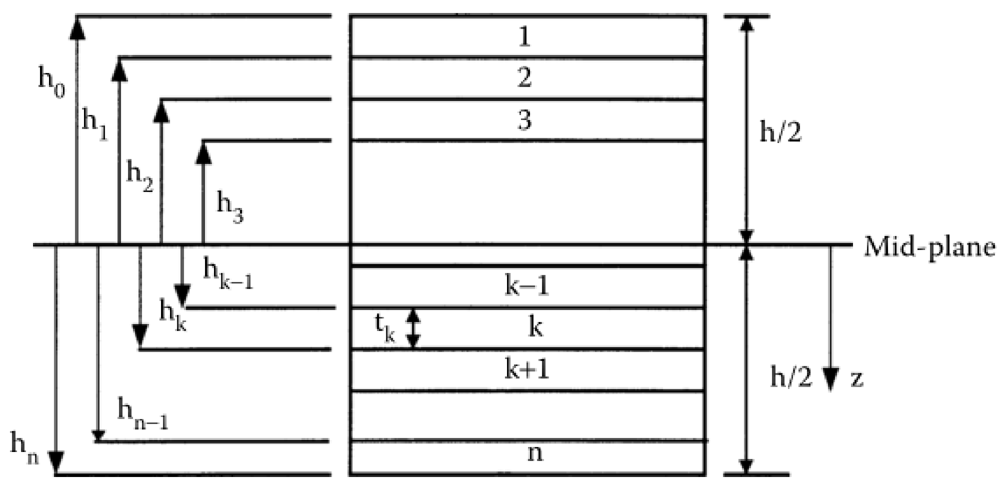


FIGURE 4.6

Coordinate locations of plies in a laminate.

h: thickness of laminate

$h_0 = -h/2$

$h_1 = h_0 + t_1$

$h_2 = h_1 + t_2$

…

$h_N = h/2$

3) Mid-plane Displacements (unknowns to be solved)

$u_o(x, y)$: membrane stretches

$v_o(x, y)$: membrane stretches

$w_o(x, y)$: lateral deflection

4) "Slopes" and Curvatures

$\phi_y(x, y) = \frac{\delta w_o}{\delta x}$: rotation about x

$\phi_x(x, y) = \frac{\delta w_o}{\delta y}$: rotation about y

$\kappa_x(x, y) = -\frac{\delta^2 w_o}{\delta x^2}$: curvature

$\kappa_y(x, y) = -\frac{\delta^2 w_o}{\delta y^2}$: curvature

$\kappa_{xy}(x, y) = -2\frac{\delta^2 w_o}{\delta x \delta y}$: twisting curvature

5) Membrane strain vector and curvature vector

$\{\varepsilon^0\} =$

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \tag{4.14}$$

$\{\kappa\} =$

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}, \tag{4.15}$$

Note: Both ε^0 and κ are based on the midplane.

6 Global displacements and strains at points off the midplane

On midplane where $z = 0$:

$$u_0(x, y),\ \ v_0(x, y),\ \ w_o(x, y)$$

Off the midplane ($z \neq 0$):

$$u = u_0 - z\frac{\partial w_0}{\partial x}. \tag{4.10}$$

$$v = v_0 - z\frac{\partial w_o}{\partial y}. \tag{4.11}$$

$$w(x, y, z) = w_0$$

And:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}. \tag{4.13}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{4.16}$$

7) Resultant Forces and Moments

Sign conventions: Figure 4.3

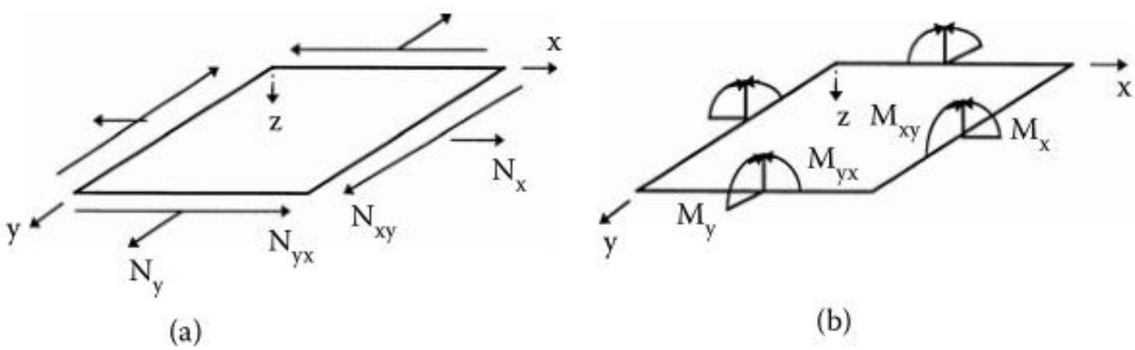


FIGURE 4.3
Resultant forces and moments on a laminate.

Definitions:

Membrane forces (per unit length)

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz, \quad (4.21a)$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y dz, \quad (4.21b)$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz, , \quad (4.21c)$$

Which can be simplified as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz, \quad (4.24a)$$

$(*)$ Unit: N/m, lb/in, … $(force/length)$

$(*)$ Integral is broken into sum over layers due to “jumps” at lamina interfaces, as per Figure 4.5.

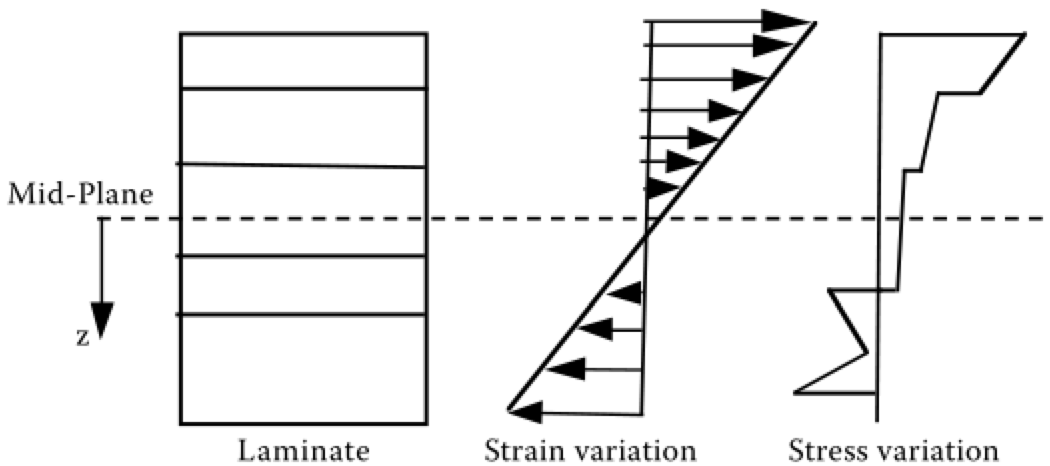


FIGURE 4.5
Strain and stress variation through the thickness of the laminate.

N_x, N_y: Normal forces per unit length
N_{xy}: shear force per unit length

Moments (per unit length)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz, \tag{4.22a}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz, \tag{4.22b}$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz, \tag{4.22c}$$

Which can be simplified as:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k z dz. \tag{4.24b}$$

$(*)$ Unit: $N \cdot m/m$, $lb \cdot in/in$, … $(moment/length)$

$(*)$ M_x, M_y: bending moments per unit length
M_{xy}: twisting moment per unit length

<u>8) Stiffness and compliance of a laminated plate</u>

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$[A]$: membrane (extensional) stiffness matrix
$Pa \cdot m$, or $psi - in$ $(stress - length)$
symmetric

$[B]$: membrane-bending coupling stiffness matrix
$Pa \cdot m^2$, or $psi \cdot in^2$ $(stress - length^2)$
symmetric

$[D]$: bending stiffness matrix
$Pa \cdot m^3$, or $psi \cdot in^3$ $(stress - length^3)$
symmetric

Derivation: $pp. 328 \sim 331$

$$A_{ij} = \sum_{k=1}^{n} [(\bar{Q}_{ij})]_k (h_k - h_{k-1}), \quad i = 1,2,6; \quad j = 1,2,6, \qquad (4.28a)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\bar{Q}_{ij})]_k (h_k^2 - h_{k-1}^2), \quad i = 1,2,6; \quad j = 1,2,6, \qquad (4.28b)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\bar{Q}_{ij})]_k (h_k^3 - h_{k-1}^3), \quad i = 1,2,6; \quad j = 1,2,6. \qquad (4.28c)$$

$[ABD]$: "stiffness matrix" of the laminate plate
$[ABD]^{-1}$: "compliance matrix" of the laminate plate obtained numerically (by inversion)
both are symmetric

As long as $[B] \neq 0$, membrane and bending are kinetically coupled.

(Eqn. 4-29) is typically written in the compact form:
$\begin{Bmatrix} N \\ M \end{Bmatrix} = [ABD] \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$

<u>9) Applications</u>
10-step procedure on p. 332
the main steps are:

a. evaluate [ABD] matrix
$k = 1, \dots, N$
$[Q]_k, \ \ [\bar{Q}]_k$
$h_{k-1}, \ \ h_k$
(Eq. 4.28a,b,c)

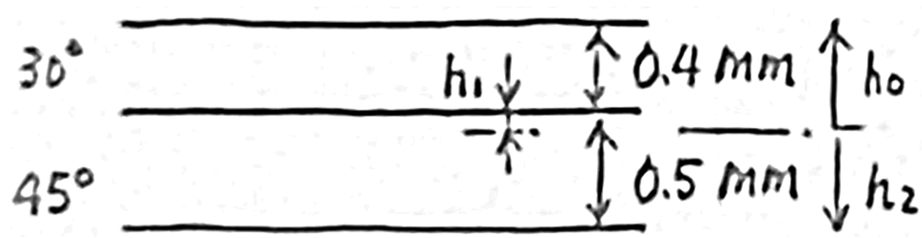
b. determine $\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$
find $N_x, N_y, \dots, M_y, M_{xy}$
inverse $[ABD]$
$[ABD]^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix}$

c. evaluate global strains $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$ via (Eqn. 4.16)
for a given z, then evaluate global stresses → local stresses (and local strains is necessary)→ SR

Examples 4.2
Examples 4.3 (the above procedure)

Example: A laminate has the layup sequence of $[30 / 45]$. The top and bottom layers are $0.4\ mm$ and $0.5\ mm$ thick. Both layers have: $E_1 = 170\ GPa$, $E_2 = 20\ GPa$, $G_{12} = 5.5\ GPa$ and $v_{!2} = 0.26$. Determine global stresses σ_x, σ_y, τ_{xy} at the top and bottom surfaces of the layers for the following loading:
$N_x = N_y = 1000\ N/m$; and
Plot stress distributions of σ_x, σ_y, τ_{xy} across the thickness.

Solution:
(1) Need [ABD] matrix, $[ABD]^{-1}$ matrix:



$h_0 = -0.00045\ m$
$h_1 = -0.00005\ m$
$h_2 = 0.00045\ m$

Then,
$[Q] \rightarrow [\bar{Q}]_1 \quad (\theta = 30°)$
$[Q] \rightarrow [\bar{Q}]_2 \quad (\theta = 45°)$

Applying (Eq. 4.28)

$$[A] = \begin{bmatrix} 6.950 & 3.653 & 3.888 \\ & 3.926 & 2.511 \\ sym & & 3.676 \end{bmatrix} \cdot (10^7) \qquad (Pa \cdot m)$$

$$[B] = \begin{bmatrix} -4.774 & 0.9940 & -1.215 \\ & 2.786 & 2.228 \\ sym & & 0.9940 \end{bmatrix} \cdot (10^3) \qquad (Pa \cdot m^2)$$

$$[D] = \begin{bmatrix} 4.850 & 2.432 & 2.665 \\ & 2.557 & 1.621 \\ sym & & 2.448 \end{bmatrix} \qquad (Pa \cdot m^3)$$

And $[ABD]^{-1} = \begin{bmatrix} A^* & B^* \\ B^* & D^* \end{bmatrix}$

$$[A^*] = \begin{bmatrix} 5.164 & -2..786 & -3.426 \\ & 6.251 & -1.260 \\ sym & & 7.731 \end{bmatrix} \cdot (10^{-8}) \qquad \left(\frac{1}{Pa \cdot m}\right)$$

$$[B^*] = \begin{bmatrix} 3.687 & -1..832 & 3.689 \\ & 0.01186 & -4.574 \\ sym & & -4.680 \end{bmatrix} \cdot (10^{-5}) \qquad \left(\frac{1}{Pa \cdot m^2}\right)$$

$$[D^*] = \begin{bmatrix} 7.468 & -4.037 & -5.257 \\ & 9.260 & -1.641 \\ sym & & 11.68 \end{bmatrix} \cdot (10^{-1}) \qquad \left(\frac{1}{Pa \cdot m^3}\right)$$

(2) Loading is $N_x = N_y = 1000\ N/m$

$$\left\{\frac{\varepsilon^0}{\kappa}\right\} = [ABD]^{-1} \begin{Bmatrix} 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 23.78 \cdot 10^{-6}\ m/m \\ 34.65 \cdot 10^{-6}\ m/m \\ -46.86 \cdot 10^{-6}\ m/m \\ 18.55 \cdot 10^{-3}\ 1/m \\ -18.20 \cdot 10^{-3}\ 1/m \\ -8.855 \cdot 10^{-3}\ 1/m \end{Bmatrix}$$

Normal forces can cause shear deformation, bending and twisting.

<u>Top layer:</u> $[\bar{Q}] = [\bar{Q}] \uparrow$

Top surface $\zeta = h_0 = -0.00045\ m$

$$\{\varepsilon\} = \{\varepsilon^0\} + \zeta[\kappa] = \begin{Bmatrix} 15.43 \\ 42.83 \\ -42.88 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = [\bar{Q}]\{\varepsilon\} = \begin{Bmatrix} 960.6 \\ 1081 \\ -78.91 \end{Bmatrix} kPa$$

Bottom surface $\zeta = h_1 = -0.00005\ m$

$$\{\varepsilon\} = \begin{Bmatrix} 22.85 \\ 35.56 \\ -46.42 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = \begin{Bmatrix} 1298 \\ 1265 \\ 53.70 \end{Bmatrix} kPa$$

<u>Bottom layer:</u> $[\bar{Q}] = [\bar{Q}] \downarrow$

Top surface $\zeta = h_1 = -0.00005\ m$

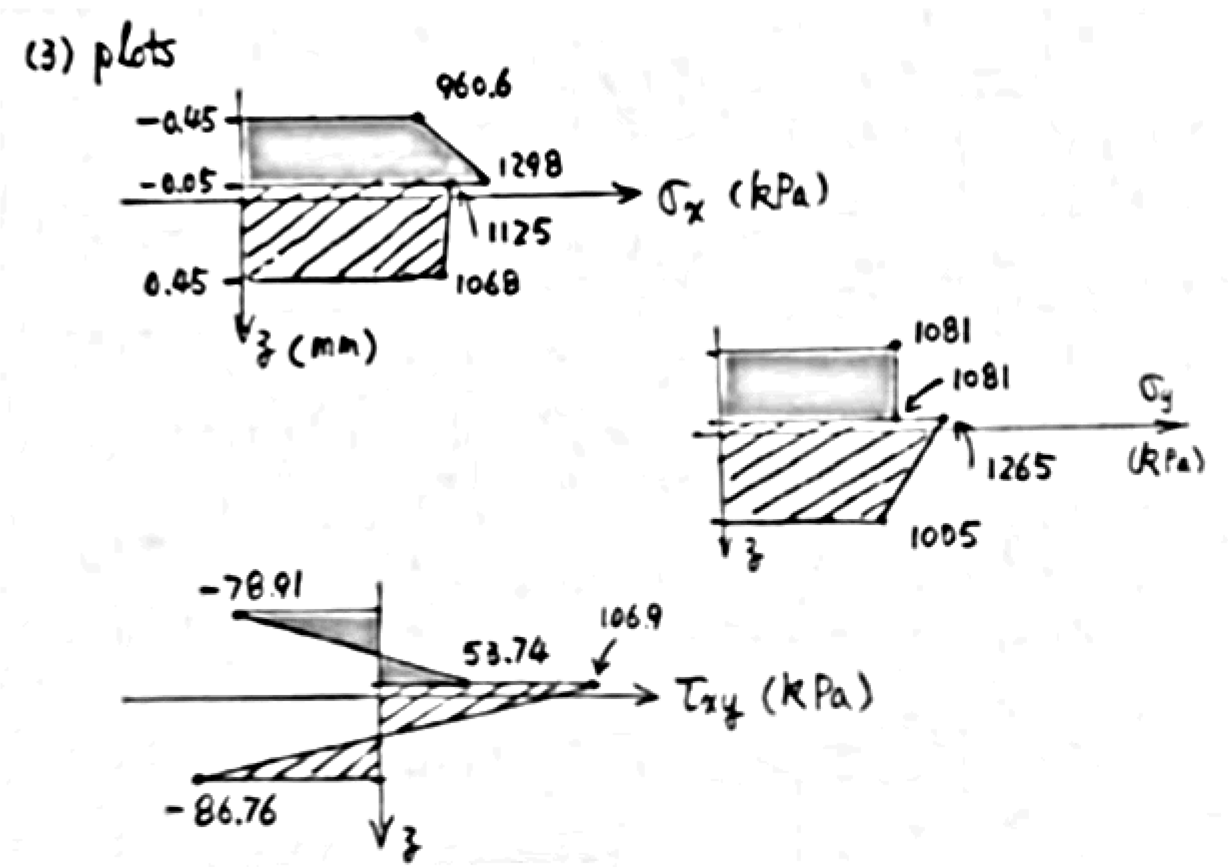
$$\{\varepsilon\} = \begin{Bmatrix} 22.85 \\ 35.56 \\ -46.42 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = \begin{Bmatrix} 1125 \\ 1265 \\ 106.9 \end{Bmatrix} kPa$$

Bottom surface $\zeta = h_2 = 0.00045\ m$

$$\{\varepsilon\} = \begin{Bmatrix} 31.13 \\ 26.46 \\ -50.84 \end{Bmatrix} \cdot 10^{-6}$$

$$\{\sigma] = \begin{Bmatrix} 1068 \\ 1005 \\ -86.76 \end{Bmatrix} kPa$$



4.4 In-Plane and Flexural Modulus of a Laminate

$[ABD] = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$

Which is a symmetric matrix. ([A], [B] and [D] are inverse as well)

$[ABD]^{-1} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}$

Where $[A^*]$, $[D^*]$ are symmetric
and $[B^*]$ may not be symmetric
and $[C^*] = [B^*]^T$

In-plane constants:

$$E_x = \frac{1}{hA_{11}^*}; \quad Effective\ in-plane\ longitudinal\ modulus$$

$$E_y = \frac{1}{hA_{22}^*}; \quad Effective\ in-plane\ transverse\ modlus$$

$$G_{xy} = \frac{1}{hA_{66}^*}; \quad Effective\ in-plane\ shear\ modulus$$

$$v_{xy} = -\frac{A_{12}^*}{A_{11}^*}; \quad Effective\ in-plane\ Poisson's\ ratio$$

$$v_{yx} = -\frac{A_{12}^*}{A_{22}^*}; \quad Effective\ in-plane\ Poisson's\ ratio$$

Note:
The larger Poisson's ratio is the <u>major</u>, and the other one is the <u>minor</u>

$$\frac{v_{xy}}{E_x} = \frac{v_{yx}}{E_y}$$

Flexural constants:

$$E_x^f = \frac{12}{h^3 D_{11}^*}; \quad effective\ flexural\ longitudinal\ modulus$$

$$E_y^f = \frac{12}{h^3 D_{22}^*}; \quad effective\ flexural\ transverse\ modulus$$

$$G_{xy}^f = \frac{12}{h^3 D_{66}^*}; \quad effective\ flexural\ shear\ modulus$$

$$v_{xy}^f = -\frac{D_{12}^*}{D_{11}^*}; \quad effective\ flexural\ Poisson's\ ratio$$

$$v_{yx}^f = -\frac{D_{12}^*}{D_{22}^*}; \quad effective\ flexural\ Poisson's\ ratio$$

Note:
The larger Poisson's ratio is the <u>major</u>, and the other one is the <u>minor</u>

$$\frac{v_{xy}^f}{E_x} = \frac{v_{yx}^f}{E_y}$$

- Example 4.4 goes through the steps above

Consider that you were given a completed [ABD] matrix, then you can use the following tools to analyze it further.

<u>Measures used to gauge how close a laminate is to an equivalent orthotropic material:</u>
In terms of membrane action:

$$r_N = \sqrt{\left(\frac{A_{26}}{A_{11}}\right)^2 + \left(\frac{A_{26}}{A_{22}}\right)^2}$$

In terms of bending action:

$$r_M = \sqrt{\left(\frac{D_{16}}{D_{11}}\right)^2 + \left(\frac{D_{26}}{D_{22}}\right)^2}$$

It is desired that $r_N \to 0, r_M \to 0$

Measures used to gauge symmetry of a laminate:

$$r_B = \frac{3}{(A_{11} + A_{22} + A_{66})h} \sqrt{\sum_i \sum_j (B_{ij})^2}$$

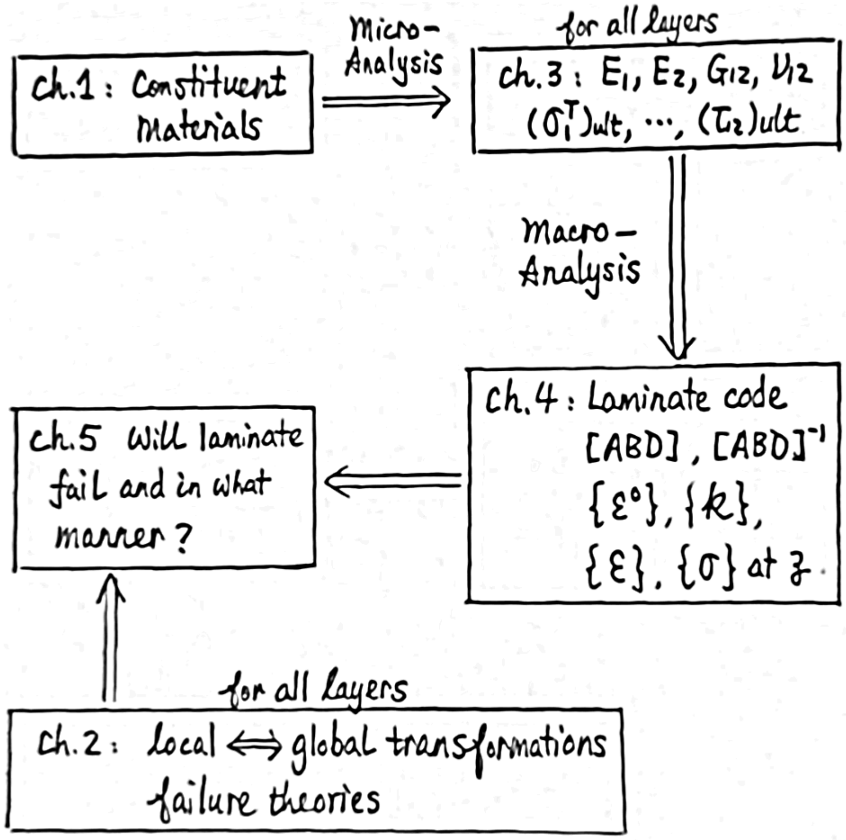
The closer r_B is to zero, the more symmetry there is

Summary:

$r_N,\ r_M,\ r_B$: the closer they are to zero, the more accurate it is to use E_x, E_y, …, G_{xy}^f and v_{xy}^f to represent the entire laminate, and to treat the laminate as an orthotropic material.

Chapter 5: Failure, Analysis and Design of Laminates

5.1: Introduction



5.2: Special Cases of Laminates

In 4.2, laminate codes were introduced. From mainly the perspective of layer orientations, cross-ply laminates, angle-ply laminates, balanced laminates, symmetric laminates, and anti-symmetric laminates were defined.

In this section, the above laminates will be defined from the perspectives of constituents and mixtures, and thicknesses, in addition to layer orientations. The effect on the $[ABD]$ matrix will be dealt with as well.

1. Symmetric Laminates

These are laminates in which fiber orientations, constituents and mixtures, and thicknesses of the top half of the laminate are mirror image of the bottom half.

A symmetric laminate can have even or odd number of layers.
→ $[B] = [0]$; as a result, membrane and bending actions are uncoupled, kinetically.

2. Cross-Ply Laminates

They are laminates in which the layers take angles of 0° and 90° only.
→ $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$

For example, a symmetric cross-ply has, $[B] = [0]$
and $A_{16} = A_{26} = D_{16} = D_{26} = 0$;

3. Angle-Ply Laminates

They are laminates that consist of pairs of layers of the same constituents and mixture, and thickness, but oriented at $+\theta$ and $-\theta$.

$\rightarrow A_{16} = A_{26} = 0$

4. Anti-Symmetric Laminates

In 4.2, anti-symmetric laminates are defined as those in which fiber orientations of the top half of the laminate are opposite those of the bottom half.

In terms of constituents and mixtures, and thicknesses, the top half and bottom half are mirror images of each other.

An anti-symmetric laminate always has even number of layers.

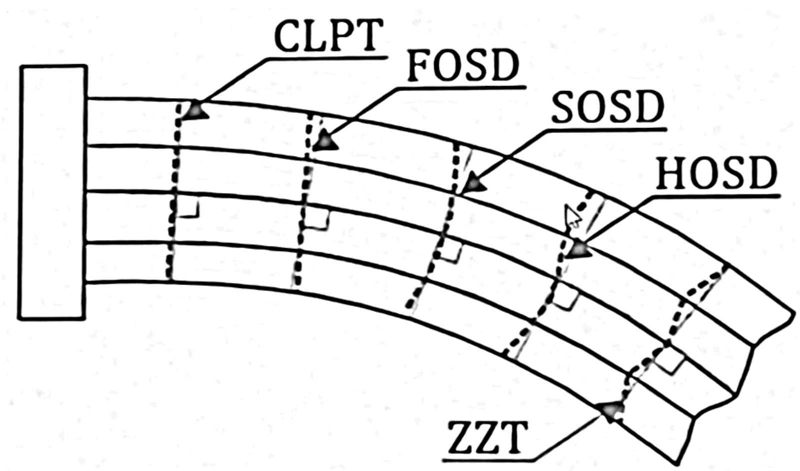
$\rightarrow A_{16} = A_{26} = D_{16} + D_{26} = 0$

5. Balanced Laminates

In 4.2, balanced laminates are defined as having pairs of "$+\theta$" and "$-\theta$" layers (θ cannot be $0°$ or $90°$).

Further, the pair of "$+\theta$" and "$-\theta$" layers must have the same constituents and mixture and thickness for a laminate to be balanced

$\rightarrow A_{16} = A_{26} = 0$



6. Quasi-Isotropic Laminates

Quasi-isotropic means behaving like an isotropic material, or independent of orientation. However, being quasi-isotropic does not mean being isotropic.

- Quasi-isotropic in terms of membrane action:
  (a) $[B] = [0]$; and
  (b) $A_{11} = A_{22}\,; A_{16} = A_{26} = 0\,; A_{66} = (A_{11} - A_{12})/2$
- Quasi-isotropic in terms of bending action:
  (a) $[B] = 0$; and
  (b) $D_{11} = D_{22}; D_{16} = D_{26} = 0; D_{66} = (D_{11} - D_{12})/2$

- Quasi-isotropic in terms of both membrane and bending actions:
  (a) [B]=[0]; and
  (b) $A_{11} = A_{22}$; $A_{16} = A_{26} = 0$; $A_{66} = (A_{11} - A_{12})/2$; and
  (c) $D_{11} = D_{22}; D_{16} = D_{26} = 0; D_{66} = (D_{11} - D_{12})/2$; or
  $$[D] = \left(\frac{h^2}{12}\right) * [A]$$

See Example 5.1 for a laminate that is quasi-isotropic in terms of membrane action.

How to make a quasi-isotropic laminate:

(1) Number of layers $N \geq 3$
(2) Orientations of two adjacent layers differ by $180°/N$.

For example, if $N = 3$, layups may be $[\ 60\ /\ 0\ /\ -80\ ]$ and $[\ 45\ /\ -15\ /\ -75\ ]$

5.3 Failure Criterion of a Laminate

1. Basic concepts and terminologies

Under the combined action of membrane and bending loads, layers will have different levels of stress, not to mention different constituents and mixtures, and giver orientations. That is, layers will have different SR's and different modes of failure.

Failure of a single layer does not lead to failure of the laminate. This is a huge advantage of lamiantes over isotropic materials.

<u>First-ply failure (FPF) and first-ply failure load:</u>
FPF refers to the phenomenon that one layer (or some layers) fails (or fail) before others.

FPF Load refers to the load level that causes FPF. This load equals the applied load times the SR of the laminate at FPF.

In general, the laminate will be able to continue to take on increased load, and more layers will fail, in a sequence (hence second-ply failure, …, and so on), unit the laminate fails, based on some pre-selected failure criteria.

<u>Ultimate-ply failure (UPF) and ultimate-ply failure load:</u>
UPF refers to when the load on the laminate is at such level that the laminate is considered failed, based on the pre-selected failure criterion. The load level that causes UPF is known as the **UPF Load.**

The process of layers in a laminate fail in some sequence as the load is increased is known as **progressive failure.**

<u>Last-ply failure (LPF) and last-ply failure load:</u>
If the progressive failure continues until the last ply (or plied) fails (or fail), the phenomenon is known as **LPF** and the corresponding load level is the **LPF Load.**

2. What determines "a laminate fails"

Termination criterion is used to determine if a laminate fails. A termination criterion can be,

- If fibers fail in tension (1T);

- If fibers fail (in tension or under compression 1T or 1C); or
- If a certain number of layers fail. Typical choice is 50% of layers, but it can be of a higher value, say 100%. Setting the value to 100% in fact gives rise to **LPF.**

3. What to do with a failed lamina?

- Originally occupied space by a failed lamina remains occupied by it; that is, the z coordinates of layers are unchanged during the progressive failure analysis.
- The failed lamina's stiffness and strengths will be discounted; the discount can be a total discount (e.g., $E_2 = 0$) or partial discount (e.g., $E_2 = 10\%$ of before-failure value).

It should be noted that answers to, (1) what termination criterion to use; and (2) how to discount a failed lamina, are not entirely technical, and far from definitive.

a) determine $[ABD]$ matrix; see Section 4.3

b) determine N_x, N_y, …, M_y, M_{xy}

c) for al plies
- for top and bottom surface of a ply
    - z → global strains → global stresses → local stresses (→ local strains is using max. strain theory)→ SR of the surface.
- SR of a ply

for top and bottom surface of a ply

SR of the laminate; also, the ply (plies) that would fail.

d) FPF load = applied loads · SR of laminate

e) Discount failed ply on plies set load level to FPF load

f) Go back to step a)
For Step c), only loop over all remaining plies for step d)

If SR of laminate ≥ 1
- 2nd ply failure load = FPF load · SR for step e), set load level to 2nd ply failure load

Otherwise
- Keep load level at FPF load discount failed plies (with SR < 1)

g) repeat step f) by progressively updating failure load and discounting failed ply or plies, until termination criterion is met, and UPF load has been determined.

Example 1: A laminate has the layup sequence of [30 / 45]. The top land bottom layers are 0.4 mm and 0.5 mm thick. Both layers have: $E_1 = 170\ GPa$, $E_2 = 20\ GPa$, $G_{12} = 5.5\ GPa$ and $v_{12} = 0.26$.

Given:
$(\sigma_1^T)_{ult} = 2990\ MPa$
$\left(\sigma_1^C\right)_{ult} = 88\ MPa$
$(\sigma_2^T)_{ult} = 45\ MPa$
$\left(\sigma_2^C\right)_{ult} = 148\ MPa$
$(\tau_{12})_{ult} = 21.5\ MPa$
$N_x = N_y = 1000\ N/m$

Determine the failure sequence of the laminate. Termination criterion is when all plies fail. Use maximum stress theory to determine SR. Discount totally the failed ply (or plies). What is the FPF (first ply failure) load? What is the LPF (last ply failure) load?

1) FPF analysis
Steps a), b): see Example in 4.3

Step c)

TOP LAYER

Top surface

global stress $= \begin{Bmatrix} 960.6 \\ 1081 \\ -78.91 \end{Bmatrix} (kPa)$

Local stress $= [T] \cdot \begin{Bmatrix} 960.6 \\ 1081 \\ -78.91 \end{Bmatrix} = \begin{Bmatrix} 922.3 \\ 1119 \\ 12.63 \end{Bmatrix} (kPa)$

$\therefore SR_1 = 3241$

$SR_2 = 40.2$

$SR_6 = 1701$

$\therefore SR = 40.2\ (2T)$

Bottom surface

Local stress $= [T] \cdot \begin{Bmatrix} 1291 \\ 1089 \\ -67.12 \end{Bmatrix} (kPa)$

$\therefore SR_1 = 2316$

$SR_2 = 41.3$

$SR_6 = 320$

$\therefore SR = 41.3\ (2T)$

$\therefore$ SR for top layer is 40.2 (2T)

BOTTOM LAYER

Top surface

$SR = 41.4\ (2T)$

Bottom surface

$SR = 40.1\ (2T)$

∴ SR for bottom later is 40.1 (2T)

d) FPF load $= (40.1) \cdot \begin{Bmatrix} 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 40.1 \\ 40.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \frac{\left(\frac{kN}{m}\right)}{\left(\frac{Nm}{m}\right)}$

e) Only top layer will be included in then next round of analysis; $N_x = N_y = 40.1\ kN/m$

2) UPF analysis

Step b):

$\{\varepsilon^0\} = \begin{Bmatrix} 8.762 \\ 21.35 \\ -21.78 \end{Bmatrix} \cdot (10^{-3})\ (m/m)$

$\{\kappa\} = \begin{Bmatrix} 28.88 \\ 70.38 \\ -71.82 \end{Bmatrix} (1/m)$

Step c):

<u>TOP LAYER</u>

<u>Top surface</u>

Local stress $= \begin{Bmatrix} -275.7 \\ -275.7 \\ 0 \end{Bmatrix} (MPa)$

$\therefore SR_1 = 0.32$

$SR_2 = 0.54$

$SR_6 => 10^{13}$

$\therefore SR = 0.32\ (1C)$

<u>Bottom surface</u>

Local stress $= [T] \cdot \begin{Bmatrix} 476.2 \\ 476.2 \\ 0 \end{Bmatrix} (MPa)$

$\therefore SR_1 = 6.3$

$SR_2 = 0.095$

$SR_6 => 10^{13}$

$\therefore SR = 0.095\ (2T)$

$\therefore SR$ for top layer is 0.095 (2T)

FPF analysis:

$Top\ layer: \dfrac{40.2(2T)}{41.3(2T)}$

$Bottom\ later: \dfrac{41.4(2T)}{40.1(2T)}$

Example 2 (probably on final)**:** A three-layered laminate is subject to an applied load of $M_x = 10\ N \cdot m/m$. Progressive failure analysis results in the following:

Failure #	Failed Layer(s)	SR	Mode of Failure
1	3	15.9	2T
2	2	1.07	2T
3	1	1.24	1T

(1) What is the UPF load?

$$M_x = (15.9)(10) = 159 \frac{Nm}{m}$$

(2) What is the UPF load, if termination criterion is fiber failure?

$$M_x = (15.9)(1.07)(1.24)(10) = 210.9612\ \frac{Nm}{m}$$

(3) What is the UPF load, if termination criterion is as long as 50% of the plies have failed?

$$M_x = (15.9)(1.07)(10) = 170.13\ \frac{Nm}{m}$$

Example 3: A 12-playered laminate has the following results from the progressive failure analysis:

Failure #	Failed Layer(s)	SR	Mode of Failure
1	1	13.9	1C
2	10	1.08	2T
3	12	0.97	6S
4	11	0.84	2T
	3	0.93	1C
5	8	0.49	2T
	9	0.54	1T

(1) What is the termination criterion used?
1T (the last failure)

(2) What is the SR (with respect to the original load) at UPF?
$(13.9)(1.08) = 15.012$

(3) Physically, in what sequence did layers fail up to the UPF?
Layer 1, 10, followed immediately by 12, 11, 3, 8 and 9;

(4) What is the LPF load?
N/A

Chapter 6

6.1: Introduction

Review of theory of isotropic beams

6.2 Symmetric beams

6.3: Non-symmetric beams

Beams are treated as special cases of plates, mainly, membrane loads are absent. Or:

$N_x = N_y = N_{xy} = 0$

The other simplifications will depend on that problem at hand. e.g.:

Symmetric beams: $\varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = 0$

Nonsymmetric beams: $\varepsilon_x^0 \neq 0,\ \varepsilon_y^0 \neq 0,\ \gamma_{xy}^0 \neq 0$

Nonsymmetric beams won't be discussed.

They bend and warp; they are also stretched or compressed, and sheared.

Symmetric beams:

$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$

Due to symmetry, $[B] = 0$

$\therefore \{N\} = [A]\{\varepsilon^0\}$

$\therefore \{M\} = [D]\{\kappa\}$

That is, because of symmetry, membrane action and bending action are decoupled.

$\because \{N\} = \{0\}$

$\therefore \{\varepsilon^0\} = \{0\}$

Nonsymmetric Beams:

$\{N\} = [A]\{\varepsilon^0\} + \{B\}\{\kappa\} = \{0\}$

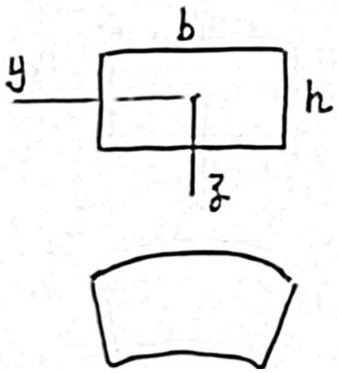
$\therefore \{\varepsilon^0\} = -[A]^{-1}[B]\{\kappa\}$

$\because \{\kappa\} \neq \{0\}$

$\therefore \{\varepsilon^0\} \neq \{0\}$

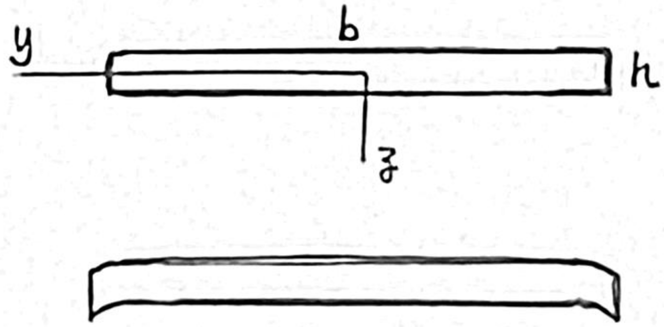
Narrow Beams versus Wide Beams

If cross-sectional dimensions are $b \cdot h$ then $b/h \geq 5$ is considered wide beams.



$\therefore \kappa_y \neq 0$
$\kappa_{xy} \neq 0$
Poisson ratio effect
(Observed in isotropic beams too)

pp. 433~440
(Beginning of 6.2 → end of Example 6.1)



$\therefore \kappa_y = 0$
$\kappa_{xy} = 0$
Edge effect
(Less in isotropic beams)

pp. 440~444
(Example 6.2)

Wide Beams
Loading: $M_x = \pm M/b$ (sign convention)
Curvatures: $\kappa_y = \kappa_{xy} = 0$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \kappa_x \\ 0 \\ 0 \end{Bmatrix}$$

$\because M_x = D_{11}\kappa_x$

$\therefore \kappa_x = \dfrac{M_x}{D_{11}}$

$M_y = D_{12}\kappa_x = \dfrac{D_{12}}{D_{11}} M_x$

$M_{xy} = D_{16}\kappa_x = \dfrac{D_{16}}{D_{11}} M_x$

Global strains at ζ:

$$[\varepsilon\} = \zeta \begin{Bmatrix} \kappa \\ 0 \\ 0 \end{Bmatrix}$$

Global stresses at ζ of layer κ:
$\{\sigma\}_\kappa = [\bar{Q}]_\kappa \{\varepsilon\}$

Flexural modulus of beam:

$$E_x^{wide} = \frac{12D_{11}}{h^3}$$

$E_x^{wide} I$ replaces EI in deflection and/or slope determination

Narrow Beams
Loading: $M_x = \pm M/b$ (sign convention)
Curvatures: $M_y = M_{xy} = 0$

$$\therefore \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = [D]^{-1} \begin{Bmatrix} M_x \\ 0 \\ 0 \end{Bmatrix}$$

Or:
$\kappa_x = D_{11}^* M_x$
$\kappa_y = D_{12}^* M_x$
$\kappa_{xy} = D_{16}^* M_x$

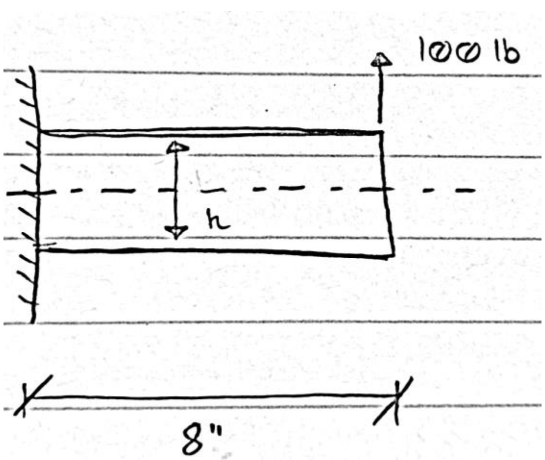
Global strains at ζ:
$\{\varepsilon\} = \zeta\{\kappa\}$

Global stresses at ζ of layer κ
$\{\sigma\}_\kappa = [\bar{Q}]_\kappa \{\varepsilon\}$

Flexural modulus of beams (same as E_x^f in 4.4):
$$E_x^{narrow} = \frac{12}{h^3 D_{11}^*}$$

$E_x^{narrow} I$ replaces EI in deflection and/or slope determination

Example: A cantilever beam has the layup sequence of [0 / 90 / 0]$_s$. The beam is 8" long and 3" wide. A load of 100 lb is applied at the free end of the beam. Each lamina is 0.1" thick, with $E_1 = 5.5\ Mpsi$, $E_2 = 1.5\ Mpsi$, $G_{12} = 0.95\ Mpsi$, and $v_{12} = 0.275$.
Determine,
(1) The maximum stress developed in the laminated beam;
(2) The deflection and slope at the free end of the beam.



Bending moment at the clamped end:
$M = 800\ lb \cdot in$
$M_x = M/b = 266.67\ lb \cdot in/in, (positive)$

$I = \frac{bh^3}{12} = 0.054\ in^4$

$\delta_{tip} = \frac{PL^3}{3EI}$

$\theta_{tip} = \frac{PL^2}{2EI}$

$b = 3.0"$

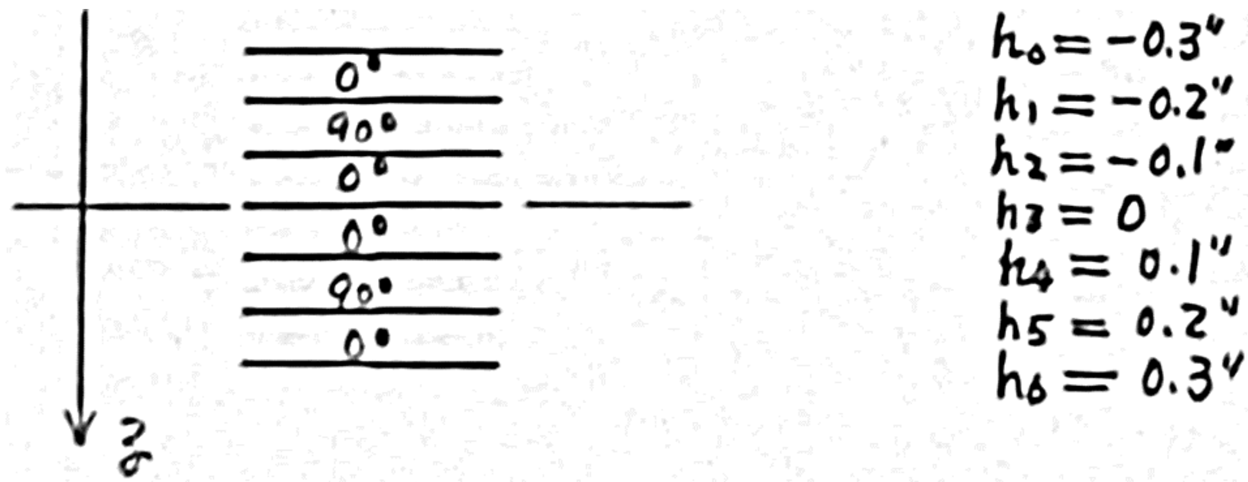
$h = 0.6"$

∴ narrow beam ($[D]^{-1}$ needed)

or wide beam ([D] needed)

$[Q]$

$[\bar{Q}]$ for 0° & 90°



$$[Q] = \begin{bmatrix} 5.616 & 0.4212 & 0 \\ & 1.523 & 0 \\ sym. & & 0.9500 \end{bmatrix} \cdot 10^6\ (psi)$$

$$[D] = \begin{bmatrix} 82.03 & 7.581 & 0 \\ & 46.63 & 0 \\ sym. & & 11.71 \end{bmatrix} \cdot 10^3\ (psi \cdot in^3)$$

$$[D]^{1} = \begin{bmatrix} 1.238 & -0.2012 & 0 \\ & 2.177 & 0 \\ sym. & & 5.848 \end{bmatrix} \cdot 10^{-5} \left(\frac{1}{psi \cdot in^3}\right)$$

$D_{11}^* = 1.238 \cdot 10^{-5} \left(\frac{1}{psi \cdot in^3}\right)$

$E_x^{narrow} = \frac{12}{h^3 D_{11}^*} = 4.488 \cdot 10^6\ (psi)$

$\kappa_x = D_{11}^* M_x = 0.003301\ (1/in)$

$\kappa_y = D_{12}^* M_x = -0.0005365\ (1/in)$

$\kappa_{xy} = D_{16}^* = D_{16}^* M_x = 0$

$$[\kappa\} = \begin{Bmatrix} 3.301 \\ -0.5365 \\ 0 \end{Bmatrix} \cdot 10^{-3} \ \left(\frac{1}{in}\right)$$

Bottom surface of laminate, $\zeta = 0.3"$

$$\{\varepsilon\} = \begin{Bmatrix} +0.9903 \\ -0.1610 \\ 0 \end{Bmatrix} \cdot 10^{3} \ \left(\frac{in}{in}\right)$$

$$\{\sigma\}_6^b = \begin{Bmatrix} +5.494 \\ +0.1705 \\ 0 \end{Bmatrix} \ (ksi)$$

$$\delta_{tip} = \frac{PL^3}{3 \cdot E_x^{narrow} I} = 0.0704"$$

$$\theta_{tip} = \frac{PL^2}{2 \cdot E_x^{narrow} I} = 0.0132 \ rad$$

$$D_{11} = 82.03 \cdot 10^3 \ (psi \cdot in^3)$$

$$E_x^{wide} = \frac{12 D_{11}}{h^3} = 4.557 \cdot 10^6 \ (psi)$$

$$\kappa_x = \frac{M_x}{D_{11}} = 0.003251 \ \left(\frac{1}{in}\right)$$

$$\{\kappa\} = \begin{Bmatrix} 3.251 \\ 0 \\ 0 \end{Bmatrix} \cdot 10^{-3} \ \left(\frac{1}{in}\right)$$

Bottom surface of laminate, $\zeta = 0.3"$

$$\{\varepsilon\} = \begin{Bmatrix} +0.9753 \\ 0 \\ 0 \end{Bmatrix} \cdot 10^{3} \ \left(\frac{in}{in}\right)$$

$$\{\sigma\}_6^b = \begin{Bmatrix} +5.494 \\ +0.4108 \\ 0 \end{Bmatrix} \ (ksi)$$

$$\delta_{tip} = \frac{PL^3}{3 \cdot E_x^{wide} I} = 0.0693"$$

$$\theta_{tip} = \frac{PL^2}{2 \cdot E_x^{wide} I} = 0.0130 \ rad$$

Emerging Composites

1. Carbon-carbon composites
2. Bio-composites
3. Nano-composites
4. Functionally graded materials (FGMs)

<u>1. Carbon-carbon composites</u>

1) Carbon fibers in a carbon matrix, hence an ultra-high temperature (up to 3300 °C) composite
2) Abrasion resistant
3) Self-lubricating
4) Aircraft brakes, steam and gas turbine engines, heat shields, rocket nozzles, nose cones, etc.
5) Can be machines, drilled, sawed
6) Lightweight

<u>2. Bio-composites</u>

1) By definition, a bio-product is one that is derived from renewable resources, stable in its intended lifetime, and bio-degradable after disposal in composting condition.
2) A bio-composite consists of biofibers and biomatrix, and is expected to be bio-degradable.
3) Biofibers
    - Wood "fibers": short fibers typically, or wood flour
    - Non-wood fibers: kenaf, flax, jute, hemp coir, and sisal (and straw and grass)

    Ranking of non-wood fibers in terms of tensile modulus and tensile strength:
    Flax (about 33% of E-glass's)
    Kenaf
    Hemp
    Sisal
    Jute
    Coir (about 5% of E-glass's)
4) Bio-polymers (Bio-resins)
    Bio-polyester (microbial polyester)
    Soy-based plastics
    Starch plastics

<u>3. Nano-composites</u>

1) Composite filled with nano-sized ($10^{-9}\ m$) particles
2) Platelets and nanotubes
3) Carbon nanotubes (CNTs):
    Young's modulus $\sim 1000\ GPa$
    Tensile strength $>\ 30\ GPa$

    Compared with PAN-based Carbon fibers:
    Young's modulus $250 \sim 550\ GPa$
    Tensile strength $1.9 \sim 6\ GPa$

4) Small amount of nano-particles will provide significant improvement in a variety of properties.
5) Applications:
   Structural components of electronic portable devices
   Auto accessories, both interior and exterior

4. Functionally Graded Materials

- First conceptualized in mid-1980 when a thermal barrier capable of withstanding a surface temperature of 2000 K ($\sim 1727\ °C$) and a surface temperature gradient of $1000\ K$ ($\sim 727\ °C$) across a section of less than $10\ mm$ was needed.
- Achieved by varying volume (or weight) fractions gradually over the volume of material.
- FGMs are not homogeneous, nor are they isotropic; in other words, E_x depends on location (x, y, z) and angle θ, for instance.
- Almost ready for commercialization.
- Applications:
  Aerospace: high thermal gradient

Final Exam Review

Chapter 1
Same as midterm (even though the course outline says it's not on the final exam…)

Chapter 2
2.3 Independent mechanical properties vs. Types of materials
e.g. Orthotropic materials, 9 constants
transversely isotropic materials, 5 constants

Chapter 3
3.2 V_f V_m W_f W_m void content
a few fibers + a few matrices + voids
V_f' V_m' V_{fmax} RVE
When an equation in the text is only valid for zero void content
3.3 Isotropic fibers + isotropic matrix
Transversely isotropic fibers + isotropic matrix

Mechanics of materials approach
Halpin-Tsai
Elasticity

3.4 $(\sigma_1^T)_{ult}$: fibers-fail-first
matrix-fails-first
equations to use, the if's
$\left(\sigma_1^C\right)_{ult}$: failure modes
$(\sigma_2^T)_{ult}$ $\left(\sigma_2^C\right)_{ult}$ $(\tau_{12})_{ult}$

Chapter 2 (again)
2.4 $[Q]$, $[S]$
2.5 $[T]$, $[\bar{Q}]$, $[\bar{S}]$
2.6 E_x E_y G_{xy} v_{xy} m_x m_y
Evaluation
Application
e.g. Given any one, find one of the remaining
Global stresses
Global strains
Local stresses
Local strains
Physical meanings of engineering constants (probably m_x and m_y)
2.8 Strength/Failure Theories of a Lamina
Based on local stresses $(\varepsilon_1, \varepsilon_2, \gamma_{12})$ or local strains $(\sigma_1, \sigma_2, \tau_{12})$
Max stress – don't' compare well with experimental data, but indicate mode of failure
Max strain – same as above
Tsai-Hill:

- Original
- Modified

Tsai-Wu:

- 3 forms on H_{12}
- Tsai-Hill, Hoffman, von Mises-Hencky (we probably get to pick what we want to use)

Tsai-Hill and Tsai-Wu:

- Compare well with experimental data but don't indicate mode of failure

Chapter 4

4.2 laminate code – shortest possible notation description of a laminate

4.3 $[ABD]$

membrane and bending coupled

membrane and bending uncoupled

Given $[ABD]^{-1}$ and $\begin{Bmatrix} N \\ M \end{Bmatrix}$

$\rightarrow \{\varepsilon^0\}\ \{\kappa\}$

$\rightarrow \{\varepsilon\}_{global}$ at certain location (i.e. ζ)

$\rightarrow \{\sigma\}_{global} \rightarrow$ plots of stresses across the thickness direction

$\rightarrow \{\sigma\}_{global} \rightarrow SR$

4.4 Given $[ABD]^{-1}$

$\rightarrow E_x\ \ E_y\ \ \ldots\ \ E_x^f\ \ E_y^f \ldots$

(Engineering constants of a laminate)

given $[ABD]$

$\rightarrow r_N\ \ r_M\ \ r_B$

Chapter 5

5.2 $[ABD]$ matrix when laminate is, for example

1) symmetric

2) quasi-isotropic

3) specially orthotropic

5.3 Progressive failure

FPF, UPF, LPF

Termination criteria

Discount on failed ply (plies)

Chapter 6

6.2 narrow vs. wide beams

Bending moment M (units, signs)

Moment resultant M_x

$E_x^{wide} I$ (to replaced EI [for isotropic beams] for deflection, and slope evaluations)

$E_x^{narrow} I$

Plates: differential elements for $\sum F_y = 0$

differential elements for$\sum M_x = 0;\ \sum M_z = 0$

(g) (h)
essential boundary conditions (satisfied?)
natural boundary conditions (satisfied?)
Given $w_o(x, y) \rightarrow M_x \;\; M_y \;\; M_{xy}$
$[ABD]^{-1}$: given if needed
Beam deflection table given from Shigley's

Emerging Composites
Material from class notes (shouldn't be anything too crazy)